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# Eigenvalue Bounds for Hill's Equation.[+]

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## Abstract

For Hill's equation

$$y'' + (\alpha + \tilde{f})\, y = 0 \text{ with } \tilde{f}(x+1) = \tilde{f}(x) \text{ and } \int_0^1 \tilde{f}\, d\,x = 0,$$

the lowest eigenvalue $\alpha_o$ of the boundary value problem $y(x+1) = y(x)$ is considered.

Introducing $L_p$ norms of the function $\tilde{f}(x)$, lower bounds for $\alpha_o$ which depend only on this norm are derived for $p = 1,2$ and $\infty$ by solving a variational principle. For these lower bounds analytical expressions are obtained. The quality of the approximations thus obtained is discussed for Mathieu's equation.

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## 1. Introduction

In the stability theory for magnetohydrostatic equilibria the following mathematical problem is encountered in the search for sufficient stability criteria [1]: what conditions must be imposed on a function $f(x)$ in order that the Riccati differential equation $z'=z^2+f$ have real, continuously differentiable solutions $z$ of period 1 for real, continuous $f(x)$ of period 1.

According to standard theorems the Riccati equation has periodic solutions if, and only if, $\int_0^1 f(x)\,dx \leqslant \alpha_o$, where $\alpha_o$ is the lowest eigenvalue of the following eigenvalue problem for Hill's equation:

$$y'' + (\alpha+\tilde{f})\,y = 0, \quad y(x+1) = y(x), \tag{1.1}$$

with

$$\tilde{f} = f - \int_0^1 f(x)\,dx \quad .$$

This is because the solutions of Riccati's equation are the logarithmic derivatives of those of Hill's equation.

Problem (1.1) is a Sturm-Liouville eigenvalue problem and has a finite, non-degenerate, lowest eigenvalue [2]

$$\alpha_o = -\,a^2 \qquad (a \text{ real}), \tag{1.2}$$

which, however can in general only be determined numerically. If $\alpha_o^{(1)}$ is a lower bound for $\alpha_o$, then $\int_0^1 f(x)\,dx \leq \alpha_o^{(1)}$ is sufficient for the Riccati equation to have periodic solutions. Therefore, it is of interest to determine analytical expressions for such lower bounds.

A problem closely related to ours has been treated by Borg [3]: in deriving so-called regions of absolute stability Borg constructed upper bounds for the eigenvalues $\alpha_o$, $\alpha_2$, $\alpha_4$, ... and lower bounds for the eigenvalues $\alpha_1$, $\alpha_3$, ... . Unfortunately, Borg's formulae for $\alpha_1$, $\alpha_3$ ... cannot be extended to $\alpha_o$, nor can the details of his derivation be directly extended to our case, since Borg explicitly uses the existence of zeroes of the higher eigenfunctions. However, it was possible to apply Borg's general idea to construct such bounds by employing a variational principle.

Accordingly, we introduce the $L_p$ norm in the space of the functions $\tilde{f}$

$$(1.3) \qquad \beta_p := ||\tilde{f}||_p = \left( \int_o^1 |\tilde{f}|^p \, d\,x \right)^{\frac{1}{p}}$$

and set

$$(1.4) \qquad g_p := \frac{\tilde{f}}{||\tilde{f}||_p}$$

Then

$$(1.5) \qquad ||g_p|| = 1, \quad \int_o^1 g(x)\, d\,x = 0.$$

Suppose $y_o$ is a solution of problem (1.1) belonging to $\alpha_o = -a^2$. Since $y_o \not\equiv 0$, we may consider

$$z_o = -\, y_o'/y_o \ ,$$

which according to (1.1) - (1.5) satisfies

$$(1.6) \qquad \beta_p\, g_p = z_o' - z_o^2 + a^2 \ ,$$

$$(1.7) \qquad z_o(x+1) = z_o(x) \ ,$$

$$(1.8) \qquad \int_o^1 z_o(x)\, d\,x = 0 \ ,$$

$$(1.9) \qquad \int_o^1 z_o^2\,(x)\, d\,x = a^2 \ .$$

Because of $||g_p||_p = 1$, eq. (1.6) implies independently of the choice of $g_p(x)$ that

(1.10) $$\beta_p > \gamma_p(a) \quad ,$$

where

(1.11) $$\gamma_p(a) = \inf_{z \in C} F_p[z,a] \quad ,$$

(1.12) $$F_p[z,a] = ||z' - z^2 + a^2||_p$$

and C consists of all functions in $C^1$, which satisfy the conditions (1.7) - (1.9) with $z_o$ replaced by z .

In this paper the variational problem (1.11) - (1.12) is solved for the cases p = 1,2 and ∞, yielding analytical functions $\gamma_p(a)$ which are monotonically increasing (see Fig. 1). Let $A_p(\gamma_p)$ be the inverse function of $\gamma_p(a)$. Then, because of this monotonicity, inequality (1.10) and eq. (1.2), the required lower bound for the lowest eigenvalue is given by

(1.13) $$\alpha_o > - [A_p(\beta_p)]^2$$

Conversely, if for specified values of p and a the solution $z_o(x)$ of the corresponding variational problem is substituted in the right-hand side of eq. (1.6), an admissible coefficient function $\tilde{f}$ is defined. For this the conditions (1.7) - (1.8) guarantee the existence of an eigenfunction $y_o$ with eigenvalue $\alpha_o = -a^2$. Hence, the lower bounds thus obtained are the best possible.

For the sake of clearness we shall first reformulate our problem and set out the results in theorems 1-3. Then, the proofs will be given in Sections 2-5. In Section 6 an application to Mathieu's equation will be presented.

We write Hill's equation in the form

$$y'' + (\alpha + \beta\, g(x))\, y = 0,$$

where admissible coefficient functions g(x) satisfy

$$g(x+1) = g(x)\ , \qquad \int_0^1 g(x)\, d\, x = 0\ .$$

The lowest eigenvalue $\alpha_o(\beta)$ is defined by the boundary value problem

$$y(x+1) = y(x)\ .$$

Lower bounds are given by the following theorems:

<u>Theorem 1:</u>

For all admissible functions g(x) with $\int_0^1 |\, g\, |\, d\, x = 1$ the best lower bound of $\alpha_o(\beta)$ is given by

$$\alpha_o^{(1)} = -\ \beta^2/16\ .$$

<u>Theorem 2:</u>

For all admissible functions g(x) with $\left(\int_0^1 g^2\, d\, x\right)^{1/2} = 1$ the best lower bound $\alpha_o^{(2)}(\beta)$ of $\alpha_o(\beta)$ is given by

$$\alpha_o^{(2)} = 16\ K(E-K),$$

$$\beta^2 = \frac{1}{3}\,(4K)^2\ \left[\ (4kK)^2 - \alpha_o^{(2)}\,(1+k^2)\ \right] - (\alpha_o^{(2)})^2\ ,$$

where $E = E(k)$ and $K = K(k)$ are complete elliptic integrals.

Theorem 3:

For all admissible functions $g(x)$ with $\max |g(x)| = 1$ the best lower bound $\alpha_o^{(\infty)}(\beta)$ of $\alpha_o(\beta)$ is given by

$$\sqrt{\beta + \alpha_o^{(\infty)}}\ \mathrm{tg}\,\frac{1}{4}\sqrt{\beta + \alpha_o^{(\infty)}} = \sqrt{\beta - \alpha_o^{(\infty)}}\ \mathrm{tgh}\,\frac{1}{4}\sqrt{\beta - \alpha_o^{(\infty)}} \quad .$$

## 2. The class of functions to be varied

It will be shown in this section that it is possible to replace C with a more suitable class of functions in our variational problem (1.11) - (1.12).

