MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

ON THE STOCHASTIC STABILITY OF
MHD EQUILIBRIA

J. Teichmann

IPP 6/187 July 1979

On leave from the University of Montreal,
Physics Department, Montreal, Canada

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

IPP 6/187 J. Teichmann On the stochastic stability of MHD equilibria

July 1979
(in English)

ABSTRACT

The stochastic stability in the large of stationary equilibria of ideal and dissipative magnetohydrodynamics under the influence of stationary random fluctuations is studied using the direct Liapunov method. Sufficient and necessary conditions for stability of the linearized Euler-Lagrangian systems are given. The destabilizing effect of stochastic fluctuations is demonstrated.

1. INTRODUCTION

The stability of static and stationary magnetohydrodynamic equilibria has been intensively studied in the past in the framework of the so-called Energy Principles based on the Lagrangian formulation[1]. The Energy Principles and similar criteria established for a non-ideal magnetohydrodynamic equilibrium configuration remain an appealing method yielding qualitative information about the stability of the equilibrium by relatively simple means without the necessity of an eigenmode analysis. In connection with these principles it was demonstrated[2-18] for a broad class of MHD equilibria that the linearized equation of motion in the Euler-Lagrange description takes the form of an evolutional equation of the second order (assuming isentropic flow):

$$A \frac{\partial^2}{\partial t^2} \xi(t) + (B+C) \frac{\partial}{\partial t} \xi(t) + (D+E) \xi(t) = 0 \tag{1}$$

Here $\xi(t)$ is the n-dimensional vector of Lagrange displacements of the fluid from its equilibrium position $\xi = 0$, A, B, C, D and E are the linear time-independent operators defined in the corresponding inner product space.

The operator A is a Hermitian positive-definite operator which represents inertia, B is a Hermitian operator describing resistive and viscous damping, C is an anti-Hermitian operator of Lorentz forces which includes the damping due to the anti-symmetric part of the viscosity tensor and the asymmetric contribution of the intrinsic acceleration of the fluid, D is a Hermitian operator of conservative forces including those due to the perturbation of the fluid pressure, and E is the anti-Hermitian operator of non-conservative forces due to resistivity and viscosity. Equation (1) encompasses several limiting cases. For B=C=E=0 eq. (1) corresponds to the static equilibrium of the ideal MHD[2,3]. The system (1) with B=E=0 corresponds to the stationary equilibrium of ideal MHD[5], for B, C $\neq$ 0 and E = 0 eq. (1) describes the static equilibrium with resistivity [13, 14, 15], and for B, C $\neq$ 0 and E = 0 the system (1) corresponds exactly to the static equilibrium with resistivity and viscosity [18].

In the previous studies [2-18] necessary and sufficient conditions for exponential stability of the deterministic system (1) were given when E = 0. The same stability analysis can be extended under several restrictive conditions on the class of the permissible displacements to the case of stationary equilibria of resistive and viscous fluid when B, C $\neq$ 0 and simultaneously E $\doteq$ 0 [18].

Sufficient and necessary conditions for the stability of the dissipative isentropic flow (general system (1) with $E \neq O$) were recently given by the author [19].

In this note we discuss the stability of the magnetohydrodynamic equilibria described by the system (1) under the influence of random fluctuations [20].

The first section summarizes the stability analysis of the deterministic system (1). We give sufficient and necessary stability conditions for eq. (1) using the direct method of Liapunov. These conditions correspond in the limiting case $B=C=E=O$ to the criteria derived from the Energy Principle for ideal MHD [2, 9].

The next section generalizes this analysis to the case when the equilibrium is subjected to random fluctuations of one or more parameters of the fluid. Based on Itô calculus, the stochastic countrepart of the Liapunov functional is found and stochastic stability conditions are established.

The last section gives an example of stochastic destabilization in some simple particular cases - ideal MHD, the one-fluid resistive MHD model and two-fluid resistive magnetohydrodynamics. The influence of stochastic fluctuations of different plasma parameters on the stability in the large of the equilibrium is demonstrated. We discuss the question of self-consistency and limitations of the stochastic model chosen.

2. DETERMINISTIC STABILITY

Let us consider the linearized system (1). The operators A, B, C, D and E are linear time-independent operators with domain and range in the n-dimensional Hilbert space $\mathcal{H}_n(\mathcal{D})$ defined on the spatial domain $\mathcal{D}$ occupied by the fluid, with the inner product $\langle \, , \, \rangle$ and norm $\| \ \|$. $\xi(t) \in \mathcal{H}_n(\mathcal{D})$, $t \in T$, $T = [0,\infty)$. The operators A, B and D are Hermitian, A is positive definite. The operators C and E are anti-Hermitian. Let A admit the positive, compact, Hermitian inverse A^{-1}. Let us introduce the congruent mapping K from $\mathcal{H}_n(\mathcal{D})$ onto $\mathcal{H}_n(\mathcal{D})$, defined as follows:

$$
\begin{aligned}
&\xi = K\hat{\xi}, \quad K^*AK = I, \quad M = K^*BK, \quad N = K^*CK, \\
&P = K^*DK, \quad Q = K^*EK, \\
&K\hat{\xi} \in \mathcal{H}_n(\mathcal{D}).
\end{aligned}
\tag{2}
$$

Using the mapping (2), eq. (1) reads

$$\ddot{\hat{\xi}}(t) + (M+N)\dot{\hat{\xi}}(t) + (P+Q)\hat{\xi}(t) = 0. \tag{3}$$

Let $\mathcal{H}_n(\mathcal{D}) \times \mathcal{H}_n(\mathcal{D})$ denote the 2n-dimensional Hilbert space. Introducing the 2n-dimensional displacement vector $\hat{\zeta} = (\hat{\xi}, \dot{\hat{\xi}})^T$, we may rewrite eq. (3) in canonical form

$$\dot{\hat{\zeta}}(t) = \begin{pmatrix} 0 & I \\ -(P+Q) & -(M+N) \end{pmatrix} \hat{\zeta}(t). \tag{4}$$

The initial conditions read $\hat{\xi}(0) = \hat{\xi}_0 \in \mathcal{H}_n(\mathcal{D})$, $\dot{\hat{\xi}}(0) = \dot{\hat{\xi}}_0 \in \mathcal{H}_n(\mathcal{D})$.

