Low-Shear Magnetohydrodynamic
Stability

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Abstract

For arbitrary toroidal magnetohydrostatic equilibria, a sufficient stability criterion is derived whose evaluation requires no more than solving a one-dimensional problem at every magnetic field line. This criterion reduces to a previously known one in the presence of shear, but is less restrictive if all field lines are closed. Comparison with a necessary criterion shows that the stability threshold is discontinuous in the limit of zero shear, low-order rational values of the rotation number being more favorable than neighboring irrational ones.

Magnetohydrodynamic stability, although one of the most extensively studied subjects in the theory of magnetic plasma confinement, is not yet fully understood. One of its puzzles arises in the low shear limit: It is well-known that the criteria for local stability take different forms depending on the topology of the magnetic field lines, being more optimistic for closed-line equilibria than they are for neighboring ones with small shear[1,2]. The question which requirement is relevant in practice for low-shear systems, that of low-shear stability or that of closed-line stability, has been decided in favor of the latter[1]. However, this question itself is not vital as long as one has not shown that the stability threshold is discontinuous. Since local stability need not be sufficient for stability, this requires considering sufficient criteria along with the necessary ones. In the present paper we derive a sufficient criterion which is stronger than previously known ones[3,4] and which shows that the stability threshold indeed behaves discontinuously in the limit of a constant rational rotation number.

The equations of ideal magnetohydrodynamics, when linearized about a static equilibrium subject to

$$\nabla P + \vec{B} \times \mathrm{curl}\, \vec{B} = 0, \tag{1}$$

$$\mathrm{div}\, \vec{B} = 0, \tag{2}$$

take the form

$$i\omega p = -\vec{u}\cdot\nabla P - \gamma P\, \mathrm{div}\, \vec{u}, \tag{3}$$

$$i\omega \vec{b} = \mathrm{curl}\, (\vec{u} \times \vec{B}), \tag{4}$$

$$i\omega \rho \vec{u} = -\nabla p + \vec{b} \times \mathrm{curl}\, \vec{B} + \vec{B} \times \mathrm{curl}\, \vec{b}, \tag{5}$$

if a time dependence $\exp i\omega t$ is assumed. Here, P and $\vec{B}$ are the unperturbed plasma pressure and magnetic field while the corresponding little letters denote perturbing quantities, ρ is the unperturbed mass density, $\vec{u}$ is the perturbing velocity, and γ is the ratio of the specific heats. (The equation for the perturbing mass density is not included because it decouples from Eqs. (3) - (5)).

The energy principle[5,6] for exponential stability arises from the system (3) - (5) if one eliminates p and $\vec{b}$ to obtain

$$\omega^2 \rho \vec{u} = \vec{F}(\vec{u}), \tag{6}$$

where

$$\vec{F}(\vec{u}) = -\nabla(\vec{u}\cdot\nabla P + \gamma P \,\mathrm{div}\,\vec{u}) \\ + \mathrm{curl}(\vec{u}\times\vec{B})\times \mathrm{curl}\,\vec{B} + \vec{B}\times \mathrm{curl}\,\mathrm{curl}(\vec{u}\times\vec{B}). \tag{7}$$

With the boundary condition $u_n = 0$ (appropriate to a toroidal, perfectly conducting, rigid wall at which $B_n = 0$) the operator $\vec{F}$ is self-adjoint. Therefore, its spectrum is real and exponential stability (viz., $\omega^2 \geq 0$) is equivalent to

$$(\vec{u}, \vec{F}(\vec{u})) \geq 0, \tag{8}$$

where $(\vec{u}_1, \vec{u}_2) = \iiint d^3r\, \vec{u}_1^{\,*}\cdot\vec{u}_2$ is the usual scalar product in the Hilbert-space of square-integrable vector fields (an asterisk denotes the complex conjugate).