Firstly, the condition (1.8) enforces at least two zeros of $z(x)$ in $[0,1]$. Since $F_p[z,a]$ and the conditions (1.7) - (1.9) are invariant with respect to a translation in x, we may assume $z(0) = 0$ and consequently $z(1) = 0$. We shall see that instead of (1.7)-(1.9) it suffices to consider the weaker conditions

(2.1) $z(0) = z(1) = 0$ ,

(2.2) $\int_0^1 z(x)\, d x = 0$ ,

(2.3) $\int_0^1 z^2\, d x = a^2$

for which inequality (1.10) holds all the more.

According to the Weierstrass approximation theorem every continuously differentiable function can be uniformly approximated by a polynomial of m-th degree $p_m(x)$ in the entire interval $[0,1]$, so that

(2.4) $|p_m(x)-z(x)| < \varepsilon(m)$, $|p_m'(x)-z'(x)| < \varepsilon(m)$

where $\varepsilon(m) \to 0$ for $m \to \infty$. It is readily seen that the approximating polynomial sequences can be chosen such that all $p_m(x)$ satisfy the conditions (2.1) - (2.3). Furthermore, it can be proved as a consequence of (2.4) that

$$\left|\ ||z'-z^2+a^2|| - ||p_m' - p_m^2 + a^2||\ \right| < \varepsilon(m)\, C,$$

C being a given constant. Thus, the infimum (1.11) can already be found within the class P of all polynomials p(x) which satisfy (2.1) - (2.3). For any given $p(x) \in P$ there exists a function $\hat{p}(x)$ with

(2.5) $$\hat{p}(x) \begin{cases} \leq 0 & \text{in } 0 \leq x \leq x^* \quad , x = x^*(p(x)) \\ \geq 0 & \text{in } x^* < x \leq 1 \end{cases}$$

such that between each two successive zeros $\hat{p}(x)$ is identical with p(x), apart from a translation in x and vice versa.

$\hat{p}(x)$ satisfies (2.1)-(2.3) and is continuous; $\hat{p}'(x)$ is continuous apart from finitely many jumps and hence is Rieman integrable. Obviously we have

$$||\hat{p}' - \hat{p}^2 + a^2|| = || p' - p^2 + a^2||$$

and therefore the class C may be replaced by the class $\hat{P}$ of all functions $\hat{p}(x)$.

## 3. The case p = 1

We require the infimum

$$\gamma_1(a) = \inf_{z \in \hat{P}} F_1\left[z,a\right] ,$$

where

(3.1) $$F_1\left[z,a\right] = \int_0^1 |z'-z^2 +a^2| \, d x .$$

From eq. (2.3) it follows that either $z_{max} \geq a$ or $z_{min} < - a$ or both are correct.

a) Case $z_{max} \geq a$ , $z_{min} \leq - a$.

From eqs. (2.1) and (2.3) it follows that

(3.2) $$\int_0^1 (z'-z^2+ a^2) \, d x = 0.$$

If $E_p\left[0,1\right]$ is the union of all intervals from $\left[0,1\right]$ in which $z'-z^2+ a^2$ is positive (for every $z \in \hat{P}$ this involves a finite number of intervals), then it follows from eq. (3.2) that

$$(3.3) \qquad \int_0^1 |z'-z^2+a^2|\,dx = 2 \int_{E_p[0,1]} (z'-z^2+a^2)\,dx .$$

Between the minimum and maximum of z(x) there is an interval $[x_1, x_2]$ with $|z(x)| < a$ for $x \in (x_1, x_2)$ and $-z(x_1) = z(x_2) = a$.

We now have $E_p[x_1, x_2] \subset E_p[0,1]$ and hence

$$\int_0^1 |z'-z^2+a^2|\,dx \geq 2 \int_{E_p[x_1, x_2]} (z'-z^2+a^2)\,dx \geq 2 \int_{x_1}^{x_2} (z'-z^2+a^2)\,dx =$$

$$= 2 \left[ z(x_2)-z(x_1)+ \int_{x_1}^{x_2} (a^2-z^2)\,dx \right] \geq 2 \left[ z(x_2)-z(x_1) \right] = 4a.$$

For later purposes it should be noted here that condition (2.2) was not used to prove this inequality.

We now consider the piecewise continuously differentiable function

$$(3.4) \quad z^{*}(\varepsilon,x) = \begin{cases} -\dfrac{ax}{\varepsilon} & \text{for } x \in [0,\varepsilon] \\ -a & \text{for } x \in [\varepsilon, \frac{1}{2}-\varepsilon] \\ \dfrac{ax}{\varepsilon} - \dfrac{a}{2\varepsilon} & \text{for } x \in [\frac{1}{2}-\varepsilon, \frac{1}{2}] \\ -z^{*}(\varepsilon,1-x) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

with $\varepsilon > 0$.

This function yields

$$\lim_{\varepsilon\to 0} \quad F_1\left[z^{*}(\varepsilon,x),a\right] = 4\,a\,.$$

Since for any $\varepsilon > 0$ the function $z^{*}(\varepsilon,x)$ can be approximated arbitrarily closely by functions $z \in \hat{P}$ , $4\,a$ is the required infimum for the functions considered in this sub-section.

For later purposes we note that the function

$$z^{*}(x) := \lim_{\varepsilon\to 0} \; z^{*}(\varepsilon,z)$$

jumps from $-a$ to $+a$ at $x = \frac{1}{2}$. If we define

$$\lim_{\varepsilon\to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} \left|z^{*\,\prime} - z^{*2}+a^2\right| \, d\,x = \lim_{\varepsilon\to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} \left|z^{*\,\prime}(\varepsilon,x)-z^{*2}(\varepsilon,x)+a^2\right| dx = 2\,a\,,$$

the contribution to $F_1\left[z^{*},a\right]$ due to the jump point is given by the magnitude of the jump of $z^{*}(x)$. Using an analogous limiting process for definition, we shall consider in section b) test functions $z(x)$ with jumps. The contribution of a jump to $F_1\left[z,a\right]$ will again be given by the magnitude of the jump of $z$.

b) Case $z_{max} > a$ and $z_{min} > -a$.

The case $z_{max} < a$, $z_{min} < -a$ is completely analogous and does not need separate consideration.

Let $x_1/\tilde{x}_3$ be the smallest/largest value of $x$ for which $z(x) - a = 0$. Then, we may construct a continuous, piecewise continuously differentiable function $\hat{p}(x)$ with $\hat{p}(x) \geq a$ only for $x_1 \leq x \leq x_3 \leq \tilde{x}_3$ such that $\hat{p}(x) \equiv z(x)$

in the intervals $[0, x_1]$ and $[\tilde{x}_3, 1]$, and that in the interval $[x_1, \tilde{x}_3]$, between each two consecutive zeros of $\hat{p}(x)-a$, $\hat{p}$ is identical with z, apart from a translation in x and vice versa. Obviously, we have $F_1(\hat{p}, a) = F_1(z, a)$. Let $\hat{p}(x)$ again be denoted by $z(x)$.