We now analyse the stability of the system (4) using the direct Liapunov method. For this purpose we have to construct a suitable Liapunov functional $V(\hat{\xi},t)$ which is positive-definite for arbitrary $\hat{\xi}$ and t. Owing to the general formulation of the system (4), the only functional which assures the desired definitivness for $V(\hat{\xi},t)$ and for the time derivative $\frac{d}{dt} V(\hat{\xi},t)$ taken along the trajectories (4) is a quadratic form in variables $\hat{\xi}$ and $\hat{\xi}^{\cdot}$. It is easy to define $V(\hat{\xi}, t)$, provided that the operator $E = 0$. In a general case, $Q \neq 0$ $(E \neq 0)$, a suitable functional $V(\hat{\xi}, t)$ has not yet been found. Nevertheless, a generalized functional transform, applied to the system (4) allows one to construct the functional $V(\hat{\xi}, t)$ having the desired properties.

Let us introduce the time-dependent linear unitary transform $L(t)$, $L(t)\hat{\xi} \in \mathcal{H}_n(\mathcal{D})$ such that [19]

$$\begin{pmatrix} \hat{\xi} \\ \hat{\xi}^{\cdot} \end{pmatrix} = \begin{pmatrix} L & 0 \\ TL & L \end{pmatrix} \begin{pmatrix} \eta \\ \eta^{\cdot} \end{pmatrix} . \tag{5}$$

Let $L(t)$ be bounded in the time interval $[0,\infty)$, have a bounded continuous derivative $L^{\cdot}(t)$ and have a lower bound $0 < m < \| L(t) \|$ for $t \geq 0$. Let the isometry $L(t)$ verify simultaneously the differential equation

$$L^{\cdot}(t) = \left\{ -(M+N)^{+}(Q+H) + [I-(M+N)^{+}(M+N)R] \right\} L(t) = \equiv T\,L(t) \tag{6}$$

and the constraint

$$L^*(t)\ L(t) = L^*(0)\ L(0) = I.$$

Let us further suppose that the operator (M+N) is bounded with closed range. Then $(M+N)^+$ is the Moore-Penrose generalized inverse and $(M+N)^+ \in \mathcal{H}_n(\mathcal{D})$. The arbitrary bounded operators H and R are defined from the conditions (6), H being Hermitian. Introducing eq. (5) into eq. (4) and premultiplying eq. (4) by $\begin{pmatrix} L^* & 0 \\ 0 & L^* \end{pmatrix}$, one obtains in the space $\mathcal{H}_n(\mathcal{D}) \times \mathcal{H}_n(\mathcal{D})$

$$\bar{\xi}^{\cdot}(t) = \begin{pmatrix} 0 & I \\ -\hat{P}(t) & -\hat{M}(t) - \hat{N}(t) \end{pmatrix} \bar{\xi}(t), \tag{7}$$

where $\bar{\xi} = (\eta, \dot{\eta})^T$. The transformed operators are now time-dependent and are defined as follows:

$$\hat{M} = \hat{M}^* = L^* ML, \ -\hat{N}^* = \hat{N} = L^*(2T+N)L, \ \hat{P}^* = \hat{P} = L^*(P+TT+S)L,$$

$$S = \frac{1}{2}(MT-TM) + \frac{1}{2}(NT+TN). \tag{8}$$

The initial conditions are now $\bar{\xi}(0) = [\eta(0), \dot{\eta}(0)]^T$. We observe that the system (7) does not have the operator $\hat{Q}$, $\hat{Q} = 0$. Thus a usual procedure may be followed in constructing the functional $V_D(\bar{\xi}, t)$. Without the limitation on the generality of our result, we can assume that the null solution of eq. (7) is $\bar{\xi}(0) = (0,0)^T$. Let us define the Liapunov functional $V_D(\bar{\xi}, t)$ to be positive-definite on $\mathcal{H}_n(\mathcal{D}) \times \mathcal{H}_n(\mathcal{D})$, continuously differentiable in t and $\bar{\xi}$, defined for all $t \geq 0$ in a set Ω, Ω: $\|\bar{\xi}\| < a$, $V_D(0,t) = 0$ for $t \geq 0$, $V_D(\bar{\xi},t) \geq U(\bar{\xi})$ for all $\bar{\xi}$ in Ω and all $t \geq 0$. These properties possesses a functional

$$V_D(\bar{\xi},t) = \langle \bar{\xi}, \hat{S}\bar{\xi} \rangle > 0, \tag{9}$$

where $\hat{S}$ is a Hermitian 2n-dimensional operator. Let us take for $\hat{S}$: $\hat{S} = \begin{pmatrix} \hat{P} & 0 \\ 0 & I \end{pmatrix}$. Then if the time derivative $\frac{d}{dt} V_D(\hat{\xi},t)$ taken along the trajectories (7)

$$\frac{d}{dt} V_D(\bar{\xi},t) = \langle \dot{\bar{\xi}}, \hat{S}\bar{\xi} \rangle + \langle \bar{\xi}, \hat{S}\dot{\bar{\xi}} \rangle + \langle \bar{\xi}, \dot{\hat{S}}\bar{\xi} \rangle \leq 0 \tag{10}$$

is negative-semi-definite, the trivial solution of eq. (7) $\bar{\xi} = 0$ is stable in the sense of Liapunov; if the derivative (10) is negative-definite, the null solution is asymptotically stable. These conditions are sufficient and necessary owing to the validity of the converse theorem. Using eqs. (7) and (8), we may write the stability conditions in the form

$$\langle \bar{\xi}, \hat{S}\bar{\xi} \rangle > 0 \quad , \quad \langle \bar{\xi}, J\bar{\xi} \rangle \leq 0 \quad , \tag{11}$$

where

$$J \equiv \begin{pmatrix} L^{*}\left[(P+S)T - T(P+S)\right] L & 0 \\ 0 & -2\, L^{*} ML \end{pmatrix} .$$

Let us note that if P and M are Hermitian, J is Hermitian. Then both $V_D(\bar{\xi},t)$ and $\frac{d}{dt} V_D(\bar{\xi},t)$ are expressed in the form of quadratic functionals in $\bar{\xi}$. The stability of the system (7) implies the stability of the system (1). In the case E = O in eq. (1) we have $L(t) = L^{*}(t) = I$, $T = O$, $V_D = V_D(\hat{\xi})$ and the stability conditions read

$$\langle \hat{\xi}, \hat{S}\hat{\xi} \rangle > 0 \quad , \quad \langle \hat{\xi}, J\hat{\xi} \rangle \leq 0 \tag{12}$$

with

$$\hat{S} \equiv \begin{pmatrix} P & O \\ O & I \end{pmatrix} \quad , \quad J = \begin{pmatrix} O & O \\ O & -2M \end{pmatrix} .$$

These conditions correspond to previous studies[2-18]. Let us note that in the case E = O the operator N is irrelevant for stability in contrast to the general case $E \neq O$. If, furthermore, M = O, as is the case with ideal MHD, the time derivative $\frac{d}{dt} V_D(\xi) = 0$ and the system (1) cannot be asymptotically stable.