We now split the operator $\vec{F}$ into two parts, $\vec{F} = \vec{S} + \vec{P}$, such that $\vec{P}$ is positive while $\vec{S}$ has a simpler structure than $\vec{F}$. This splitting arises in a natural way from the equations of motion in their primitive form (3) - (5) [rather than from Eq. (6)] if one writes these as a system for the unknowns $\vec{u}$ and $\pi = i\omega(p + \vec{B}\cdot\vec{b})$. Thus, replacing p by π , we rewrite these equations as

$$\pi = i\omega\vec{B}\cdot\vec{b} - \vec{u}\cdot\nabla P - \gamma P \,\mathrm{div}\,\vec{u}, \tag{9}$$

$$i\omega\vec{b} = -(\vec{u}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{u} - \vec{B}\,\mathrm{div}\,\vec{u}, \tag{10}$$

$$\omega^2\rho\vec{u} = \nabla\pi - i\omega[(\vec{b}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{b}]. \tag{11}$$

Next, we dot Eq. (10) with $\vec{B}$, and substitute the result into Eq. (9) to obtain

$$\frac{\pi + \vec{B}\cdot[(\vec{u}\cdot\nabla)\vec{B} - (\vec{B}\cdot\nabla)\vec{u}] + \vec{u}\cdot\nabla P}{B_*^2} + \mathrm{div}\,\vec{u} = 0, \tag{12}$$

where $B_*^2 = |\vec{B}|^2 + \gamma P$. Then, solving Eq. (12) for $\mathrm{div}\,\vec{u}$, and substituting this into Eq. (10), we find

$$i\omega\vec{b} = \frac{\pi + \vec{B}\cdot[(\vec{u}\cdot\nabla)\vec{B} - (\vec{B}\cdot\nabla)\vec{u}] + \vec{u}\cdot\nabla P}{B_*^2}\vec{B} - (\vec{u}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{u}. \tag{13}$$

The system (3)-(5) is equivalent to the system (11)-(13). If $\vec{b}$ is eliminated substituting Eq. (13) into Eq. (11), this system takes the form

$$M(\vec{u}) + \pi/B_*^2 = 0, \quad \vec{S}(\vec{u}) - \vec{M}^*(\pi) = \omega^2\rho\vec{u}, \tag{14}$$

where $\vec{S}$ is a self-adjoint operator, and $\vec{M}^*$ is the adjoint of the operator M. Finally, eliminating π yields Eq. (6) with $\vec{F} = \vec{S} + \vec{P}$ and $\vec{P} = \vec{M}^* B_*^2 M$. Since $(\vec{u}, \vec{P}(\vec{u})) = \| B_* M(\vec{u})\|^2 \geq 0$,

$$(\vec{u}, \vec{S}(\vec{u})) \geq 0 \tag{15}$$

is a sufficient criterion for stability.

The operator $\vec{S}$ is given by

$$\vec{S}(\vec{u}) = (\vec{a} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{a}, \tag{16}$$

$$\vec{a} = \vec{c} - (\vec{B} \cdot \vec{c} + \vec{u} \cdot \nabla P)\vec{B}/B_*^2, \tag{17}$$

$$\vec{c} = (\vec{u} \cdot \nabla)\vec{B} - (\vec{B} \cdot \nabla)\vec{u}, \tag{18}$$

thus being an ordinary differential operator with derivatives only along magnetic field lines. To exploit this, we write $(\vec{u}, \vec{S}(\vec{u})) = \int dV \oiint d^2\sigma\, \vec{u}^* \cdot \vec{S}(\vec{u})/|\nabla V|^2$, where V is the volume enclosed by a pressure surface while $d^2\sigma$ is a surface element, and conclude that the criterion (15) is satisfied if, and only if

$$\oiint d^2\sigma\, \vec{u}^* \cdot \vec{S}(\vec{u})/|\nabla V|^2 \geq 0 \tag{19}$$

at every pressure surface. To further evaluate this, we now assume that there are closed field lines, thus ignoring the case of a constant irrational rotation number. This leaves us with two cases: Either the rotation number depends on V, and there is a dense set of rational surfaces at which the field lines are

closed; or the rotation number is a rational constant, and all surfaces are rational. In either case the condition (19) is satisfied at every surface if, and only if, it is satisfied at every rational surface. At such a surface, we write $\oiint d^2\sigma\, \vec{u}^*\cdot\vec{S}(\vec{u})/|\nabla V|^2$ $= \int d\theta \langle \vec{u}^*\cdot\vec{S}(\vec{u})\rangle$,
where θ is a coordinate labeling the field lines, and