Construction of $z_1(x)$ with $F_1[z_1(x), a] \leq F_1[z(x), a]$

For every such $z(x)$ we now construct a function $z_1(x)$ with the property

(3.5) $$F_1[z_1(x), a] \leq F_1[z(x), a] \quad .$$

For this purpose we consider the functions

(3.6) $$u(x - \eta) = - a \operatorname{tgh} a(x - \eta)$$

and

(3.7) $$v(x - \chi) = - a \operatorname{tgh}^{-1} a(x - \chi),$$

which are solutions of the differential equation

(3.8) $$\zeta' - \zeta^2 + a^2 = 0 ,$$

$\eta$ and $\chi$ being parameters. It holds that

(3.9) $$|u(x - \eta)| < a, |v(x - \chi)| > a \qquad \text{for all } x, \eta, \chi .$$

Since $-a < z(x) \leq 0$ in $[0, x^*]$, an $\eta_1$ may be found such that for all $x \in [0, x^*]$ we have $u(x-\eta_1) \leq z(x)$, and that for a certain $x_o \in [0, x^*]$ we get $u(x_o - \eta_1) = z(x_o)$. With this we define

$$(3.10)\quad z_1(x) = \begin{cases} u\,(x-\eta_1) & \text{for } x \in (0,x^*) \\ a & \text{for } x \in (x_2,x_3) \\ z\,(x) & \text{else} \end{cases}$$

The point $x_2 \in [x_1, x_3]$ introduced in (3.10) is fixed by the condition

$$(3.11)\qquad \int_0^1 z_1^2(x)\,d\,x = \int_0^1 z^2(x)\,d\,x = a^2 \quad .$$

Since $|u(x-\eta_1)| < a$ and since $z(x) \geq a$ only in $[x_1, x_3]$, it is always possible to find a (unique) point $x_2$ such that eq. (3.11) is satisfied. Because of $z_1 \leq z$ we have

$$(3.12)\qquad \int_0^1 z_1(x)\,d\,x \leq 0.$$

We shall now show that $z_1\,(x)$ satisfies the inequality (3.5). As $z_1(x)$ differs from $z(x)$ only in the intervals $(0,x^*)$ and $(x_2, x_3)$, all that matters are the contributions to the integral (3.1) from these intervals. We have the inequalities

$$(3.13)\qquad \int_0^{x^*} |z'-z^2+a^2|\,d\,x + \int_{x_2}^{x_3} |z'-z^2+a^2|\,d\,x \geq$$

$$\geq -\int_0^{x_o} (z'-z^2+a^2)\,d\,x + \int_{x_o}^{x^*} (z'-z^2+a^2)\,d\,x - \int_{x_2}^{x_3} (z'-z^2+a^2)\,d\,x \geq$$

$$\geq -\int_0^{x_o} (z_1'-z_1^2+a^2)\,d\,x + \int_{x_o}^{x^*} (z_1'-z_1^2+a^2)\,d\,x - \int_{x_2}^{x_3} (z_1'-z_1^2+a^2)\,d\,x =$$

$$= \int_0^{x^*} |z_1'-z_1^2+a^2|\,d\,x + \int_{x_2}^{x_3} |z_1'-z_1^2+a^2|\,d\,x \,.$$

In the second inequality use has been made of the fact, firstly, that

$$-\int_0^{x_o} z'\,dx + \int_{x_o}^{x^*} z'\,dx = -2\,z(x_o) = -2\,z_1(x_o) = -\int_0^{x_o} z_1'\,dx + \int_{x_o}^{x^*} z_1'\,dx$$

and, secondly, that as a result of eq. (3.11)

$$\int_0^{x_o} z^2 dx + \int_{x_2}^{x_3} z^2 dx = \int_0^{x_o} z_1^2 dx + \int_{x_2}^{x_3} z_1^2\,dx + \int_{x_o}^{x^*} (z_1^2 - z^2)\,dx > \int_0^{x_o} z_1^2 dx + \int_{x_2}^{x_3} z_1^2 dx\,.$$

Furthermore, because $z_1 = u(x-\eta_1) \leq z \leq 0$ in $[x_o, x^*]$

it holds that

$$-\int_{x_o}^{x^*} z^2 dx \geq -\int_{x_o}^{x^*} z_1^2\,dx\,.$$

The equality sign of the last step in (3.13) is valid for the following reason: $z_1$ satisfies eq. (3.8) inside the integration limits and hence only makes contributions in the jumppoints $x = 0$, $x^*$, and $x_2$. On both sides these contributions are the magnitudes of the jumps of $z_1$. The inequality (3.5) follows direct from (3.13).

Construction of $z(x,\tilde{\chi})$ with $4a \leq F_1[z(x,\tilde{\chi}),\, a] \leq F_1[z_1(x), a]$

We shall now consider a set of functions $z(x,\chi)$ defined by

$$(3.14)\quad z(x,\chi) = \begin{cases} u(x-\eta(\chi)) & \text{for } x \in (0,x^*) \\ v(x-\chi) & \text{for } x \in \bigcup_{i=1}^{n} (\xi_{2i-1}, \xi_{2i}) \\ z_1(x) & \text{else} \end{cases}\,.$$

The $\xi_i$, $\xi_1 < \xi_2 < .. < \xi_{2n}$, are the zeros of the function $v(x-\chi) - z_1(x)$ in whose vicinity this function changes sign. As $z_1(x)$ is represented by polynomials for $z_1(x) > a$, the number of $\xi_i$ is finite and even. For $\chi \leq \tilde{\chi}$,

$\tilde{\chi}$ being defined below, the function $\eta(\chi)$ uniquely determined by the condition

(3.15) $$\int_0^1 z^2(x,\chi)\, d\, x = \int_0^1 z_1^2(x)\, d\, x = a^2 \quad ,$$

as is explained in the following. Let the largest $\chi$ for which no $\xi_i$ exist be denoted by $\chi_1$ (according to (3.7) we have $\chi_1 > x_1$). Thus, for $\chi \leq \chi_1$ we must put $\eta = \eta_1$ in order to satisfy equation (3.15). As $\chi$ increases over $\chi_1$, $\int_{x_1}^{x_2} z^2(x,\chi)\, d\, x$ decreases monotonically with $\chi$. On the other hand, $\int_0^{x^*} u^2(x-\eta)\, d\, x$ increases monotonically with-$\eta$, which may thus uniquely be chosen such that eq. (3.15) is satisfied. Since for $\chi \to \infty$ we have $z(x,\chi) \to a$ in $[x_1, x_2]$, the only interval where $|z(x,\chi)| \geq a$, there is a $\tilde{\chi} < \infty$ which we get $z(x,\tilde{\chi}) = -a$ in $(0,x^*)$ (i.e. $\eta = \infty$) from (3.15). $z(x,\chi)$ is a continuous and piecewise continuously differentiable function of both x and $\chi$. Because of $z(x,\chi) \leq z_1(x)$ and (3.12) it obeys

(3.16) $$\int_0^1 z(x,\chi)\, d\, x \leq 0$$

Since $z(x,\tilde{\chi}) = -a$ in $(0,x^*)$ and $z(x,\tilde{\chi}) \geq a$ in $[x_1, x_2]$, $z(x,\tilde{\chi})$ belongs to the functions considered in a) and from there we obtain

(3.17) $$F_1 \left[ z(x,\tilde{\chi}),\ a \right] \geq 4a \ .$$

In view of (3.16) if may be recalled that (2.2) was not used in deriving (3.17).

