Unstable Continuous Spectrum and Ballooning Mode Growth Rates in Closed-Line Magnetohydrostatic Equilibria

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Abstract

In an equilibrium with all magnetic field lines closed, every such line gives rise to part of the continuous spectrum, depending on the orientation of the singularity in the eigenfunction, and given by the eigenvalues of a fourth-order ordinary differential operator. Unlike the Alfven- and Cusp continua which correspond to the special orientation given by the pressure gradient, these new continua may be unstable. The criterion for their stability is identical with that for stability to the so-called "ballooning modes". Therefore, the growth rates of these modes, being in the continuum, are obtainable from a fourth (rather than second)-order problem.

1. Introduction

The spectrum of ideal magnetohydrodynamics unifies various aspects of the problem of magnetic plasma confinement, such as heating, stability, bifurcation, islation, diffusion, and other non-ideal effects. The continuous part of this spectrum is of particular interest to the stability problem. Stable continua (viz., the Alfven- and Cusp continua [1 - 3]) are related [1, 4, 5] to stability in that unstable point eigenvalues accumulate at their edge (viz., the origin) whenever the criterion [6 - 11] for stability to perturbations localized at a pressure surface is violated. The corresponding growth rates are small, and the eigenfunctions have many radial nodes. Unstable continua arising from dense sets of point eigenvalues were shown [12] to be possible in closed-line equilibria. The corresponding growth rates are finite, and the eigenfunctions have many azimuthal nodes. In the special case of axial symmetry and purely poloidal magnetic fields the criterion for the stability of these continua was shown [5] to be identical with the so-called "ballooning mode criterion" (which happens to be the criterion for absolute stability in this case [13]).

In the present paper we show that the same is true in arbitrary closed-line equilibria. Thus, we derive the equations governing the continua corresponding to

arbitrarily oriented singularities in the eigenfunctions. These equations constitute an eigenvalue problem with a fourth-order ordinary differential operator. The criterion for the stability of these continua is then shown to be identical with the closed-line version [14] of the "ballooning mode criterion". Therefore, the determination of the corresponding growth rates requires solving a fourth-order system of ordinary differential equations.

We anticipate that a similar relationship exists between the continuous spectrum and "ballooning modes" in sheared systems [15, 16]. In particular, we anticipate that the determination of ballooning mode growth rates in sheared systems also requires solving a fourth-order problem (rather than the second-order problem which was obtained [15] by neglecting the longitudinal kinetic energy, and approximation which, although not affecting stability, severely distorts the spectrum).

2. Equations of motion

When linearized about a static equilibrium,

$$\nabla P + \vec{B} \times \mathrm{curl}\, \vec{B} = 0, \quad \mathrm{div}\, \vec{B} = 0, \tag{1}$$

where $\vec{B}$ and P are the magnetic field and the plasma pressure, and when Fourier decomposed in time, the equations of ideal magnetohydrodynamics can be written [17] in the particularly useful form

$$\pi / B_*^2 + M(\vec{u}) = 0, \quad -M^*(\pi) + S(\vec{u}) = \omega^2 \rho \vec{u}. \tag{2}$$

Here, $\vec{u}$ is the perturbing velocity, π is the perturbing total pressure, $B_*^2 = B^2 + \gamma P$ (where $B = |\vec{B}|$, and γ is the ratio of the specific heats), ρ is the equilibrium mass density, and ω is the frequency. The operators S, M, and M^* are given by

$$S(\vec{u}) = (\vec{a} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{a}, \tag{3}$$

$$M(\vec{u}) = \mathrm{div}\, \vec{u} + (\vec{u} \cdot \nabla P + \vec{B} \cdot \vec{c}) / B_*^2, \tag{4}$$

$$M^*(\pi) = -\nabla \pi + 2(\pi / B_*^2)(\vec{B} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla \pi / B_*^2)\vec{B}, \tag{5}$$