$$\langle\cdots\rangle = \oint \frac{dl}{|\vec{B}|}\cdots \Big/ \oint \frac{dl}{|\vec{B}|} \tag{20}$$

is the usual field line average. As a consequence, our sufficient criterion is satisfied if, and only if

$$\langle \vec{u}^*\cdot\vec{S}(\vec{u})\rangle \geq 0 \tag{21}$$

at every closed field line. The evaluation of this condition amounts to solving a sixth-order system of ordinary differential equations at every closed field line.

A more illuminating version of the criterion (21) is obtained by minimizing the functional $\langle \vec{u}^*\cdot\vec{S}(\vec{u})\rangle$ with respect to the components of $\vec{u}$ in a pressure surface, thus expressing it in terms of the normal component. This calculation, though rather tedious, is straightforward and yields

$$\left\langle \frac{|\vec{B}\cdot\nabla X|^2}{|\nabla P|^2} + f|X|^2 \right\rangle + \tag{22}$$

$$\frac{|\langle gX\rangle|^2 + \gamma P\left[\langle fX\rangle\langle (af-bg)X\rangle^* - \langle gX\rangle\langle (bf-cg)X\rangle^*\right]}{a+\gamma P(ac-b^2)} \geq 0.$$

Here, $X = \vec{u}\cdot\nabla P$ is the only remaining test function, and

$$f = \frac{\vec{B}\times\nabla P}{|\nabla P|^2}\cdot \mathrm{curl}\, \frac{\vec{J}\times\nabla P}{|\nabla P|^2} - \frac{|\vec{J}|^2}{|\nabla P|^2}, \tag{23}$$

$$g = \frac{\vec{B}\times\nabla P}{|\nabla P|^2}\cdot \mathrm{curl}\, \frac{\vec{B}\times\nabla P}{|\nabla P|^2} - \frac{\vec{J}\cdot\vec{B}}{|\nabla P|^2}, \tag{24}$$

$$a = \left\langle \frac{|\vec{B}|^2}{|\nabla P|^2} \right\rangle, \; b = \left\langle \frac{\vec{J}\cdot\vec{B}}{|\nabla P|^2} \right\rangle, \; c = \left\langle \frac{|\vec{J}|^2}{|\nabla P|^2} \right\rangle, \tag{25}$$

where $\vec{J} = \mathrm{curl}\,\vec{B}$ is the current density. The evaluation of the condition (22) amounts to solving a second-order ordinary integro-differential equation at every closed field line.

For constant test functions, $\vec{B}\cdot\nabla X = 0$, the condition (22) reduces to

$$\langle f\rangle + \frac{\langle g\rangle^2 + \gamma P\left(a\langle f\rangle^2 - 2b\langle f\rangle\langle g\rangle + c\langle g\rangle^2\right)}{a+\gamma P(ac-b^2)} \geq 0. \tag{26}$$

Being necessary for a sufficient criterion to be satisfied, this condition, in general, is neither necessary nor sufficient for stability. However, it turns out to be identical with the necessary and sufficient criterion for stability to perturbations which are localized at a pressure surface[2] if all field lines are closed, thus being necessary for stability in this case. (In the presence of shear, it is not related to any known necessary stability criterion.)

To compare the criterion (30) with previously known sufficient criteria, we note that the second term is greater than $\gamma P|\langle \rho X\rangle|^2/(1+\gamma Pc)$, which, in turn, is positive. Therefore, either of the two conditions

$$\left\langle \frac{|\vec{B}\cdot\nabla X|^2}{|\nabla P|^2} + \rho|X|^2 \right\rangle + \frac{\gamma P|\langle \rho X\rangle|^2}{1+\gamma Pc} \geq 0 \tag{27}$$

and

$$\left\langle \frac{|\vec{B}\cdot\nabla X|^2}{|\nabla P|^2} + \rho|X|^2 \right\rangle \geq 0 \tag{28}$$

is sufficient for stability, but the condition (28) (which was previously derived in Ref. 3) is more pessimistic than the condition (27), and the latter (which was previously derived in Ref. 4) is more pessimistic than our new condition (22). However, our condition reduces to the condition (27) in equilibria with reflection symmetry[7] (in which all field lines are closed), and it even reduces to the condition (28) in equilibria with shear, so that a genuine improvement has been achieved only for closed-line equilibria without reflection symmetry.