In the remaining part of this section it will be proved that

$$(3.18)\qquad \frac{d}{d\chi} F_1 \left[ z(x,\chi), a \right] \leq 0$$

is valid. Assuming the validity of (3.18) for the moment, it follows together with (3.17) and (3.5) that

$$F_1 \left[ z(x), a \right] \geq F_1 \left[ z_1(x), a \right] = F_1 \left[ z(x,\chi_1), a \right] \geq F_1 \left[ z(x,\tilde{\chi}), a \right] \geq 4\, a.$$

It remains to prove the inequality (3.18). First we define

$$(3.19)\qquad q_1\ (\eta(\chi)) = \int_o^{x^*} z^2 (x,\chi) d\, x$$

$$(3.20)\qquad q_2\ (\chi) = \int_{x_1}^{x_2} z^2 (x,\chi)\, d\, x.$$

As $z(x,\chi)$ only varies with $\chi$ in the intervals $(0,x^*)$ and $\left[ x_1,\ x_2 \right]$, differentiation of (3.15) with respect to $\chi$ yields the relation

$$(3.21)\qquad \frac{dq_1}{d\eta} \frac{d\eta}{d\chi} + \frac{dq_2}{d\chi} = 0 \quad .$$

In accordance with (3.19), (3.14) and since $u\ (x-\eta)$ satisfies eq. (3.8), $\frac{dq_1}{d\eta}$ is given by

$$(3.22)\qquad \frac{dq_1}{d\eta} = \frac{d}{d\eta} \int_o^{x^*} (u'+a^2) d\, x = -\ \dot{u}\ (x^* - \eta) + \dot{u}\ (-\eta)$$

where ' denotes the derivative with respect to x and $\dot{}$ the derivative with respect to the whole argument.

Irrespective of the shape of $z_1(x)$ it can be shown that

$$(3.23) \qquad \frac{dq_2}{d\chi} = \sum_{i=1}^{n} \left[ \dot{v}\,(\xi_{2i-1} - \chi) - \dot{v}\,(\xi_{2i} - \chi) \right] .$$

This is also valid when the function $v(x-\chi)-z_1(x)$ vanishes to higher order at the points $\xi_i$.

From (3.21) - (3.23) it follows that

$$(3.24) \qquad \frac{d\eta}{d\chi} = - \frac{\sum_{i=1}^{n} \left[ \dot{v}(\xi_{2i-1}-\chi) - \dot{v}(\xi_{2i}-\chi) \right]}{\dot{u}\,(-\eta) - \dot{u}\,(x^{*}-\eta)} .$$

We now introduce further definitions:

$$(3.25) \qquad F_1(\eta(\chi)) = \int_{0}^{x^{*}} \left| z'(x,\chi)-z^2(x,\chi) + a^2 \right| d\,x ,$$

$$(3.26) \qquad F_2(\chi) = \int_{x_1}^{x_2} \left| z'(x,\chi)-z^2(x,\chi) + a^2 \right| d\,x .$$

We have

$$(3.27) \qquad \frac{d}{d\chi} F_1\left[ z(x,\chi),a \right] = \frac{dF_1}{d\eta}\frac{d\eta}{d\chi} + \frac{dF_2}{d\chi} .$$

In accordance with the definition (3.14) and since $u(x-\eta)$ satisfies eq. (3.8) we get for $\frac{dF_1}{d\eta}$

$$(3.28) \qquad \frac{dF_1}{d\eta} = \frac{d}{d\eta}\left[ -u(-\eta)-u(x^{*}-\eta) \right] = \dot{u}(-\eta)+\dot{u}(x^{*}-\eta) .$$

Irrespective of the shape of $z_1(x)$ and irrespective of whether the function $v(x-\chi)-z_1(x)$ vanishes to higher order in the zeros $\xi_i$ we get for $\frac{dF_2}{d\chi}$

$$(3.29)\quad \frac{dF_2}{d\chi} = -\sum_{i=1}^{n}\left[\dot{v}\,(\xi_{2i-1}-\chi) + \dot{v}\,(\xi_{2i}-\chi)\right] .$$

From (3.24) and (3.27) - (3.29) it follows that

$$(3.30)\quad \frac{d}{d\chi} F_1\left[z(x,\chi),a\right] = 2\left[\dot{u}(-\eta)-\dot{u}(x^{*}-\eta)\right]^{-1}\left\{\sum_{i=1}^{n}\left[\dot{v}(\xi_{2i}-\chi)\dot{u}(x^{*}-\eta) - \dot{v}(\xi_{2i-1}-\chi)\dot{u}(-\eta)\right]\right\}.$$

Because $x^{*} > 0$ we find according to (3.6)

$$(3.31)\quad \left[\dot{u}(-\eta) - \dot{u}\,(x^{*}-\eta)\right]^{-1} < 0 .$$

We now make use of the relation (3.16). As $z(x,\chi)$ is only negative in the interval $[0, x^{*}]$, it follows from the definition (3.14) that, in particular,

$$(3.32)\quad \int_{0}^{x^{*}} u(x-\eta)\, d\, x + \int_{\xi_{2i-1}}^{\xi_{2i}} v(x-\chi)\, d\, x \leq 0,\ i = 1,2,\dots, n .$$

According to (3.6) we get

$$(3.33)\quad \int_{0}^{x^{*}} u(x-\eta)\, d\, x = -\ln \frac{\cosh a(x^{*}-\eta)}{\cosh a(-\eta)}$$

and according to (3.7)

$$(3.34)\quad \int_{\xi_{2i-1}}^{\xi_{2i}} v(x-\chi) d\, x = -\ln \frac{\sinh a(\xi_{2i}-\chi)}{\sinh a(\xi_{2i-1}-\chi)},\ i = 1,2,\dots, n$$

and hence with (3.32)

$$(3.35)\qquad \cosh^2 a\,(x^*-\eta)\sinh^2 a(\xi_{2i}-\chi)-\cosh^2 a(-\eta)\sinh^2 a(\xi_{2i-1}-\chi) \geq 0.$$

$$i = 1,2,\dots, n$$

From (3.6) and 3.7) in conjunction with (3.35) it follows that

$$(3.36)\qquad \dot v\,(\xi_{2i}-\chi)\,\dot u\,(x^*-\eta)-\dot v(\xi_{2i-1}-\chi)\dot u(-\eta) \geq 0$$

$$i = 1,2,\dots,n \quad ,$$

and (3.30) in conjunction with (3.31) and (3.36) yields the assertion (3.18)

c) Final result

Summarizing the results of subsections a) and b), we have the final result

$$(3.37)\qquad \gamma_1(a) = 4a$$

The infimizing function is given by

$$(3.38)\qquad z(x) = \begin{cases} 0 & \text{for } x = 0,1/2,1 \\ -a & \text{for } x \in (0,1/2) \\ a & \text{for } x \in (1/2,1) \end{cases} .$$

## 4. The case p = 2

We require the infimum

$$\gamma_2(a) = \inf_{z \in \hat P} F_2[z,a] \quad ,$$

where

$$(4.1)\qquad F_2^{\,2}[z,a] = \int_0^1 (z'-z^2+a^2)^2 d\,x \quad .$$

We note that for any continuous and piecewise continuously differentiable function z(x) which satisfies the conditions (2.1) and (2.3) we have

$$(4.2)\qquad F_2^{\,2}[z,\alpha] = \int_0^1 \left[z'^2+(\alpha^2-z^2)^2\right] d\,x$$

and

(4.3) $$F_2^{\,2}[z,a] = F_2^{\,2}[z,\alpha] - (a^2-\alpha^2)^2 ,$$

where $\alpha$ is an arbitrary real auxiliary parameter.