where

$$\vec{a} = \vec{c} - (\vec{u}\cdot\nabla P + \vec{B}\cdot\vec{c})\vec{B}/B_*^2, \tag{6}$$

$$\vec{c} = (\vec{u}\cdot\nabla)\vec{B} - (\vec{B}\cdot\nabla)\vec{u}. \tag{7}$$

With the boundary condition that $\vec{u}$ be tangential, M^* is the adjoint of the operator M, and S is selfadjoint, provided $\vec{B}$ is tangential, too. As usually, the discrete spectrum consists of those values of ω for which the system (2) has square-integrable solutions $\vec{u}$, and the continuous spectrum corresponds to singular solutions.

In arbitrary curvilinear coordinates (x^1, x^2, x^3) we write $\vec{u} = u^i \vec{e}_i$, where $\vec{e}_i$ is a covariant basis vector, and the summation convention is used. The system (2) then takes the form

$$\pi/B_*^2 + M_j u^j = 0, \quad -M_i^* \pi + S_{ij} u^j = \omega^2 \rho g_{ij} u^j, \tag{8}$$

where $S_{ij}\lambda = \vec{e}_i \cdot S(\lambda \vec{e}_j)$, $M_j \lambda = M(\lambda \vec{e}_j)$, $M_i^* \lambda = \vec{e}_i \cdot M^*(\lambda)$, and $g_{ij} = \vec{e}_i \cdot \vec{e}_j$ is the metric tensor.

Explicitly,

$$S_{ij} = \vec{B}\cdot\nabla\left[\vec{e}_i\cdot\left(\vec{\alpha}_j - \frac{\vec{B}\cdot\vec{\alpha}_j + P_{/j} + \gamma P \operatorname{div}\vec{e}_j}{B_*^2}\vec{B}\right)\right.$$

$$\left. -\left(g_{ij} - \frac{B_i B_j}{B_*^2}\right)\vec{B}\cdot\nabla\right] + \left(\vec{\alpha}_i + \vec{e}_i \times \operatorname{curl}\vec{B} - \vec{B}\operatorname{div}\vec{e}_i\right)$$

$$\cdot\left[\vec{\alpha}_j - \frac{\vec{B}\cdot\vec{\alpha}_j + P_{/j} + \gamma P \operatorname{div}\vec{e}_j}{B_*^2}\vec{B} - \left(\vec{e}_j - \frac{B_j}{B_*^2}\vec{B}\right)\vec{B}\cdot\nabla\right], \quad (9)$$

$$M_i = \frac{\partial}{\partial x^i} - \frac{B_i}{B_*^2}\vec{B}\cdot\nabla + \frac{\vec{B}\cdot\vec{\alpha}_i + P_{/i} + \gamma P \operatorname{div}\vec{e}_i}{B_*^2}, \quad (10)$$

$$M_i^* = -\frac{\partial}{\partial x^i} + \vec{B}\cdot\nabla\frac{B_i}{B_*^2} + \frac{\vec{B}\cdot\vec{\alpha}_i + P_{/i} + \gamma P \operatorname{div}\vec{e}_i}{B_*^2}, \quad (11)$$

where $B_i = \vec{e}_i\cdot\vec{B}$, $P_{/i} = \partial P/\partial x^i$, and $\vec{\alpha}_i = \operatorname{curl}(\vec{B}\times\vec{e}_i)$.