Let us note that the operator A is nonsingular in correctly defined fluid equations. In the case of singular A, the intrinsic symmetry between the dynamic variables and the corresponding electromagnetic field is violated (e.g. by neglecting the displacement current). In dissipative fluids, the displacement current must not be neglected in the field equation, even if apparently small. In the limiting case of ideal MHD, the operator A becomes the identity operator I.

3. STOCHASTIC STABILITY

Let us write the deterministic system (4) in the form

$$\frac{d}{dt}\hat{\xi}(t) = F(\hat{\xi}, t)\,, \quad 0 \leq t \leq T. \tag{13}$$

The operator F is a function of equilibrium fluid parameters such as density, magnetic field, etc. Under the influence of MHD turbulence, turbulence due to kinetic effects, fluctuations of external fields and forces, one or more of the fluid parameters may vary randomly around a certain mean value with relatively small amplitudes. We may thus assume that the operator F is in general a function of one or more stochastic variables.

Random variables are tacitly referred to an underlying probability space $(\Omega, \mathcal{F}, P)$, P being the probability measure on a σ-algebra $\mathcal{F}$ of subsets of Ω. $t \in T$, T is a linear index set.

Assuming that eq. (13) admits a unique solution $\hat{\xi}(t, \hat{\xi}_o)$, $t \in T$ for every initial state $\hat{\xi}_o(0)$, we write for the randomly perturbed evolutional equation (13)

$$\frac{d}{dt}\hat{\xi}(t) = F(\hat{\xi}, t) + G(t, \hat{\xi})\,\nu(t). \tag{14}$$

G is the 2nxm matrix valued functional of $\hat{\xi}$, t and represents the m-dimensional Gaussian process. The drift term F ($\hat{\xi}$, t) is additively perturbed by a disturbance ν(t), which is a stationary random process normally distributed with zero mean having the spectral density approximativly constant up to frequencies which are high relative to the time scale of (13). The effective disturbance magnitude may depend on t and $\hat{\xi}$, the resulting paths $\hat{\xi}$ (t) are continuous. To be able to benefit from the powerful apparatus of the stochastic calculus, we have to use instead of eq. (14) a precisely defined mathematical model. A suitable model equation is Itô's stochastic differential equation [21]

$$d\hat{\xi}(t) = F(\hat{\xi}, t)\,dt + G(\hat{\xi}, t)\,dW_t \quad , t \in T, \tag{15}$$

where W_t is the m-dimensional vector-valued standard Wiener process. $W_t(t) = [\, W_1(t), \ldots W_m(t) \,]^T$, where $W_j(t)$ are mutually independent scalar-valued Wiener processes such that

$$\mathcal{E}\{ W_j(t) - W_j(s) \} = 0 \qquad , j = 1, \cdots n$$

$$\mathcal{E}\{ [W_j(t) - W_j(s)]^2 \} = |t - s|$$

for all t, s $\in$ T. $\mathcal{E}\{\ \}$ denotes the expectation operator. It is assumed that F and G are measurable in t, $\hat{\xi}$ for t $\in$ T and $\hat{\xi} \in \mathcal{H}_{2n}(\mathcal{D})$, G being a non-anticipating operator. With a suitable hypothesis on F, G and $\hat{\xi}$ the solution process of eq. (15) $\hat{\xi}$ (t) is unique and is a separable measurable Markov process. We also assume that the sample paths of the process $\hat{\xi}(t)$ are defined for all $t \in T$ i.e. we suppose that the killing time is equal to infinity with probability one [22].

The solution process $\hat{\xi}(t)$ of eq. (15) is associated with the differential generator [22]

$$\mathcal{L} \equiv \frac{\partial}{\partial t} + \sum_{i=1}^{2n} F_i(\hat{\xi},t)\frac{\partial}{\partial \hat{\xi}_i} + \\ + \frac{1}{2}\sum_{i,j=1}^{2n} \left[G(\hat{\xi},t)\, G^T(\hat{\xi},t) \right]_{ij} \frac{\partial^2}{\partial \hat{\xi}_i \partial \hat{\xi}_j} \qquad (16)$$

The first two terms of the generator (16) represent the Eulerian time derivative along the deterministic paths of eq. (15). The last term containing the Hessian $\nabla^2_{\hat{\xi}\hat{\xi}}$ expresses the diffusion due to the Wiener process W_t.

By analogy with the deterministic case we discuss the stability of eq. (15) using the stochastic direct Liapunov method. The stochastic analogy with the deterministic time derivative $\frac{d}{dt} V_D(\hat{\xi},t)$ is now the backward diffusion operator $\mathcal{L} V_s(\hat{\xi},t)$.

Using the solution process $\hat{\xi}(t)$ of eq. (15), we define the stochastic Liapunov functional

$$V_s(\hat{\xi},t) = \langle \hat{\xi}, \hat{S}\hat{\xi} \rangle \qquad (17)$$

with the Hermitian operator $\hat{S}$. The functional $V_s(\hat{\xi},t)$

with compact support is supposed to be non-negative, continuous in both $\hat{\xi}$ and t and posssessing a partial derivative in t and partial derivatives in $\hat{\xi}$ up to the second order. V_s is defined in the domain of the operator $\mathcal{L}$. The stochastic stability can be defined in many ways [22, 23]. For our purpose, we use only the definition of the stability in probability, which requires that for an equilibrium solution $\hat{\xi}(t) = 0$

$$P\left\{ \sup_{0 \leq t < \infty} \| \hat{\xi}(\hat{\xi}_0) \| \geq \varepsilon \right\} = 0 \; , \quad \varepsilon > 0, \; \| \hat{\xi}_0 \| < \delta \, .$$