<u>Construction of $\tilde{z}(x)$ with $F_2[\tilde{z},\alpha] = F_2[z,\alpha]$ and $\tilde{z}(\frac{1}{2}) = 0$ etc.</u>

Starting with a function $z \in \hat{P}$, we shall now construct a function $\tilde{z}$ with the properties (2.1), (2.3),

(4.4) $$F_2[\tilde{z},\alpha] = F_2[z,\alpha] ,$$

and

(4.5) $$\tilde{z} \begin{cases} \leq 0 & \text{for } x \in [0,1/2] \\ = 0 & \text{for } x = 1/2 \\ \geq 0 & \text{for } x \in [1/2,1] \end{cases}$$

It can be assumed that $x^*[z] \geq 1/2$ because otherwise we may consider the function $\bar{z}(x) = -z(1-x)$, which satisfies $z \in \hat{P}$, $F_2[\bar{z},\alpha] = F_2[z,\alpha]$ and $x^*[\bar{z}] = 1-x^*[z]$ .

Let us first introduce the function

(4.6) $$g(x) = -z(x^* - x + 1/2).$$

In $[x^*, 1]$ both $g(x)$ and $z(x)$ are non-negative. According to (4.6) and (2.2), in view of the definition of $x^*$, and since $x^* \geq 1/2$ it follows that

$$\int_{x^*}^{1} g(x)\,dx = \int_{x^*-1/2}^{1/2} [-z(x)]\,dx \leq \int_{o}^{x^*} [-z(x)]\,dx = \int_{x^*}^{1} z(x)\,dx .$$

From this and because of $z(x^*) = z(1) = 0$ it follows that there is an $x_s \in [x^*, 1]$ with

(4.7) $$g(x_s) = z(x_s).$$

We now set

(4.8) $$\tilde{z}(x) = \begin{cases} g(x), & x \in [1/2 + x^* - x_s, x_s] \\ z(x) & \text{else} \end{cases}$$

Because of (4.7) $\tilde{z}(x)$ is continuous at $1/2 + x^* - x_s$ and at $x_s$. The validity of the relations (2.1), (2.3), (4.4) and (4.5) for $\tilde{z}$ is easily derived from the definitions of $\tilde{z}(x)$, $g(x)$, $x^*[z]$ and from (4.2).

Estimate of $F_2 [\tilde{z},\alpha]$ using the Hilbert integral

$F_2[\tilde{z},\alpha]$ is now estimated with the Hilbert integral of the variational calculus. For this purpose we consider the Euler equation corresponding to the integral (4.2)

(4.9) $$y'' - 2 y^3 + 2\alpha^2 y = 0$$

and its solutions passing through the point $x = 0$, $y = 0$, with $y'(o) \leq 0$. After integration (4.9) yields

(4.10) $$y'^2 = y^4 - 2\alpha^2 y^2 + C^2$$

with the solution $y(x,C)$, where

$$C = |y'(o)|.$$

For $0 < C \leq \alpha^2$ the required solutions can be expressed by means of the Jacobian elliptic function $u = \text{sn } v$ [1)]

1) For the reader's convenience the known properties of $u = \text{sn } v$ that are used in the following are set out here:

$$\dot{u}^2 = k^2u^4-(1+k^2)u^2+ 1, \; k \in [0,1] ,$$

$$\text{sn}(o) = \text{sn}(2K) = 0, \; \text{sn } v > 0 \text{ for } v \in (0,2K) ,$$

$$\text{sn}(K-v) = \text{sn}(K+v) = - \text{sn}(v-K) ,$$

where $$K = K(k) = \int_o^{\pi/2} (1-k^2\sin^2\phi)^{-1/2} \, d\phi.$$

With $E = E(k) = \int_o^{\pi/2} (1-k^2\sin^2\phi)^{1/2} \, d\phi$ the integral formulae $\int_o^K \text{sn}^2 v \, d v = \frac{1}{k^2} (K-E)$, $\int_o^K \text{sn}^4 v \, d v = \frac{2}{3} \frac{1+k^2}{k^4} (K-E) - \frac{K}{3k^2}$ are valid.

(4.11) $$y(x,C) = - \sqrt{k\,C}\ \ \mathrm{sn}\sqrt{\frac{C}{k}}\ x \quad ,$$

where

(4.12) $$k = \frac{1}{C}\ (\ \alpha^2 - \sqrt{\alpha^4 - C^2})\ .$$

For $C > \alpha^2$ the solutions $y(x,C)$ go monotonically from $y = 0$ to $y = - \infty$ . If we restrict the (periodic) solutions (4.11) to $x \in [o,\ \mathrm{Min}(2\ K \sqrt{\frac{k}{C}}\ ,\ \frac{1}{2}]$ the strip $S:\ 0 < x < \frac{1}{2}\ ,\ - \infty < y \leq 0$ is simply covered by the set of all solutions with $y'(0) < 0$ passing through the point $(0,0)$. $y(x,C)$ has the unique inverse function $C(x,y),\ x,y \in S$.

At every point $x,y$ of $S$ the direction

(4.13) $$p(x,y) = \frac{\partial}{\partial x}\, y\ \ (x,C(x,y))\ \Big|_C$$

is uniquely defined by the curve $y(x,C)$ passing through the point $x,y$. Because of (4.9) and (4.13) the following relation holds for the directional field

(4.14) $$\frac{d}{dx}\, p(x,y(x,C)) = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\, p = 2\ y^3 - 2\alpha^2 y\ .$$

For all continuous and piecewise continuously differentiable functions $z(x)$ which are wholly in $S$ and which satisfy

(4.15) $$z(o) = z(\tfrac{1}{2}) = 0$$

we now define the Hilbert integral

(4.16) $$H\left[z(x)\right] = \int_o^{1/2} \left[(\alpha^2 - z^2)^2 - p^2(x,z) + 2\,\frac{dz}{dx}\, p(x,z)\right] dx.$$

From (4.13) and (4.14) it follows that $H[z(x)]$ has the same value for all $z(x)$ considered here.

The function $\tilde{z}(x)$ given by (4.8) is entirely in $S \cup \partial S$ and satisfies (4.15). Thus, $H[\tilde{z}(x)]$ is well defined and it is found that

(4.17) $$\int_0^{1/2} [\tilde{z}'^2 + (\alpha^2 - \tilde{z}^2)^2] d\,x - H[\tilde{z}] = \int_0^{1/2} [\tilde{z}' - p(x, \tilde{z}(x))]^2 d\,x \geq 0.$$

In order to calculate $H[\tilde{z}(x)]$, we consider that function $y(x,C)$ which satisfies (4.15). According to footnote 1) it is given by

(4.18) $$y\,(x, C^*) = -4\, K^* k^* \operatorname{sn} 4\, K^* x\ ,$$

where $k^* \in [0,1]$ is arbitrary and

(4.19) $$C^* = 16\, K^{*2}\, k^*\ , \qquad K^* = K(k^*)\ ,$$

if the auxiliary parameter $\alpha$ is utilized in such a way that eq. (4.12) or

(4.20) $$\alpha^2 = 8\, K^{*2}\, (1 + k^{*2})$$

is valid.

