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COALESCENCE OF MAGNETIC ISLANDS

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Abstract

The process of coalescence of magnetic islands is investigated. For intermediate values of the resistivity η the reconnection rate M_s is independent of η , although the reconnection process is basically different from a Petschek type model. For very small values of η the dynamics of the coalescence process is more complex with reconnection being significantly slower.

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I. INTRODUCTION

It is well known that in a current carrying plasma tearing modes may lead to magnetic islands of a certain size forming a new geometrically more complex equilibrium configuration. Naturally, there arises the question about the stability of this configuration. The most fundamental possibly unstable modes correspond to either the break-up of the islands into smaller ones or the coalescence of two islands into a larger one, depending on the wavelength of the original islands. The latter process is the more frequent one, since short wavelength tearing modes usually have larger growth rates leading to islands that subsequently tend to coalesce. For a collisionless plasma the coalescence instability has been predicted¹⁾ with a much larger growth rate than the collisionless tearing mode. In a later investigation²⁾ an even more rapid MHD instability was found. No threshold island size should exist³⁾, which has very recently been confirmed numerically⁴⁾.

The nonlinear development of the coalescence process is of particular interest, as it is connected with magnetic reconnection or field line merging across an x-type neutral point, which is believed to play a crucial role in various explosive magnetic events such as internal $m = 1$ disruptions in tokamaks, solar

flares and magnetospheric substorms. Several stationary models of fast reconnection have been proposed, the most prominent one being Petschek's model⁵⁾. In contrast to these models, which assume a fixed asymptotic plasma flow and magnetic field, the coalescence of magnetic islands represents a selfconsistent process in a closed system. As we shall see, its behavior cannot be described by a Petschek type model.

The organization of this paper is as follows. In section II we give the basic equations. Because of the complex geometry involved a numerical treatment is required. Section III describes the nonlinear behavior of the coalescence process for intermediate values of the resistivity, while section IV deals with the asymptotic behavior as $\eta \rightarrow 0$. In section V we compare the present results with the models of Sweet and Parker⁶⁾ and of Petschek. Section VI summarizes our results and outlines some applications in particular the internal disruption in tokamaks.

II. BASIC EQUATIONS

We consider the two-dimensional resistive MHD equations in the incompressible limit which is well justified in the presence of a strong magnetic field as for instance in tokamaks. The dynamical equations are

$$(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla) \psi = \eta j - E_0 \quad (1)$$

$$(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla) \nabla^2 \phi = \vec{B} \cdot \nabla j \quad (2)$$

with

$$\vec{v} = \vec{e}_z \times \nabla \phi, \quad \vec{B} = \vec{e}_z \times \nabla \psi, \quad \nabla^2 \psi = j$$

In (1) E_0 is the constant equilibrium electric field. The density is assumed to be constant $\rho = 1$. For convenience only plane geometry is considered. There is a class of exact periodic equilibria⁷⁾

$$\psi_0(x, y) = \ln(\cosh x + \alpha \cos y) \quad (3)$$

satisfying the equilibrium equation

$$\nabla^2 \psi_0 = j(\psi_0) = (1 - \alpha^2) e^{-2\psi_0}$$

This solution can be interpreted as the marginally stable tearing mode $k_y = 1$ of finite amplitude α superimposed (nonlinearly) on the one-dimensional $B_y = \tanh x$ sheet pinch configuration. We have used the following units: the asymptotic magnetic field $B_0 = B_y(x \rightarrow \infty)$, the corresponding Alfvén speed v_{A0} , the half-width a of the current sheet. In these units η is the inverse magnetic Reynold's number $S^{-1} = \eta c^2 / 4\pi v_{A0} a$. The island width W_0 is given by the parameter α in (3), $W_0 = 4\sqrt{\alpha}$ for $\alpha \ll 1$. The equilibrium configuration is shown in Fig. 1 for $\alpha = 0.2$. In the computations we choose $\alpha = 0.1, 0.2$, and the value of η is varied between 2×10^{-3} and 10^{-4} . The system size is $-4 \leq x \leq 4$,

$-2\pi \leq y \leq 2\pi$ with fixed boundary conditions at $x = \pm 4$ and periodicity in y -direction (only a quadrant of this system is actually computed). Rather fine grid point spacing is required in y -direction for adequate resolution of the diffusion region around the x -point.