Similarly, the asymptotic stability in probability is defined by

$$\lim_{\xi_0 \to 0} P\left\{ \lim_{t \to \infty} \hat{\xi}(\hat{\xi}_0) = 0 \right\} = 1 \, .$$

The sufficient conditions for stochastic stability (asymptotic stability) are thus[22]

$$\mathcal{E}\{ d V_s(\hat{\xi}, t) \} \leq 0 \, . \tag{18}$$

In the case of the eq. (15), the process $V_s(\hat{\xi}, t)$ is again associated with the differential generator eq. (16). Because of

$$\mathcal{E}\{ d \bar{V}_s(\hat{\xi}, t) \} = \mathcal{E}\{ \mathcal{L} \bar{V}_s(\hat{\xi}, t) \, dt \}$$

the condition (18) reads

$$W_s(\hat{\xi}, t) \equiv \mathcal{L} V_s(\hat{\xi}, t) \leq 0 \, , \; t \leq T. \qquad (19)$$

This condition expresses simultaneously the supermartingale property [22] of the process $V_s(\hat{\xi}, t)$.

In the case of a stochastic analogue to the deterministic evolutional equation (7) we write Itô's equation (15) in the form

$$d\bar{\xi}(t) = Q\bar{\xi}(t)\,dt + G(\bar{\xi}, t)\,d\bar{W}_t \, , \quad t \in T. \qquad (20)$$

Then, the conditions for stability (asymptotic stability) in probability read

$$\langle \bar{\xi}, \hat{S}\bar{\xi} \rangle > 0,$$

$$\mathcal{E}\{\langle Q\bar{\xi}, \hat{S}\bar{\xi}\rangle + \langle \bar{\xi}, \hat{S}Q\bar{\xi}\rangle + \langle \bar{\xi}, \hat{S}^{\bullet}\bar{\xi}\rangle +$$

$$+\frac{1}{2}\langle \sum_{\alpha,\beta,\gamma=1}^{2n} G_{\alpha\beta}(\bar{\xi}) G_{\beta\alpha}(\bar{\xi}) \frac{\partial}{\partial \bar{\xi}_\alpha} \frac{\partial}{\partial \bar{\xi}_\beta} (\bar{\xi}, \hat{S}\bar{\xi})\rangle\} \leq 0.$$

(21)

Here the symbol (,) defines the inner product without integration in space.

Owing to the definition of the inner product the conditions (11), (12) and (21) describe the stability in the large.

Let us consider in more detail the properties of the system (7). We observe that stochastic perturbations affect the operators $\hat{P}$, $\hat{M}$, and $\hat{N}$ in eq. (7). Thus in the diffusion matrix operator G $(\hat{\xi}, t)$ all elements for which $0 \leq 1, j \leq n$ vanish. The Hessian in eq. (16) then operates only on that part of the Liapunov functional which is proportional to velocities $\dot{\eta}$. Taking for the drift operator F in eq. (15) the form (7) and defining the operator $\hat{S}$ in eq. (17) identically as was done in eq. (9), we obtain the sufficient conditions for stochastic stability (21) in the form

$$V_s(\bar{\xi},t) \equiv \langle \bar{\xi}, \hat{S}\bar{\xi} \rangle > 0$$

$$\bar{W}_s(\bar{\xi},t) \equiv \langle \bar{\xi}, J\bar{\xi} \rangle + \langle \mathrm{Sp}[\hat{G}(\bar{\xi},t)\, \hat{G}^T(\bar{\xi},t)] \rangle \leq 0 \tag{22}$$

The operator J is defined for different forms of the deterministic equation (1) in eqs. (11) and (12). The operator $\hat{G}$ $(\bar{\xi},t)$ is obtained from G $(\hat{\xi},t)$ as defined in eq. (15) with the help of the transform (5).

If we consider the functional W_s $(\bar{\xi},t)$ as a total time derivative of the process $\bar{V}_s(\bar{\xi},t)$, we may solve the system (20) with respect to the functional $\bar{V}_s(\bar{\xi},t)$ to obtain

$$\langle \bar{\xi}, \hat{S}\bar{\xi} \rangle_t = \langle \bar{\xi}, \hat{S}\bar{\xi} \rangle_{t=t_0} \exp \int_{t_0}^{t} d\tau \frac{\bar{W}_s(\bar{\xi},\tau)}{V_s(\bar{\xi},\tau)} . \tag{23}$$

This relation may serve as a definition of the growth of the Lagrangian deplacement $\bar{\xi}$ in the case of instability.

Let us note that owing to the fact that for ideal MHD equilibria the operator J = O and the term

$$\langle Sp \quad G(\hat{\xi},t)\; G^T(\hat{\xi},t)\rangle$$

is positive-definite, the stochastic fluctuations of any plasma parameter in the ideal MHD lead to instability. We observe also that for dissipative fluid at rest, i.e. when in eq. (1) the operator E = O, any random disturbation leading to the perturbed operator P_s results in unstable growth of $\hat{\xi}$ (t).

4. EXAMPLES

As a first example we discuss the stochastic perturbation of a static equilibrium of an ideal magnetohydrodynamic fluid. The corresponding fluid equations are

$$\rho \frac{d}{dt} \underset{\sim}{v} = -\nabla p + \underset{\sim}{J} \times \underset{\sim}{B} \quad ,$$

$$\frac{d}{dt} \rho + \nabla \cdot (\rho \underset{\sim}{v}) = 0 \quad , \tag{24}$$

$$\frac{d}{dt} (p \rho^{-\gamma}) = 0$$

The perturbation algorithm is formalized by introducing two independent small parameters δ and σ. δ corresponds to the small parameter of Euler-Lagrangian displacements and measures the departure from the equilibrium state. This equilibrium depends on time owing to the random modulation of the basic fluid quantities. The construction of this state involves an other small parameter σ, which expresses the rate of modulation imposed on fluid parameters. Let us introduce the constraint $\mathcal{E}\{\underset{\sim}{v}(t)\}$ = 0, i.e. the equilibrium randomly unmodulated is a static equilibrium $\underset{\sim}{v}_0$ = 0. For the equilibrium quantities, we introduce

$$\begin{aligned} \underset{\sim}{B} &= \underset{\sim}{B}_0 \sum_{n=0}^{\infty} \sigma^n b_n \varphi_n^{(B)}(t), \\ \underset{\sim}{v} &= \sigma \sum_{n=0}^{\infty} \sigma^n \underset{\sim}{v}_n \varphi_n^{(v)}(t), \\ &\vdots \end{aligned} \tag{25}$$