Assuming now that all field lines of $\vec{B}$ are closed, we can choose the coordinates x^i such that both x^1 and x^2 are constant along the field lines while x^3 increases, and such that physical quantities have simple periodicity properties. Specifically, we choose Hamada's coordinates [18, 9, 19] relative to a poloidal cut surface which is not intersected by field lines. Thus, we put $(x^1, x^2, x^3) = (V, \Theta, \zeta)$, where V is the volume of a pressure surface, Θ and ζ are angle-like variables with period unity (each pressure surface is mapped onto a unit square in the (Θ, ζ) plane), the Jacobian is unity, and $\vec{B} = q^{-1}\vec{e}_\zeta$, where $q(V) = \oint dl/B$. The quantities Θ and ζ are

unique only within additive functions of V corresponding to the arbitrariness of the choice of the origin in each pressure surface. This is significant because it enables us to give the surfaces Θ = const (which form a family of magnetic surfaces different from the pressure surfaces) any orientation relative to the pressure gradient. We now have div $\vec{e}_i = 0$, and $\vec{\alpha}_V = -(\dot{q}/q)\vec{B}$, $\vec{\alpha}_\theta = \vec{\alpha}_\zeta = 0$ (dots denote derivatives with respect to V). Therefore, the system (8) becomes particularly simple in these coordinates.

3. Continuous spectrum

Only the operators M_V and M_V^* contain derivatives with respect to V, and only the operators M_θ and M_θ^* contain derivatives with respect to Θ. This leads us to write the system (8) such that either of the derivatives $\partial/\partial V$ or $\partial/\partial\theta$ is isolated. The first possibility yields the familiar Alfven- and Cusp continua; the second possibility yields the new continua which we are interested in (and which correspond to the dense sets of point eigenvalues which one obtains if one assumes a θ-dependence in the form $\exp im\int d\theta\,\varphi$ with large m).

Thus, we group the unknowns π and u^θ into a 2-vector $\vec{x}$, and the remaining unknowns u^V and u^ζ into a 2-vector $\vec{y}$. The system (8) then becomes

$$\left(\varepsilon\frac{\partial}{\partial\theta}+\tau\right)\vec{x}+\mu\vec{y}=0,\quad \nu\vec{x}+\sigma\vec{y}=0, \tag{12}$$

where

$$\varepsilon=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{13}$$

$$\tau=\begin{pmatrix}1/B_*^2 & M_\theta-\frac{\partial}{\partial\theta}\\ -M_\theta^*-\frac{\partial}{\partial\theta} & S_{\theta\theta}-\omega^2\rho g_{\theta\theta}\end{pmatrix}, \tag{14}$$

$$\mu=\begin{pmatrix}M_V & M_\zeta\\ S_{\theta V}-\omega^2\rho g_{\theta V} & S_{\theta\zeta}-\omega^2\rho g_{\theta\zeta}\end{pmatrix}, \tag{15}$$

$$\nu = \begin{pmatrix} -M_V^* & S_{V\theta} - \omega^2 \rho g_{V\theta} \\ -M_\zeta^* & S_{\zeta\theta} - \omega^2 \rho g_{\zeta\theta} \end{pmatrix}, \tag{16}$$

$$\sigma = \begin{pmatrix} S_{VV} - \omega^2 \rho g_{VV} & S_{V\zeta} - \omega^2 \rho g_{V\zeta} \\ S_{\zeta V} - \omega^2 \rho g_{\zeta V} & S_{\zeta\zeta} - \omega^2 \rho g_{\zeta\zeta} \end{pmatrix}. \tag{17}$$

Both τ and σ are ordinary differential operators in the coordinate ζ; μ and ν also contain derivates with respect to V. Elimination of $\vec{y}$ from the system (12) now yields an equation for $\vec{x}$ which is singular whenever the operator σ has no bounded inverse. Therefore, the solution has a singularity wherever the subsystem $\sigma\vec{y} = 0$ has a non-trivial periodic solution, and the eigenvalues ω^2 of this equation belong to the continuous spectrum.