III. RECONNECTION PROCESS FOR INTERMEDIATE VALUES OF η

The nonlinear process of island coalescence can roughly be divided into i) an MHD phase, where the plasma within the islands gains a finite momentum toward the x -point leading to compression of the magnetic field and generation of a quasi-neutral layer between the islands; ii) a reconnection phase, where the accumulated magnetic flux is transported across the x -point. The reconnection process is necessarily associated with some nonideal effect such as resistivity, in the electron equation of motion, i.e. in Ohm's law, untying the electrons from the field lines. Since these effects are usually very weak, the most interesting question is the time scale of the reconnection process. As indicated in eq. (1), only resistivity will be considered.

In this section we restrict ourselves to an intermediate regime of η values, small enough to give rise to field compression and formation of a quasi neutral layer, but larger than a certain

value η_c , the meaning of which will become clear later. For an equilibrium with $\alpha = 0.2$ this range is $5 \times 10^{-3} < \eta < 5 \times 10^{-5}$. Figure 2 shows plots of the entire system as well as enlarged sections thereof during the reconnection phase. Around the x-point the boundary of the diffusion region is plotted defined by $|\vec{v} \cdot \nabla \psi| = |\eta j|$. Since the magnetic field B_x in the islands in front of the neutral layer - the upstream field - is not stationary, it is not very useful to consider the reconnection rate $M_0 = u/B_x$, i.e. the upstream velocity $v_y = u$ in terms of the upstream Alfvén speed $v_A = B_x$ in our units, which is used in the stationary models⁵⁾. Instead we are interested to know how fast reconnection proceeds in terms of the external time scale, i.e. the asymptotic Alfvén speed $v_{Ao} = 1$ in our units. A direct measure of the reconnection rate is given by the change of the magnetic flux ψ_s enclosed by the separatrix

$$M_s = \frac{\partial \psi_s}{\partial t} = \eta j_s - E_0 \sim \eta j_s \quad (4)$$

where j_s is the current density at the x-point. Varying η within the range defined above we observe that M_s is practically independent of η ,

$$\eta j_s \sim 1 \quad (5)$$

The scaling of the lateral width δ of the diffusion region is

$$\delta \sim \eta^{2/3} \quad (6)$$

while the upstream magnetic field B_x scales as

$$B_x \sim \eta^{-1/3}, \quad (7)$$

(6) and (7) being consistent with (5), since $j_s \sim B_x/\delta$.

Because of the compression work performed by the plasma in the islands the kinetic energy is reduced; the upstream velocity is found to scale as

$$u \sim \eta^{1/3} \quad (8)$$

The velocity $v_x = v$ of the plasma leaving the diffusion region is close to the upstream Alfvén speed

$$M_1 = \frac{v}{B_x} \sim 1,$$

hence

$$v \sim \eta^{-1/3} \quad (9)$$

The longitudinal dimension Δ of the diffusion region should be independent of η

$$\Delta \sim 1 \quad (10)$$

because of mass conservation $u\Delta \sim v\delta$, which is in fact observed in the numerical computations. The scaling laws (5) - (10)

suggest a similarity behavior of the system in the vicinity of the diffusion region. Let us introduce the coordinates x' , y' , t'

$$x = x' , \quad y = \eta^{2/3} y' , \quad t = t' \quad (11)$$

Observing that

$$\psi = \psi_s(t) + \Delta\psi(\vec{x}, t) , \quad \psi_s \sim 1 \quad (12)$$

we define functions ψ' , ϕ' by

$$\begin{aligned} \Delta\psi &= \eta^{1/3} \psi' \\ \phi &= \eta^{1/3} \phi' \end{aligned} \quad (13)$$

so that

$$B_x = - \frac{\partial \Delta\psi}{\partial y} = \eta^{-1/3} B'_x \quad (14)$$

$$B_y = \frac{\partial \Delta\psi}{\partial x} = \eta^{1/3} B'_y$$

$$u = \frac{\partial \phi}{\partial x} = \eta^{1/3} v'_y \quad (15)$$

$$v = - \frac{\partial \phi}{\partial y} = \eta^{-1/3} v'_x$$

Since

$$\vec{v} \cdot \nabla = \eta^{-1/3} \vec{v}' \cdot \nabla' \gg \frac{\partial}{\partial t}$$

the primed quantities obey the equations

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \vec{v}' \cdot \nabla' \psi' &= j' \\ \vec{v}' \cdot \nabla' \nabla'^2 \phi' &= \vec{B}' \cdot \nabla' j' \end{aligned} \quad (16)$$