According to the definition of $H[z]$ we have

(4.21) $$H[\tilde{z}] = H[y(x, C^*)].$$

The undetermined value of $k^*$ is now fixed by requiring that

(4.22) $$\int_0^1 y^2(x, C\ )\, d\,x = 16 K^* \ (K^* - E^*\ ) = a^2, \ E^* = E(k^*).$$

(The first equality sign follows from the properties of sn v set out in the footnote.) As $K^*(K^* - E^*)$ assumes all values between 0 and $\infty$ for $k^* \in [0,1]$ monotonic in $k^*$, there is a unique value $k^*$ for every $a^2$.

With (4.10), (4.13), (4.16), (4.18), (4.19), (4.20), (4.22), and footnote 1) it is found that

(4.23) $$H\left[y(x,C^*)\right] = \frac{1}{3}(4k^* K^*)^2 \left[8K^{*2} - \frac{1+k^{*2}}{k^{*2}} a^2\right] + \frac{(4K^*)^4}{8}(1+k^{*2})^2$$

For the value of $\alpha$ fixed by (4.20) and (4.22) it follows from (4.17) and (4.21) that

(4.24) $$\int_0^{\frac{1}{2}} \left[\tilde{z}'^2 + (\alpha^2-\tilde{z}^2)\right] dx \geq H\left[y(x,C^*)\right]$$

For the contribution of $\tilde{z}$ to $F_2\left[\tilde{z}, \alpha\right]$ from the interval $\left[\frac{1}{2}, 1\right]$ the same estimate is obtained by transformation to the interval $\left[0,\frac{1}{2}\right]$ with $\tilde{\tilde{z}} = -\tilde{z}\,(1-x)$. Thus we have

(4.25) $$\int_0^1 \left[\tilde{z}'^2 + (\alpha^2-\tilde{z}^2)^2\right] dx > 2H\left[y(x,C^*)\right].$$

and from (4.2), (4.3), (4.4), (4.20), (4.23), and (4.25) it finally follows that

(4.26) $$F_2^2\left[z,a\right] \geq \frac{1}{3}(4K^*)^2\left[(4k^* K^*)^2 + (1+k^{*2})a^2\right] - a^4.$$

The function (4.18) satisfies all subsidiary conditions (2.1) - (2.3) when $k^*$ is given by (4.22). If $z(x) \equiv y(x,C^*)$ from the very outset, the equality sign is valid for all inequalities in this section. Consequently, the right-hand side of (4.26) yields a minimum for $F_2^2\left[z,a\right]$.

Result:

To summarize, the result is

(4.27) $$\gamma_2(a) = \left\{\frac{1}{3}(4K)^2\left[(4kK)^2 + (1+k^2)a^2\right] - a^4\right\}^{1/2},$$

where $k$ satisfies the equation

(4.28) $$16\,K(K-E) = a^2.$$

The minimizing function is

(4.29) $$z = -4\,K\,k\,\mathrm{sn}\,4\,K\,x.$$

For $\gamma_2(a)$ the following approximations are valied:

(4.30) $$a \to 0: \quad \gamma_2(a) \to 2\pi a\left(1 + \frac{a^2}{16\pi^2}\right),$$

(4.31) $$a \to \infty: \quad \gamma_2(a) \to \frac{4}{\sqrt{3}}\, a^{3/2} + \ldots \quad .$$

## 5. The case $p = \infty$

We require the infimum

$$\gamma_\infty(a) = \inf_{z \in \hat{P}} F_\infty[z,a],$$

where

(5.1) $$F_\infty[z,a] = \max_{x \in [0,1]} |z' - z^2 + a^2|$$

Construction of $z_2(x)$ with $F_\infty[z_2,a] = F_\infty[z_1,a]$

and $z_2^2(x) \geq z_1^2(x)$.

We start with a function $z_1 \in \hat{P}$ and may again assume $x^*(z_1) \geq \frac{1}{2}$ as in Section 3. From this we construct a function $z_2$ with

(5.2) $$F_\infty[z_2,a] = F_\infty[z_1,a]$$

by solving the differential equations

(5.3) $$\begin{cases} z_2' - z_2^2 + a^2 = \begin{cases} -F_\infty[z_1,a], & x \in [0,1-\xi] \cup [\xi.1] \\ +F_\infty[z_1,a], & x \in [2x^* - \xi, \xi] \end{cases} \\ z_2' = 0, \quad x \in [1-\xi, 2x^* - \xi] \end{cases}$$

and determining $\xi \in [x^*, 1]$ so that $z_2$ is continuous, piecewise continuously differentiable and satisfies

$$(5.4) \qquad z_2(0) = z_2(x^*) = z_2(1) = 0.$$

All of these conditions are satisfied by

$$(5.5) \quad z_2 = \begin{cases} -q \operatorname{tgh} q x & \text{for } x \in [0, 1-\xi] \\ -q \operatorname{tgh} q(1-\xi) & \text{for } x \in [1-\xi, 2x^*-\xi] \\ p \operatorname{tg} p(x-x^*) & \text{for } x \in [2x^*-\xi, \xi] \\ -q \operatorname{tgh} q(x-1) & \text{for } x \in [\xi, 1] \end{cases}$$

where

$$(5.6) \quad \begin{cases} q = \sqrt{F_\infty [z_1, a] + a^2} \\ p = \sqrt{F_\infty [z_1, a] - a^2} \end{cases}$$

and $\xi$ is the solution of

$$(5.7) \qquad -q \operatorname{tgh} q(\xi-1) = p \operatorname{tg} p(\xi-x^*), \quad \xi \in [x^*, 1].$$

p is real because of

$$F_\infty [z_1, a] \geq |z_1'(x^*+0) - z_1^2(x^*) + a^2| = |z_1'(x^*+0) + a^2| \geq a^2$$

Furthermore, we assert that the inequality

$$(5.8) \qquad |z_2(x)| \geq |z_1(x)|, \quad x \in [0, 1]$$

is valid. For the intervals $[0, 1-\xi]$, $[2x^*-\xi, \xi]$ and $[\xi, 1]$ the assertion follows from (5.3), (5.4), from

$$(5.9) \qquad |z_1' - z_1^2 + a^2| \leq F_\infty [z_1, a]$$

and from the fact that both $z_1$ and $z_2$ are $\leq 0$ in $[0,x^*]$ and $\geq 0$ in $[x^*,1]$. If it were violated in $[1-\xi,\ 2x^*-\xi]$, there would exist an $x_o$ with $-z_1(x_o) > z_2(x_o) = z_2(\xi)$, and because of (5.9) we had $-\int_o^{x^*} z_1 dx > \int_{x^*}^1 z_2 dx \geq \int_{x^*}^1 z_1 dx$ or $\int_o^1 z_1 dx < 0$, thus violating the condition (2.2).

The inequality (5.8) results in

$$(5.10)\qquad \int_o^1 z_2^2\, d\,x \geq \int_o^1 z_1^2\, d\,x\,.$$

Construction of $z_3(x)$ with $F_\infty[z_3,a] = F_\infty[z_2,a]$, $\int_o^1 z_3 dx = 0$ and $\int_o^1 z_3^2\, dx = \int_o^1 z_2^2\, dx$

From the function $z_2$ we then proceed to a function $z_3$

$$(5.11)\quad z_3 = \begin{cases} p\ \mathrm{tg}\ p\ (x-\frac{1}{2}) & \text{for } x \in [\frac{1}{2}-\xi+x^*,\ \frac{1}{2}+\xi-x^*] \\ -q\ \mathrm{tgh}\ q\ (\xi-1) & \text{for } x \in [\frac{1}{2}+\xi-x^*,\ \xi] \\ z_2 & \text{else.} \end{cases}$$

This obviously satisfies

$$(5.12)\quad \begin{cases} \int_o^1 z_3\, d\,x = 0 \\ F_\infty[z_3,a] = F_\infty[z_2,a]\,. \end{cases}$$

Furthermore, the property

$$(5.13)\qquad \int_o^1 z_3^2\, d\,x = \int_o^1 z_2^2\, d\,x$$

can be verified, e.g. geometrically.