where $\varphi_n(t)$ are the random processes depending on the parameter $t \in T$, having zero mean values. The Euler--Lagrangian displacements will be developed around the equilibrium (25). In the framework of the linearized theory, we keep only terms of the orders σ, δ and $\sigma\delta$. Completing the system (24) by the Maxwell equations, we observe that the zeroth-order equilibrium is

$$0 = -\nabla p_0 + \underset{\sim}{J}_0 \times \underset{\sim}{B}_0$$

$$\nabla \times \underset{\sim}{B}_0 = \mu_0 \underset{\sim}{J}_0$$

The first-order equilibrium equations are then

$$\rho_0 \frac{\partial}{\partial t} \underset{\sim}{v}_1 = -\nabla p_1 + (\underset{\sim}{J}_0 \times \underset{\sim}{B}_1) + (\underset{\sim}{J}_1 \times \underset{\sim}{B}_0)$$

$$\frac{\partial}{\partial t} \rho_1 + \nabla \cdot (\rho_0 \underset{\sim}{v}_1) = 0 \,, \quad \frac{\partial}{\partial t} \underset{\sim}{B}_1 = \nabla \times (\underset{\sim}{v}_1 \times \underset{\sim}{B}_0) \,.$$

These equations allow us to find the first two terms in series eq. (25). We have

$$B = B_0 + \sigma B_0 \varphi(t) + \mathcal{O}(\sigma^2)$$

$$\rho = \rho_0 + \sigma \rho_1 \varphi(t) + \mathcal{O}(\sigma^2)$$

$$\underset{\sim}{v} = \sigma \underset{\sim}{v}_1 \dot{\varphi}(t) + \mathcal{O}(\sigma^2)$$

$$\nabla p = \nabla p_0 + \sigma (\nabla p_{11}) \varphi(t) + \sigma (\nabla p_{12}) \ddot{\varphi}(t) + \mathcal{O}(\sigma^2)$$

where

$$\nabla \times (\underset{\sim}{v}_1 \times \underset{\sim}{B}_0) = \underset{\sim}{B}_0 \,, \quad \rho_1 = -\nabla \cdot (\rho_0 \underset{\sim}{v}_1)$$

$$\nabla p_{11} = \frac{2}{\mu_0} (\nabla \times \underset{\sim}{B}_0) \times \underset{\sim}{B}_0 \,, \quad (\nabla p_{12}) = -\rho_0 \underset{\sim}{v}_1 \,.$$

The Euler-Lagrangian displacements then around the time-dependent equilibrium are now

$$(\delta\rho) = -\nabla\cdot(\rho_0\underset{\sim}{\xi}) - \sigma\varphi\,\nabla\cdot(\rho_1\underset{\sim}{\xi})\,,$$

$$(\delta p) = -\gamma p_0(\nabla\cdot\underset{\sim}{\xi}) - \underset{\sim}{\xi}\cdot\nabla p_0 - \sigma\{\varphi\gamma p_{11}(\nabla\cdot\underset{\sim}{\xi}) - \ddot{\varphi}\gamma p_{12}(\nabla\cdot\underset{\sim}{\xi})$$
$$- \varphi\,\underset{\sim}{\xi}\cdot\nabla p_{11} - \ddot{\varphi}\,\underset{\sim}{\xi}\cdot\nabla p_{12}\}$$

$$(\delta\underset{\sim}{v}) = \dot{\underset{\sim}{\xi}} + \sigma\{\dot{\varphi}(\underset{\sim}{v}_1\cdot\nabla)\underset{\sim}{\xi} - \dot{\varphi}(\underset{\sim}{\xi}\cdot\nabla)\underset{\sim}{v}_1\},$$

$$(\delta\underset{\sim}{B}) = \nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0) + \sigma\varphi\,\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0)\,. \qquad (24)$$

With these displacements, the momentum transfer equation in (24) yields

$$\ddot{\underset{\sim}{\xi}} + \frac{1}{\rho_0}D\underset{\sim}{\xi} + \sigma\dot{\varphi}Q_1\dot{\underset{\sim}{\xi}} + \sigma\varphi Q_2\underset{\sim}{\xi} + \sigma\ddot{\varphi}Q_3\underset{\sim}{\xi} = 0 \qquad (27)$$

where D is the usual force operator [2]:

$$D\underset{\sim}{\xi} = -\gamma\nabla[p_0(\nabla\cdot\underset{\sim}{\xi})] - \nabla(\underset{\sim}{\xi}\cdot\nabla p_0) - \frac{1}{\mu_0}(\nabla\times\underset{\sim}{B}_0)\times[\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0)]$$
$$- \frac{1}{\mu_0}[\nabla\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0))]\times\underset{\sim}{B}_0$$

and

$$Q_1 \dot{\underset{\sim}{\xi}} = 2\,(\underset{\sim}{v}_1 \cdot \nabla)\,\dot{\underset{\sim}{\xi}}\,,$$

$$Q_2 \underset{\sim}{\xi} = -\frac{\rho_1}{\rho_0^2} D\underset{\sim}{\xi} - \frac{\gamma}{\rho_0} \nabla[P_{11}(\nabla\cdot\underset{\sim}{\xi})] - \frac{1}{\rho_0}\nabla(\underset{\sim}{\xi}\cdot\nabla P_{11}) -$$

$$-\frac{2}{\mu_0}\frac{1}{\rho_0}\{[(\nabla\times\underset{\sim}{B}_0)\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0))] + [\nabla\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0))]\times\underset{\sim}{B}_0\}\,,$$

$$Q_3 \underset{\sim}{\xi} = (\underset{\sim}{v}_1\cdot\nabla)\underset{\sim}{\xi} - (\underset{\sim}{\xi}\cdot\nabla)\underset{\sim}{v}_1 - \frac{1}{\rho_0}\underset{\sim}{v}_1\nabla\cdot(\rho_0\underset{\sim}{\xi}) - \frac{\gamma}{\rho_0}\nabla[P_{12}(\nabla\cdot\underset{\sim}{\xi})] - \frac{1}{\rho_0}\nabla(\underset{\sim}{\xi}\cdot\nabla P_{12})\,.$$