The numerical solutions in fact obey this similarity law in the vicinity of the diffusion layer. Also the time variation proceeds in a selfsimilar way. The length of the diffusion layer $\Delta(t)$ is a measure of the instantaneous island width. Writing the basic quantities ψ, ϕ as

$$\phi(x, y, t) = \Delta(t) \hat{\phi}(\hat{x}, y)$$

$$\psi(x, y, t) = \Delta(t) \hat{\psi}(\hat{x}, y)$$

with $\hat{x} = x/\Delta(t)$, we find immediately the time dependence

$$\begin{aligned} v &\propto \Delta, \quad B_x \propto \Delta, \quad j \propto \Delta \\ u(\hat{x}, y) &= \text{const} \end{aligned} \quad (17)$$

Figure 3 gives plots of ηj_s , of the upstream velocity u taken at the position of the maximum upstream field $B_x = B_m$, of the maximum value of the downstream velocity v , and of Δ .

The scaling law eqs. (5) - (10) has a remarkable property regarding the rate of energy dissipation. In flowing across the diffusion region the plasma is accelerated as well as heated. The power consumed in the acceleration process is

$$W_a \sim \Delta \delta u v^2, \quad (18)$$

while the Ohmic heating power is

$$W_\eta \sim \Delta \delta \eta j^2 \quad (19)$$

Consider the most general scaling consistent with i) reconnection rate $M_s = \eta j_s \sim 1$; ii) Alfvén Mach number of the downstream flow $v/B_x \simeq 1$; iii) finite length of the diffusion layer $\Delta \sim 1$. If $u \sim \eta^\nu$ one obtains from these conditions and quasi stationarity that $B_x \sim \eta^{-\nu}$, $v \sim \eta^{-\nu}$, $\delta \sim \eta^{-2\nu}$, $W_a \sim \eta^{-\nu}$, $W_\eta \sim \eta^{-1+2\nu}$. Hence for the observed value $\nu = 1/3$, eq. (8), both acceleration and dissipation rates are equal in η . This result can also be associated with an extremum principle: For $\nu = 1/3$ the total power consumption in the reconnection process $W_a + W_\eta$ is minimal.

IV. ASYMPTOTIC BEHAVIOR FOR $\eta \rightarrow 0$

Since the free energy of the equilibrium is finite, it is clear for instance from the dissipation rate $W \sim \eta^{-1/3}$, that the picture of the reconnection process as described in the previous section cannot be valid for arbitrarily small η . A change in the scaling has to take place for values of η such that B_x comes close to B_∞ , the maximum value of B_x in an ideal MHD process where no reconnection occurs. If $B_x \simeq B_\infty$ the kinetic energy will be transformed entirely into magnetic energy which will push the plasma back into the islands. The value of $\eta = \eta_c$, where the transition in scaling should occur, depends on the original island size. Let us define - somewhat arbitrarily - that $\eta = \eta_c$, if $B_x = 0.75 B_\infty$. Numerically we find $\eta_c \simeq 8 \times 10^{-5}$ for $\alpha = 0.2$ and $\eta_c \simeq 4 \times 10^{-5}$ for $\alpha = 0.1$.

For $\eta < \eta_c$ the process of island coalescence is more complex. Only a fraction of the magnetic flux inside the separatrix can be reconnected until the plasma motion is reversed. This leads to a sloshing motion of the plasma in the islands with reconnection taking place only during the phases of compressed flux. We may estimate the time scale for total reconnection. Since in addition to Δ now B_x and v are essentially independent of η , conditions correspond to the reconnection model of Sweet and Parker with $u \sim M_s \sim \eta^{1/2}$.

For very small values of η the reconnection process becomes rather slow. The numerical simulations show an even greater complexity of the coalescence process due to growth of tearing modes in the neutral sheet between the islands, which for $\eta < \eta_c$ are no longer stabilized by the plasma flow along the neutral sheet, see Fig. 4.

V. COMPARISON WITH STATIONARY RECONNECTION MODELS

The two most widely discussed concepts for stationary reconnection are the Petschek model ⁵⁾ and the Sweet-Parker model ⁶⁾ (A review of the different reconnection models is given in Ref. 8). Both models assume the upstream magnetic field to be essentially determined by the external boundary conditions independent of η . The reconnection rate M_0 is defined by the asymptotic upstream plasma velocity in terms of the upstream Alfvén speed. While in Petschek's model practically any value $M_0 \ll 1$ is allowed with the diffusion region adjusting itself appropriately, only one value $M_0 = R \sim \eta^{1/2}$ is possible in the Sweet-Parker model. This is, however, no real restriction, since in a time dependent system the magnetic field and the plasma velocity may change until $M_0 \simeq R$.