Construction of $z_4(x)$ with $F_\infty[z_4,a] = F_\infty[z_3,a]$, $\int_o^1 z_4^2\, dx \geq a^2$ and estimate of $F_\infty[z_4,a]$.

If we determine $\eta = \eta\ (p)$ as the solution of the equation

(5.14) $$-q \operatorname{tgh} q \eta = p \operatorname{tg} p (\eta - \tfrac{1}{2}), \quad \eta \in [0, \tfrac{1}{2}],$$

where

(5.15) $$q^2 = p^2 + 2 a^2$$

according to (5.6), we can finally define a function $z_4 = z_4 (x,p)$ by

(5.16) $$z_4 = \begin{cases} -q \operatorname{tgh} q x & \text{for } x \in [0,\eta] \\ p \operatorname{tg} p (x-\tfrac{1}{2}) & \text{for } x \in [\eta, 1-\eta] \\ -q \operatorname{tgh} q (x-1) & \text{for } x \in [1-\eta, 1]. \end{cases}$$

$z_4$ satisfies (2.1), (2.2),

(5.17) $$F_\infty [z_4,a] = F_\infty [z_1,a] = p^2 + a^2 ,$$

(5.18) $$\int_0^1 z_4^2 \, d x = 4 (p^2 + a^2 )\eta - p^2 ,$$

and finally because $\frac{\partial}{\partial p} z_4^2(x,p) \geq 0$

(5.19) $$\frac{d}{dp} \int_0^1 z_4^2 \, d x \geq 0.$$

Because of $|z_4| > |z_3|$, (5.13), (5.10), and (2.3) p is restricted so that

(5.20) $$\int_0^1 z_4^2 \, d x \geq a^2.$$

According to (5.17) $F_\infty [z_1,a]$ increases monotonically with p, and therefore it follows from (5.19) and (5.20) that we obtain a lower bound for $F_\infty [z,a]$ if we set $\int_0^1 z_4^2 \, d x = a^2$ or, according to (5.18), if we set $\eta = \frac{1}{4}$. That this lower bound can be approximated as closely as desired follows from the fact that the function $z = z_4\ (x,p^*)$, $\eta\ (p^*) = \frac{1}{4}$, which does not belong to $\hat{P}$, can be uniformly approximated by functions $\in\ \hat{P}$.

Result:

Inserting $\eta(p^*) = \frac{1}{4}$, (5.15), and $p^* = \sqrt{\gamma_\infty - a^2}$ from (5.6) in (5.14) yields the implicit equation for $\gamma_\infty(a)$:

$$(5.21)\quad \sqrt{\gamma_\infty - a^2}\ \mathrm{tg}\ \frac{1}{4}\sqrt{\gamma_\infty - a^2} - \sqrt{\gamma_\infty + a^2}\ \mathrm{tgh}\ \frac{1}{4}\sqrt{\gamma_\infty + a^2} = 0.$$

The infimizing function is given by

$$(5.22)\qquad z = \begin{cases} -\sqrt{\gamma_\infty + a^2}\ \mathrm{tgh}\ (\sqrt{\gamma_\infty + a^2}\ x)\ , & x \in [0, \frac{1}{4}] \\ \sqrt{\gamma_\infty - a^2}\ \mathrm{tg}\ (\sqrt{\gamma_\infty - a^2}\ (x - \frac{1}{2})]\ , & x \in [\frac{1}{4}, \frac{3}{4}] \\ -\sqrt{\gamma_\infty + a^2}\ \mathrm{tgh}\ [\sqrt{\gamma_\infty + a^2}\ (x-1)]\ , & x \in [\frac{3}{4}, 1] \end{cases}$$

From (5.21) the following approximations are obtained

$$(5.23)\qquad a \to 0 : \quad \gamma_\infty(a) \to 4\sqrt{3}\ a$$

$$(5.24)\qquad a \to \infty : \quad \gamma_\infty(a) \to a^2 + 4\pi^2$$

## 6. An example of application

In this section we evaluate the eigenvalue bounds for the case of Mathieu's equation

$$\frac{d^2y}{dt^2} + (\lambda - 2h^2 \cos 2\ t)\ y = 0,\ y(x) = y\ (x+\pi),\ h^2 \text{ real.}$$

With

$$t = \pi\ x$$

we have

$$\tilde{f} = -\ 2\pi^2\ h^2\ \cos\ 2\ \pi x,$$

$$\alpha = \pi^2\ \lambda\ ,$$

and

$$\beta_1 = 4\pi^2|h^2| \ , \ \beta_2 = \sqrt{2}\ \pi^2|h^2|, \ \beta_\infty = 2\pi^2|h^2| .$$

Thus, according to inequality (1.13), we have the bounds

$$\lambda_o^{(1)} = -\frac{1}{\pi^2}\left[A_1\ (4\pi^2|h^2|)\right]^2 ,$$

$$\lambda_o^{(2)} = -\frac{1}{\pi^2}\left[A_2\ (\sqrt{2}\,\pi^2|h^2|\right]^2,$$

$$\lambda_o^{(\infty)} = -\frac{1}{\pi^2}\left[A_\infty\ (2\pi^2|h^2|)\right]^2$$

for the lowest eigenvalue $\lambda_o = a_o\ (h^2)$ , which are shown in Fig. 2. If the asymptotic formulae

$$a_o(h^2) \to -\frac{1}{2}\ h^4 + \frac{7}{128}\ h^8 \ , \ h \to 0,$$

$$a_o(h^2) \to -\ 2\ h^2 + 2\ h \qquad , \ h \to \infty$$

are compared with

$$\lambda_o^{(2)} \to -\frac{1}{2}\ h^4 + \frac{1}{32}\ h^8 \qquad \lambda_o^{(\infty)} \to -\frac{\pi^2}{12}\ h^4 \ , \quad h \to 0 \quad ,$$

$$\lambda_o^{(2)} \to -\left(\frac{9}{62}\ \pi^2 h^8\right)^{\frac{1}{3}} , \qquad \lambda_o^{(\infty)} \to -\ 2\ h^2 + 4, \ h \to \infty \quad ,$$

it is seen that $\lambda_o^{(2)}$ yields a good approximation for small h, while $\lambda_o^{(\infty)}$ is good for large h.