To substantiate our claim regarding reflection symmetry, we note that both f and $|\nabla P|^2$ are even. Therefore, the condition (27) is violated for some even test function X if it can be violated at all. Since g is odd, $\langle gX \rangle = 0$ for this test function, and the condition (22) is also violated because $b = 0$ ($\vec{J}\cdot\vec{B}$ is odd).

We do not give here a general proof of our claim regarding shear, but merely demonstrate it for topologically toroidal circular cylinders. Thus, we assume that scalar equilibrium quantities are functions of r, but not of φ and z, where (r, φ, z) are cylindrical coordinates, and consider test functions of the form $X = x(r)\exp i(m\varphi + nz/R)$, where m and n are integers, and $2\pi R$ is the length of the cylinder. Then, $\vec{B}\cdot\nabla X = iHX$, where $H = mB_\varphi/r + nB_z/R$, and $\langle X \rangle = X$ if $H = 0$, but $\langle X \rangle = 0$ if $H \neq 0$. Therefore, our criterion (22) requires that

$$f + \frac{g^2 + \gamma P(af^2 - 2bfg + cg^2)}{a + \gamma P(ac - b^2)} \geq 0 \tag{29}$$

and

$$f + H^2/|\nabla P|^2 \geq 0 \tag{30}$$

for $H \neq 0$, while the criterion (28) requires that (30) holds for all values of H . At the rational surface with rotation number $\mu = RB_\varphi/rB_z = K/L$ (K and L are integers with no common divisor), $H = (mK + nL)B_z/RL$, so that H takes on all multiples of the quantity B_z/RL . Hence our criterion (22) is equivalent to (29) and (30) with $H = B_z/RL$, while the criterion (28) is equivalent to $f \geq 0$. If the equilibrium has shear, we consider an arbitrary irrational surface $r = r_0$, along with a sequence of rational surfaces $r = r_\ell$ such that $r_\ell \rightarrow r_0$ as $\ell \rightarrow \infty$. Since $L \rightarrow \infty$ as $\ell \rightarrow \infty$, the criterion (22) thus requires that $f \geq 0$ at every irrational surface, and hence everywhere, as does the criterion (28). In contrast, L has the same value on all surfaces if μ is a rational constant, and the criterion (22) retains the form (29) and (30), thus being more optimistic than the criterion (28).

Let us finally demonstrate that the stability threshold is discontinuous in the low-shear limit. We consider again circular cylinders, but assume now that μ is a constant. If μ is irrational, $dP/d\tau \geq 0$ is necessary for stability[1]. (The same is true in the limit $d\mu/d\tau \to 0$, no matter whether μ approaches an irrational value or not). If μ is rational, our sufficient criterion, from (29) and (30) with $H = B_z/RL$, takes the form

$$\frac{dP}{d\tau} + \frac{2\gamma P \mu^2 \tau B_z^2}{B_*^2 R^2} \geq 0 \tag{31}$$

and

$$\frac{dP}{d\tau} + \frac{B_z^2}{2\tau}\left(\frac{1}{K^2} + \frac{\tau^2}{R^2 L^2} - 4\right) \geq 0 . \tag{32}$$

From this, a family of equilibria depending on the parameter μ can be constructed such that irrational μ implies instability while $\mu = K/L$ with sufficiently small K and L implies stability. In other words, the stability threshold, when viewed as a bound upon some parameter (e.g., the plasma beta) is a function of μ which is discontinuous at every rational value of μ .

Extrapolating from the cylindrical case, we conclude that a closed-line equilibrium can be stable even if neighboring low-shear equilibria are unstable, that necessary stability criteria which were derived assuming shear indeed need not be relevant in practice for low-shear systems, and that it is worthwhile to evaluate our new criterion even for unstable low-shear systems, provided their rotation number is close to a low-order rational number.

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