In table 1 the scaling is given of the various quantities in the stationary reconnection models and in the island coalescence process discussed in sections III and IV. While for

	stationary models		island coalescence	
	Petschek	S. - P.	$\eta > \eta_c$	$\eta < \eta_c$
B_x	1	1	$\eta^{-1/3}$	1
u	1	$\eta^{1/2}$	$\eta^{1/3}$	$\eta^{1/2}$
v	1	1	$\eta^{-1/3}$	1
δ	η	$\eta^{1/2}$	$\eta^{2/3}$	$\eta^{1/2}$
Δ	η	1	1	1
M_o	≤ 1	$\eta^{1/2}$	$\eta^{2/3}$	$\eta^{1/2}$
M_s			1	$\eta^{1/2}$
W_a	1	$\eta^{1/2}$	$\eta^{-1/3}$	$\eta^{1/2}$
W_η	η	$\eta^{1/2}$	$\eta^{-1/3}$	$\eta^{1/2}$

Table 1

$\eta < \eta_c$ the Sweet-Parker model is realized, Petschek's model never applies, though $M_s \sim 1$ (not $M_o!$) in the regime $\eta > \eta_c$. It might be argued that the slow shocks, the main ingredients in Petschek's concept, are excluded in our low- β plasma model. Petschek does not assume a strong normal magnetic field B_o , but he, too, treats the two-dimensional resistive MHD equations in the incompressible limit which leads to precisely the same equations (1), (2). According to Petschek compressibility should only make the reconnection process slightly weaker.

We might tentatively conclude, that Petschek's stationary process either requires very special boundary conditions or is not stable and hence not realizable.

VI. CONCLUSIONS

We have studied the process of coalescence of magnetic islands in a simple periodic equilibrium using an incompressible resistive MHD model. The numerical simulations reveal rather simple scaling laws of the various quantities in particular the rate of magnetic reconnection. For intermediate values of η , $\eta > \eta_c$, the scaling is determined by the compression of the magnetic field $B_x \sim \eta^{-1/3}$, which leads to a reconnection rate M_s independent of η . Nevertheless the behavior is basically different from a Petschek type reconnection process. The main difference is that in Petschek's model the external solution is independent of η , while in the selfconsistent process of island coalescence all quantities depend on η . The length of the diffusion region does not become small compared with the macroscopic dimensions.

For $\eta < \eta_c$ the magnetic field is essentially constant $B \sim B_\infty$, B_∞ being determined by the free energy of the system, and the reconnection rate is rather slow $M_s \sim \eta^{1/2}$. Since in astrophysical applications, e.g. solar flares, η is usually in this regime,

time scales for magnetic energy release would be too long. One has, however, to keep in mind, that the extremely high current densities $j \sim \eta^{-1/2}$ are in general not tolerated in a plasma, which responds by excitation of turbulence giving rise to a strongly enhanced resistivity. This effect has recently been taken into account phenomenologically⁹⁾.

The current density generated during an internal disruption in a tokamak will probably not be sufficiently high for a resistivity enhancement. The present results therefore suggest, that for very small values of η the reconnection of helical field should become quite slow. This might explain recent observations of the internal disruption in the PLT tokamak, which have been interpreted in the way that instead of sweeping across the whole volume interior to the $q = 1$ surface the $m = 1$ island only grows to a finite size and subsequently decays.

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Figure captions

- Fig. 1 Equilibrium configuration ψ_0 - contours
with $\alpha = 0.2$
- Fig. 2 Reconnection phase, $\eta = 5 \times 10^{-4}$
a) ψ - contours of entire system
b) ψ - contours and ϕ - contours of
enlarged sections
- Fig. 3 Time development of ηj_s , u , v , Δ for
 $\eta = 10^{-3}, 5 \times 10^{-4}$
- Fig. 4 Tearing instability of the neutral layer
for $\eta < \eta_c$ producing secondary islands