In all practical situations the process $\varphi(t)$ will be Gaussian. In this case $\dot{\varphi}(t)$ is independent of $\varphi(t)$ and $\ddot{\varphi}(t)$ but $\ddot{\varphi}(t)$ depends on $\varphi(t)$. We assume that $\varphi(t)$ is a continuous time process. In order to make the transition from the evolutional equation (27) to the desired mathematical model (15), we assume that $\varphi(t)$, $\dot{\varphi}(t)$, $\ddot{\varphi}(t)$ are represented by three Gaussian white noise processes $\nu_1(t)$, $\nu_2(t)$, $\nu_3(t)$, so that

$$\mathcal{E}\{\nu_\alpha(t)\} = 0\,, \quad \mathcal{E}\{\nu_\alpha(t)\,\nu_\beta(s)\} = \delta(t-s)\,C_{\alpha\beta}(t)\,,$$

where δ is the Dirac delta function, $C_{\alpha\beta}$ the nonsingular, positive, symmetric cross-covariance matrix with elements of the order 1. Assuming that $\nu_1(t)$ correlates with $\nu_3(t)$, and that $\nu_2(t)$ correlates neither with $\nu_1(t)$ nor with $\nu_3(t)$, the cross-covariance matrix reads

$$C_{\alpha\beta}(t) = \begin{pmatrix} c_1^2 & 0 & 0 \\ 0 & c_2^2 & c_2 c_3 \\ 0 & c_2 c_3 & c_3^2 \end{pmatrix}.$$

Let us construct the process Y (t) which has ν_α (t) as components

$$dY(t) = [\nu_1(t), \nu_2(t), \nu_3(t)]^T dt,$$

and which verifies the relation

$$\mathcal{E}\{Y(t)\, Y^T(s)\} = \int_{t_0}^{\min(t,s)} du\, C(u) \quad , \; C \equiv C_{\alpha\beta} .$$

We can express Y(t) using the Wiener process W_s

$$dY(t) = C^{1/2}(s)\, dW_s$$

For a positive-definite nonsingular matrix it is always possible to extract a square root. Thus

$$[C^{1/2}(s)]_{11} = C_1 \; , \; [C^{1/2}(s)]_{\alpha\beta} = \frac{C_\alpha C_\beta}{\sqrt{C_2^2 + C_3^2}} \quad , \; \alpha,\beta = 2,3 .$$

The process W_s has now three independent components, $dW_s = [\, dW_1,\, dW_2,\, dW_3 \,]^T$. Itô's model evolutional equation related to the eq. (23) takes the form

$$d\xi(t) = \begin{pmatrix} 0 & I \\ -\frac{1}{\rho_0} D & 0 \end{pmatrix} \xi(t) + G(\xi,t)\, dW_s \tag{28}$$

where

$$G(\xi,t) = \sigma \begin{pmatrix} 0 & 0 & 0 \\ -C_1 Q_1 \underset{\sim}{\xi} & -\frac{C_2^2}{d} Q_2 \underset{\sim}{\xi} - \frac{C_2 C_3}{d} Q_3 \underset{\sim}{\xi} & -\frac{C_2 C_3}{d} Q_2 \underset{\sim}{\xi} - \frac{C_3^2}{d} Q_3 \underset{\sim}{\xi} \end{pmatrix},$$

$$d = \sqrt{C_2^2 + C_3^2} .$$

With these definitions we construct the Liapunov functional $V_s(\xi, t)$ according to eq. (17) as a process

$$V_s(\xi, t) = (\xi, \hat{S}\xi) \quad \text{with} \quad \hat{S} = \begin{pmatrix} \frac{1}{\rho_0} D & 0 \\ 0 & I \end{pmatrix}.$$

The stability conditions (22) now read

$$\langle \underset{\sim}{\xi}, \frac{1}{\rho_0} D \underset{\sim}{\xi} \rangle + \langle \underset{\sim}{\dot{\xi}}, \underset{\sim}{\dot{\xi}} \rangle > 0 ,$$

$$\sigma^2 \Big\{ C_1^2 \langle Q_1 \underset{\sim}{\dot{\xi}}, Q_1 \underset{\sim}{\dot{\xi}} \rangle + \frac{1}{C_1^2 + C_2^2} \langle (C_2^2 Q_2 \underset{\sim}{\xi} + C_2 C_3 Q_3 \underset{\sim}{\xi}), (C_2^2 Q_2 \underset{\sim}{\xi} + C_2 C_3 Q_3 \underset{\sim}{\xi}) \rangle$$

$$+ \frac{1}{C_2^2 + C_3^2} \langle (C_2 C_3 Q_2 \underset{\sim}{\xi} + C_2^2 Q_3 \underset{\sim}{\xi}), (C_2 C_3 Q_2 \underset{\sim}{\xi} + C_2^2 Q_3 \underset{\sim}{\xi}) \rangle \Big\} \leq 0 \tag{29}$$

Evidently, the second stability condition is not satisfied for $\sigma \neq 0$. Thus the stochastic perturbation of an equilibrium of ideal MHD has a destabilizing influence. The temporal growth of the Lagrangian displacement $\xi(t)$ is given by eq. (23).

The same results are obtained for a non-correlated white noise Gaussian processes. In this case C = I and

$$Sp\left[G(\xi, t)\, G^T(\xi, t)\right] = \sigma^2 \Big\{ (Q_1 \underset{\sim}{\dot{\xi}}, Q_1 \underset{\sim}{\dot{\xi}}) + (Q_2 \underset{\sim}{\xi}, Q_2 \underset{\sim}{\xi}) + (Q_3 \underset{\sim}{\xi}, Q_3 \underset{\sim}{\xi}) \Big\}.$$

Again, $\mathcal{L} V_s(\xi, t) \geqq 0$ for $\sigma \neq 0$ and the equilibrium is unstable.