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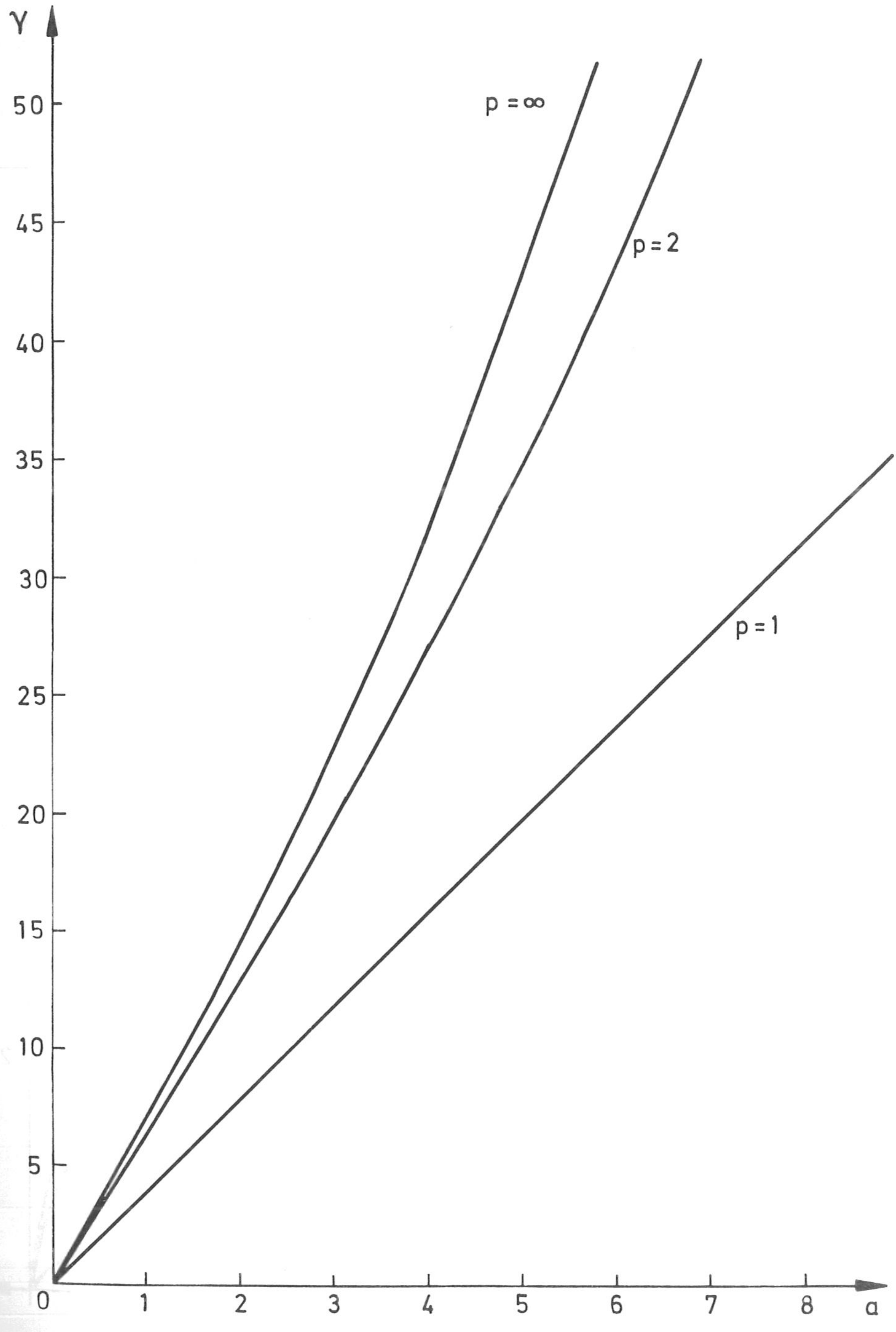
Figure captions

Fig. 1: Solutions $\gamma(a)$ of the variational problem for $p = 1, 2,$ and $\infty$.

Fig. 2: Lowest eigenvalue $a_o(h^2)$ of Mathieu's equation and lower bounds $\lambda_o^{(1)}$, $\lambda_o^{(2)}$ and $\lambda_o^{(\infty)}$.



γ
50
45
40
35
30
25
20
15
10
5
0
1
2
3
4
5
6
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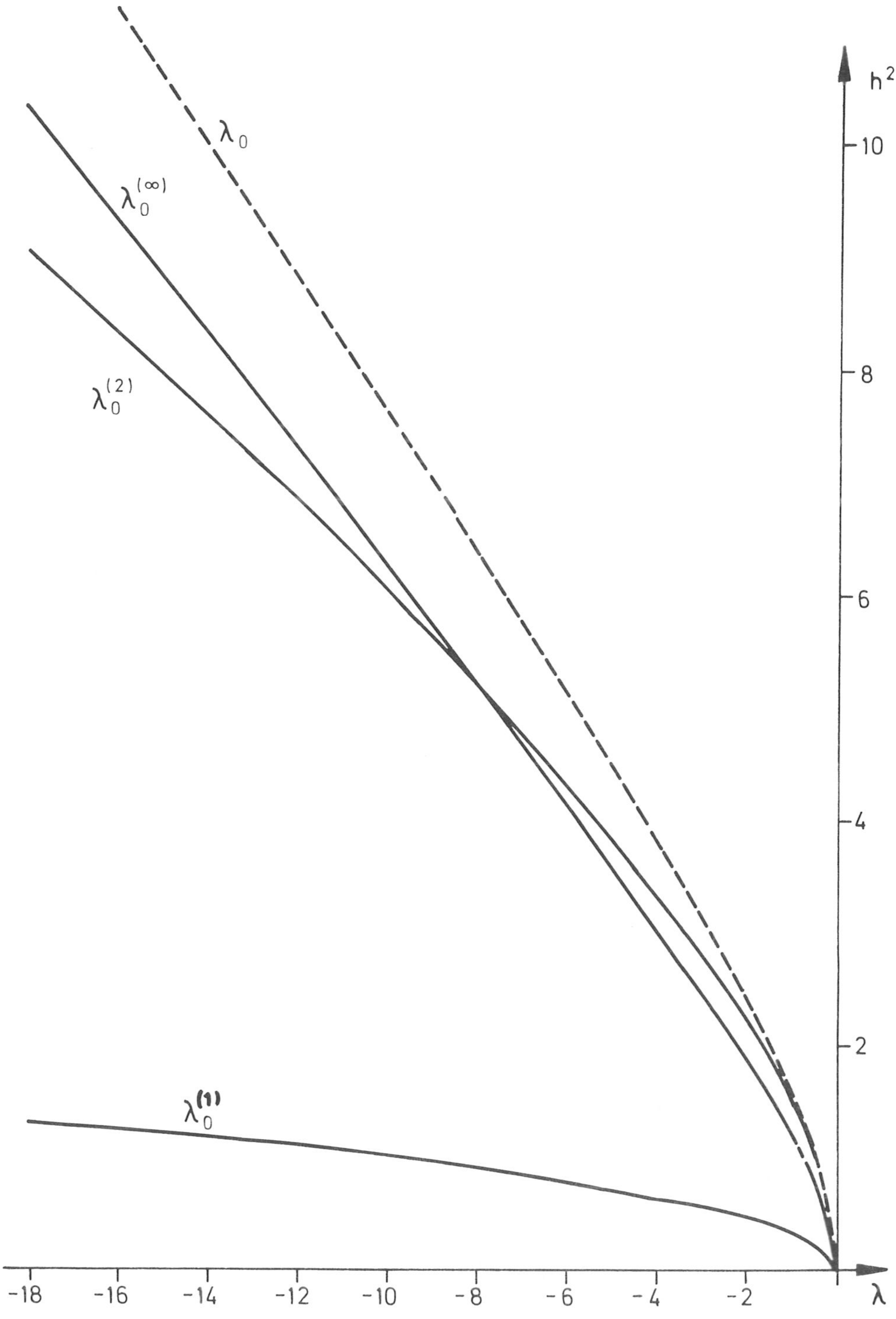
8
a
p = ∞
p = 2
p = 1



h²
10
8
6
4
2
λ₀
λ₀^(∞)
λ₀^(2)
λ₀^(1)
-18
-16
-14
-12
-10
-8
-6
-4
-2
λ