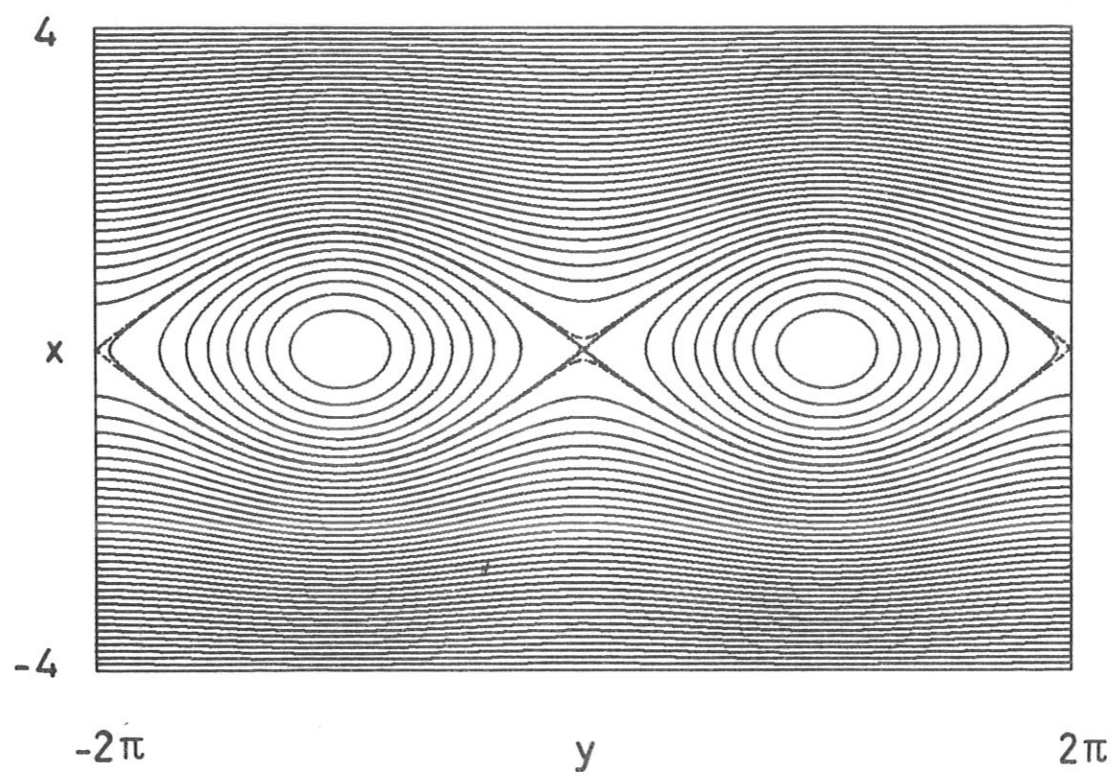


Fig. 1