To generalize the previous discussion, we assume that the magnetostatic field $\underset{\sim}{B}_o$ is perturbed by two stochastic components - one in the direction of $\underset{\sim}{B}_o$, the other one in a direction perpendicular to $\underset{\sim}{B}_o$. The equilibrium quantities up to the order σ^2 are now

$$\underset{\sim}{B} = \underset{\sim}{B}_o + \sigma_{\parallel} \varphi_{\parallel}(t) \underset{\sim}{B}_o + \sigma_{\perp} \varphi_{\perp}(t) \underset{\sim}{B}_{o\perp}$$

$$\rho = \rho_o + \sigma_{\parallel} \varphi_{\parallel}(t) \rho_1 + \sigma_{\perp} \varphi_{\perp}(t) \rho_2$$

$$\underset{\sim}{v} = \qquad \sigma_{\parallel} \dot{\varphi}_{\parallel}(t) \underset{\sim}{v}_1 + \sigma_{\perp} \dot{\varphi}_{\perp}(t) \underset{\sim}{v}_2$$

$$p = p_o + \sigma_{\parallel} [\varphi_{\parallel}(t) p_{11} + \ddot{\varphi}_{\parallel}(t) p_{12}] + \sigma_{\perp} [\varphi_{\perp}(t) p_{21} + \ddot{\varphi}_{\perp}(t) p_{22}],$$

where ρ_j, p_{ij} and $\underset{\sim}{v}_j$ are defined from the equilibrium equations. Substantially the same calculations as before give the evolutional equation

$$\begin{aligned} \ddot{\underset{\sim}{\xi}} + \frac{1}{\rho_o} D \underset{\sim}{\xi} + (\sigma_{\parallel} \dot{\varphi}_{\parallel} Q_{1,\parallel} + \sigma_{\perp} \dot{\varphi}_{\perp} Q_{1,\perp}) \dot{\underset{\sim}{\xi}} + (\sigma_{\parallel} \varphi_{\parallel} Q_{2,\parallel} + \sigma_{\perp} \varphi_{\perp} Q_{2,\perp}) \underset{\sim}{\xi} \\ + (\sigma_{\parallel} \ddot{\varphi}_{\parallel} Q_{3,\parallel} + \sigma_{\perp} \ddot{\varphi}_{\perp} Q_{3,\perp}) \underset{\sim}{\xi} = 0. \end{aligned} \tag{30}$$

The operators $Q_{j,\parallel}$, $Q_{j,\perp}$ are obtained from the operators $Q_{j,\parallel}$ defined in eq. (26) by replacing the subscript 1 by 2, with the exception of the last term in the operator $Q_{2,\perp}$, which now reads

$$-\frac{1}{\mu_0}\frac{1}{\rho_0}\Big\{(\nabla\times\underset{\sim}{B}_0)\times[\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_{0\perp})]+(\nabla\times\underset{\sim}{B}_{0\perp})\times[\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0)]$$

$$+[\nabla\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_{0\perp}))]\times\underset{\sim}{B}_0+[\nabla\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0))]\times\underset{\sim}{B}_{0\perp}\Big\}$$

The evolutional equation (30) is now modelled by Itô's equation with a Wiener process having 6 components. If one assumes for simplicity that all Wiener processes W_j are uncorrelated, the additional term in the second stability condition (22) is

$$\langle Sp\, G(\xi)\, G^T(\xi)\rangle = \sigma_{\parallel}^2\Big\{\langle Q_{1,\parallel}\dot{\underset{\sim}{\xi}},\, Q_{1,\parallel}\underset{\sim}{\xi}\rangle+\langle Q_{2,\parallel}\underset{\sim}{\xi},\, Q_{2,\parallel}\underset{\sim}{\xi}\rangle$$
$$+\langle Q_{3,\parallel}\underset{\sim}{\xi},\, Q_{3,\parallel}\underset{\sim}{\xi}\rangle\Big\}$$
$$+\sigma_{\perp}^2\Big\{\langle Q_{1,\perp}\dot{\underset{\sim}{\xi}},\, Q_{1,\perp}\dot{\xi}\rangle+\langle Q_{2,\perp}\underset{\sim}{\xi},\, Q_{2,\perp}\underset{\sim}{\xi}\rangle+\langle Q_{3,\perp}\underset{\sim}{\xi},\, Q_{3,\perp}\underset{\sim}{\xi}\rangle\Big\} \tag{31}$$

Again, the stability conditions (22) are not verified for $\sigma_{\parallel}$, $\sigma_{\perp} \neq 0$.

The destabilizing influence of the random gravitational force will be illustrated on the model of one-fluid dissipative MHD, as described in Appendix A.

In contrast to the previous example, we assume that the gravitational force varies independently of other plasma parameters. In the evolutional equation (A.2) the stochastic term is given by

$$D_s \hat{\xi} = \begin{pmatrix} -\nabla\cdot(\rho\, \underset{\sim}{\xi})\nabla \Psi_s \\ 0 \end{pmatrix} \sigma \varphi(t).$$

The corresponding Itô's equation encompasses only the one-dimensional scalar Wiener process. Transforming the system (A.2) with the help of the congruent mapping $K = A^{-1/2}$, defining the functional $V_s(\hat{\zeta})$ in the form (17) and performing the operation $\mathcal{L}\, V_s(\hat{\zeta})$, we obtain after returning to the original variabless $\underset{\sim}{\xi}$, $\dot{\underset{\sim}{\xi}}$

$$\bar{V}_s(\xi) = \langle \underset{\sim}{\xi}, D\underset{\sim}{\xi}\rangle + \langle \dot{\underset{\sim}{\xi}}, A\dot{\underset{\sim}{\xi}}\rangle$$

$$\bar{W}_s(\xi) \equiv \mathcal{L}\,\bar{V}_s(\xi) = -2\langle \dot{\underset{\sim}{\xi}}, B\dot{\underset{\sim}{\xi}}\rangle + \langle D_s \underset{\sim}{\xi}, A^{-1} D_s \underset{\sim}{\xi}\rangle . \tag{32}$$