When written out, the sub-system $\sigma\vec{y} = 0$ is

$$s\vec{y} = \omega^2 \rho g \vec{y}, \tag{18}$$

where

$$S = \begin{pmatrix} -\vec{B}\cdot\nabla e_W \vec{B}\cdot\nabla + F & -(\vec{B}\cdot\nabla e_{V\zeta} + G)\vec{B}\cdot\nabla \\ -\vec{B}\cdot\nabla(e_{\zeta V}\vec{B}\cdot\nabla - G) & -\vec{B}\cdot\nabla e_{\zeta\zeta}\vec{B}\cdot\nabla \end{pmatrix}, \tag{19}$$

$$g = \begin{pmatrix} g_{vv} & g_{v\zeta} \\ g_{\zeta v} & g_{\zeta\zeta} \end{pmatrix}, \tag{20}$$

with $e_{ij} = g_{ij} - B_i B_j / B_*^2$, and

$$F = \frac{\gamma P B^2 \dot{q}^2}{B_*^2 q^2} + \frac{\dot{P}\dot{q}}{q}\left(2\frac{B^2}{B_*^2} - 1\right) - \frac{\dot{P}^2}{B_*^2}$$

$$-\vec{B}\cdot\nabla\left[\frac{B_v}{B_*^2}\left(\frac{\gamma P \dot{q}}{q} + \dot{P}\right)\right], \tag{21}$$

$$G = \frac{\gamma P}{B_*^2}\left(q\dot{P} - B^2\dot{q}\right). \tag{22}$$

Since S any g are self-adjoint, the eigenvalues ω^2 are real. They depend upon the field line (through the parameters V and Θ), and also upon the orientation of the singularity (through the choice of the coordinate Θ).

We remark that for $\gamma = 0$ (isobaric motions), $G = 0$, $F = \dot{P}A$, with

$$A = -2\vec{e}_v \cdot \vec{K} \tag{23}$$

($\vec{K}$ is the curvature vector of a field line), and $e_{v\zeta} = e_{\zeta v} = e_{\zeta\zeta} = 0$, but $e_{vv} = |\nabla\Theta|^2 / q^2 B^2$. Therefore, the fourth-order system (18) reduces to the second-order equation

$$\left(-\vec{B}\cdot\nabla\frac{|\nabla\Theta|^2}{q^2 B^2}\vec{B}\cdot\nabla + \dot{P}A\right)\mu^v = \omega^2 g\, g_{vv}\mu^v \tag{24}$$

in this case.

4. Stability

The stability of the continuum ($\omega^2 \geq 0$) is equivalent to the positivity of S , or explicitly

$$\left\langle S_{vv}(\mu^v)^2+(S_{v\zeta}+S_{\zeta v})\mu^v\mu^\zeta+S_{\zeta\zeta}(\mu^\zeta)^2\right\rangle \geq 0 \qquad (25)$$

where $\langle\cdots\rangle=\int d\zeta\cdots=q^{-1}\oint d\ell\cdots/B$ is the usual field line average. Minimizing the functional (25) with respect to μ^ζ leads to the Euler equation

$$\vec{B}\cdot\nabla(G\mu^v-e_{v\zeta}\vec{B}\cdot\nabla\mu^v+e_{\zeta\zeta}\vec{B}\cdot\nabla\mu^\zeta)=0, \qquad (26)$$

which can be solved to yield

$$\vec{B}\cdot\nabla\mu^\zeta=(G\mu^v-e_{v\zeta}\vec{B}\cdot\nabla\mu^v)/e_{\zeta\zeta}$$

$$-\langle(G\mu^v-e_{v\zeta}\vec{B}\cdot\nabla\mu^v)/e_{\zeta\zeta}\rangle(e_{\zeta\zeta}\langle e_{\zeta\zeta}^{-1}\rangle)^{-1}. \qquad (27)$$

Upon substitution into (25), we find

$$\left\langle\frac{|\nabla\theta|^2}{q^2B^2}(\vec{B}\cdot\nabla\mu^v)^2+\dot{P}A(\mu^v)^2\right\rangle+\frac{\gamma P\langle A\mu^v\rangle^2}{1+\gamma P\langle B^{-2}\rangle}\geq 0. \qquad (28)$$

The criterion (28) is identical with the closed-line version [14] of the ballooning mode criterion.

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