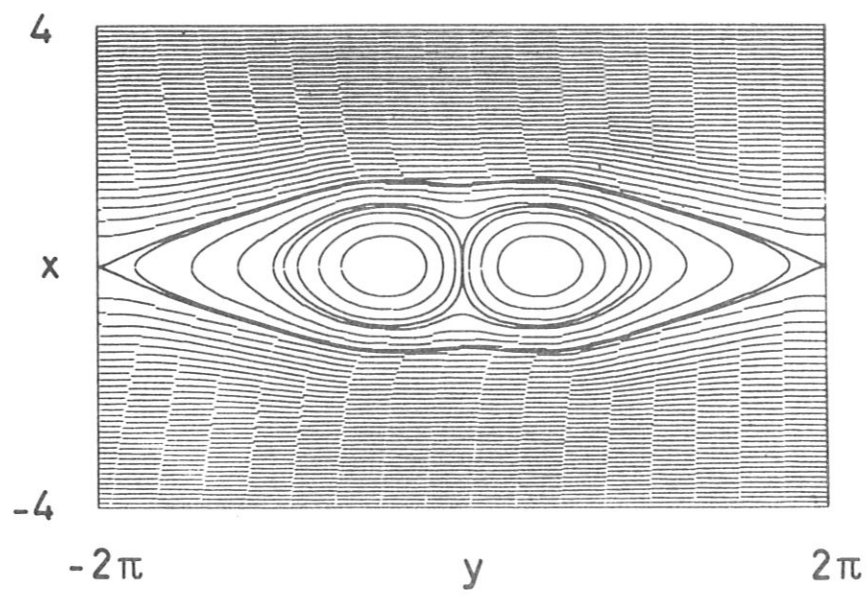
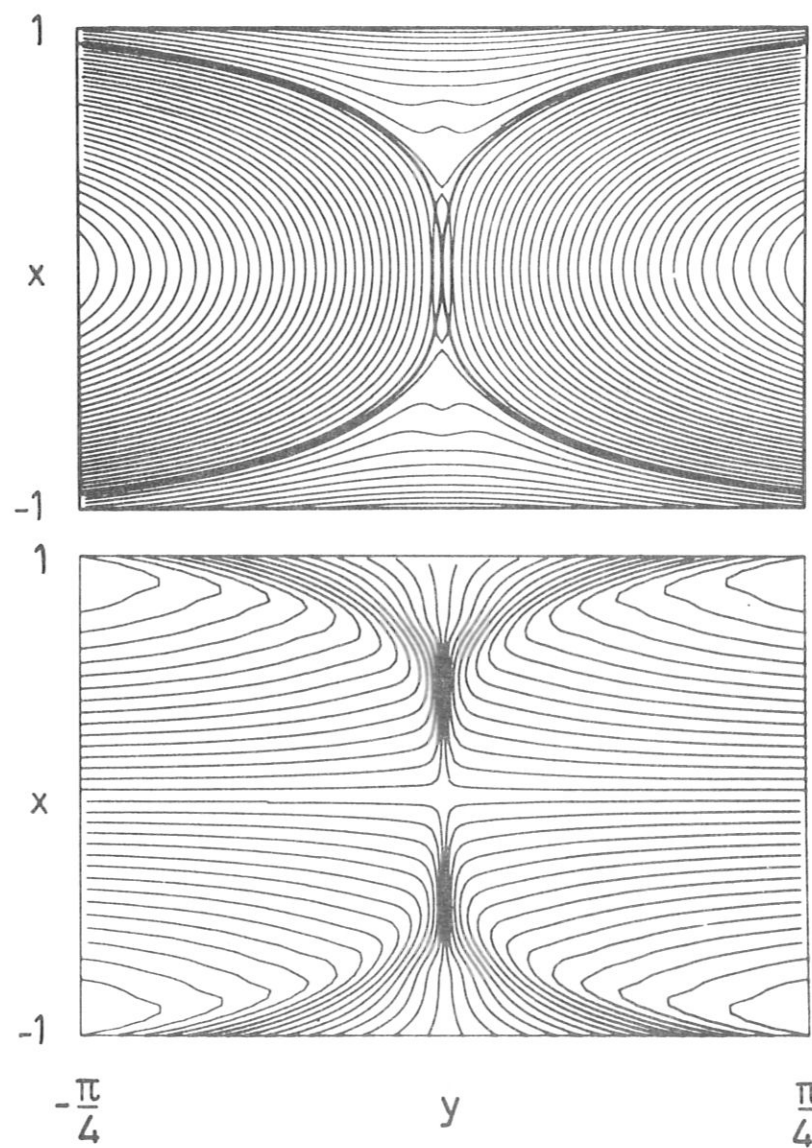


Fig. 2



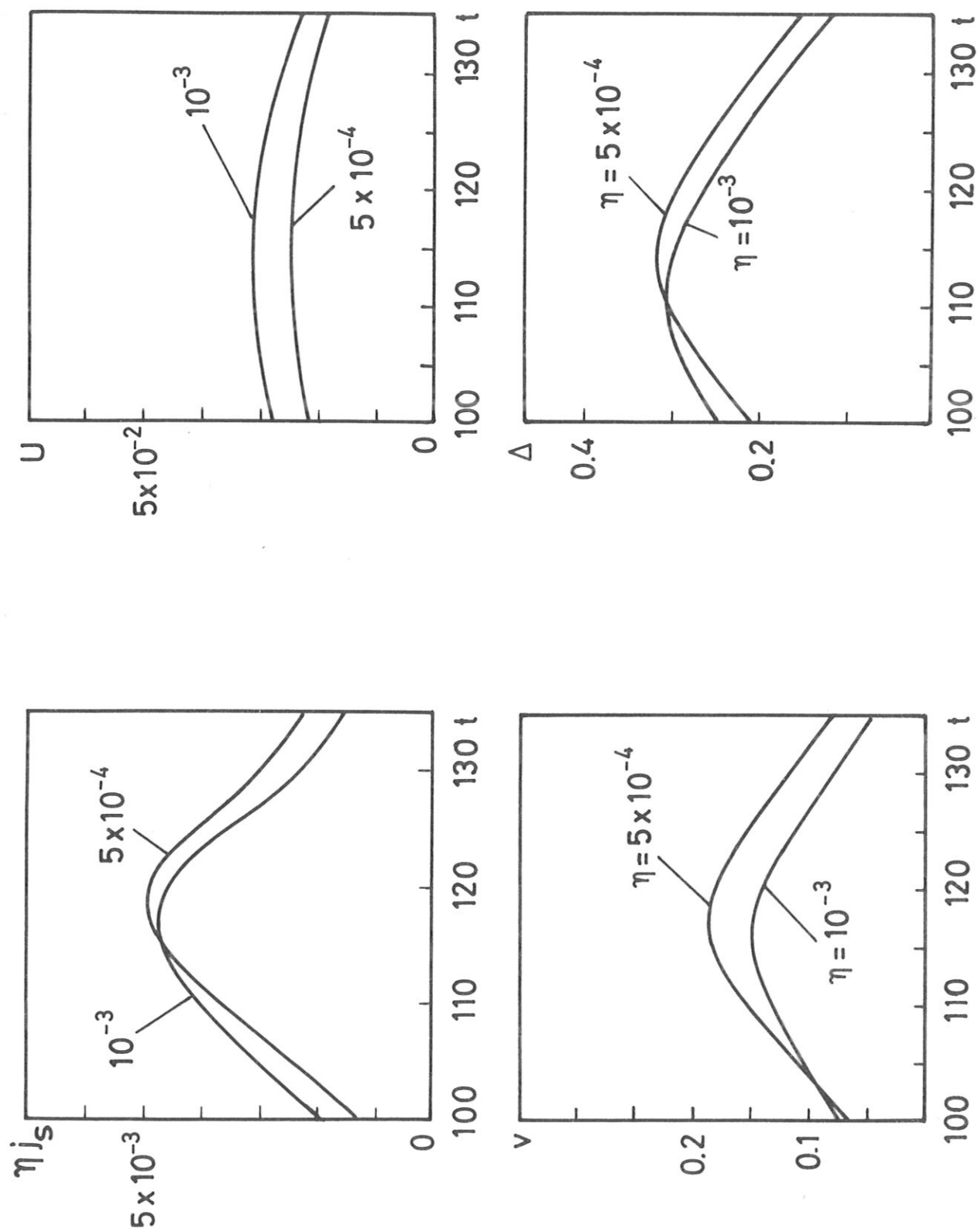


Fig. 3

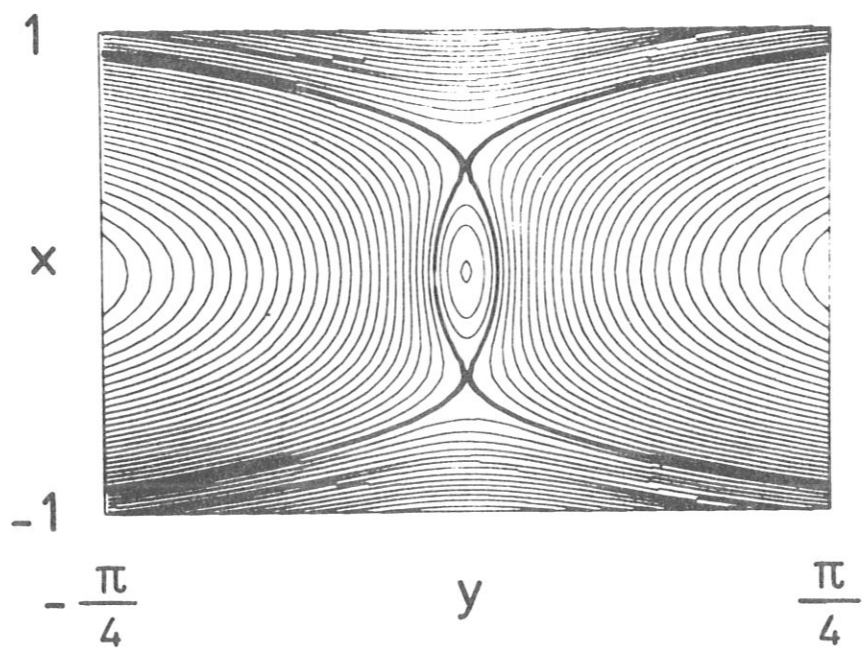


Fig. 4