The stability conditions read explicitly $\left(\underset{\sim}{\beta} \equiv \int dt\, (\sigma \underset{\sim}{B})\right)$:

$$\int_V dV \Big\{ -\underset{\sim}{\xi}\cdot\Big[\nabla(\underset{\sim}{\xi}\cdot\nabla p) + \gamma\nabla(p(\nabla\cdot\underset{\sim}{\xi})) + \frac{1}{\mu_0}(\nabla\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B})))\times\underset{\sim}{B}\Big]$$
$$+\frac{1}{\mu_0^2}\underset{\sim}{\xi}\cdot(\nabla\times[\nabla\times(\eta\nabla\times\underset{\sim}{\beta})])\times\underset{\sim}{B}_0 - \frac{1}{\mu_0^2}\underset{\sim}{\beta}\cdot\nabla\times[\eta\nabla\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}))]$$
$$+\frac{1}{\mu_0^3}\underset{\sim}{\beta}\cdot\nabla\times[\eta\nabla\times(\nabla\times(\eta\nabla\times\underset{\sim}{\beta}))]\Big\}$$
$$+\int_V dV\Big\{\rho_0\,\dot{\underset{\sim}{\xi}}\cdot\dot{\underset{\sim}{\xi}} + \frac{\varepsilon_0}{\mu_0^2}\eta\,\dot{\underset{\sim}{\beta}}\,\nabla\times[\eta\nabla\times\dot{\underset{\sim}{\beta}}]\Big\} > 0,$$

$$-2\int_V dV\Big\{\dot{\underset{\sim}{\xi}}\cdot\nabla\cdot\underset{\approx}{\Pi}^{H}(\dot{\underset{\sim}{\xi}}) + \frac{1}{\mu_0^2}\dot{\underset{\sim}{\beta}}\,\nabla\times[\eta\nabla\times\dot{\underset{\sim}{\beta}}]\Big\}$$
$$+\sigma^2\int_V dV\,\frac{1}{\rho^2}[\nabla\cdot(\rho\underset{\sim}{\xi})\nabla\Psi_s]\cdot[\nabla\cdot(\rho\underset{\sim}{\xi})\nabla\Psi_s] \le 0$$

(33)

Here again the term proportional to σ^2 leads to instability, with the growth given by eq. (23).

For a third example we take the two-fluid magnetohydrodynamic equilibrium perturbed by an electrostatic turbulence with the field

$$\underset{\sim}{E}_T(t) = -\nabla \Phi_T(t) = \sigma \varphi(t) \underset{\sim}{E}_T$$

Let us suppose that the equilibrium is characterized by $\underset{\sim}{E}_o = 0$. Then Itô's model equation (15) reads

$$d\hat{\xi} = \begin{pmatrix} 0 & I \\ -P & -(M+N) \end{pmatrix} \hat{\xi} + (G\hat{\xi})\, dW_t^{(1)}$$

The operators M, N and P are given in the Appendix B, eq. (B.1). The Wiener process is here only a one-dimensional scalar process. We have neglected the self-consistent random fluctuations of all parameters but the field E_T. This approximation is only permissible if the turbulent field is low enough that its influence on other plasma parameters can be neglected in the equilibrium equations. The operator G has the components

$$G = \begin{pmatrix} 0 & 0 \\ P_s & 0 \end{pmatrix},$$

where $$P_s \underset{\sim}{\xi}_\alpha = \frac{e_\alpha}{m_\alpha \sqrt{n_\alpha}} \left(\nabla \cdot \sqrt{n_\alpha}\, \hat{\underset{\sim}{\xi}}_\alpha\right) \underset{\sim}{E}_T .$$

Then

$$Sp\{(G\hat{\xi})(G\hat{\xi})^T\} = \sum_\alpha \frac{e_\alpha^2}{m_\alpha^2 n_\alpha}\{(\nabla\cdot\sqrt{n_\alpha}\,\underset{\sim}{\hat{\xi}}_\alpha)E_T\}^2 .$$

The stability conditions in coordinates $\underset{\sim}{\xi}$ are

$$\int_V dV\Big\{-\sum_\alpha \underset{\sim}{\xi}_\alpha\cdot\nabla(\underset{\sim}{\xi}_\alpha\cdot\nabla p_\alpha) + \underset{\sim}{\xi}_\alpha\cdot\nabla[\gamma p_\alpha(\nabla\cdot\underset{\sim}{\xi}_\alpha)] +$$

$$m_\alpha \underset{\sim}{\xi}_\alpha\cdot(\nabla\cdot\ \underset{\sim}{\xi}_\alpha)\nabla\Psi_0 \ + \frac{1}{\mu_0}\underset{\sim}{A}\cdot[\nabla\times(\nabla\times\underset{\sim}{A})]\Big\}$$

$$+\int_V dV\Big\{\sum_\alpha m_\alpha n_\alpha \dot{\underset{\sim}{\xi}}_\alpha\cdot\dot{\underset{\sim}{\xi}}_\alpha + \frac{1}{\mu_0}\frac{1}{c^2}\dot{\underset{\sim}{A}}\cdot\dot{\underset{\sim}{A}}\Big\} > 0 ,$$

$$-2\int_V dV\Big\{\sum_\alpha\Big[\eta\frac{e^2}{c}n_\alpha^2\,\dot{\underset{\sim}{\xi}}_\alpha\cdot\dot{\underset{\sim}{\xi}}_\alpha + \dot{\underset{\sim}{\xi}}_\alpha\cdot\nabla\cdot\underset{\approx}{\Pi}_\alpha(\dot{\underset{\sim}{\xi}}_\alpha)\Big] - 2\eta\frac{e^2}{c}n_e n_i\,\dot{\underset{\sim}{\xi}}_e\cdot\dot{\underset{\sim}{\xi}}_i\Big\}$$

$$+\sigma^2\int_V dV\sum_\alpha\frac{e_\alpha^2}{m_\alpha n_\alpha}\{(\nabla\cdot n_\alpha\underset{\sim}{\xi}_\alpha)\underset{\sim}{E}_T\}^2 \le 0 .$$

(34)

Similarly to previous examples, the second condition (34) cannot be satisfied for all $\sigma \neq 0$ and all t. The electrostatic turbulence also drives the equilibrium to the unstable regime.

When instead of the fluctuating electrostatic field the gravitational force randomly oscillates, then under the assumption that the stochastic gravitational force

$$\Psi_s = \sigma\,\Psi\,\varphi(t)$$

does not influence the equilibrium, the second term in eq. (34) reads

$$\sigma^2 \int_V dV \sum_\alpha \frac{m_\alpha}{n_\alpha} \left[(\nabla \cdot n_\alpha \underset{\sim}{\xi}_\alpha) \nabla \Psi \right]^2$$

Again, the equilibrium is unstable in probability.

We also observe that the non-self-consistent fluctuations of the magnetic field $B_o \left[1 + \sigma \varphi(t) \right]$ lead to the term

$$\sigma^2 \int_V dV \sum_\alpha \frac{e_\alpha^2 n_\alpha}{m_\alpha} (\underset{\sim}{B}_o \times \dot{\underset{\sim}{\xi}}_\alpha) \cdot (\underset{\sim}{B}_o \times \dot{\underset{\sim}{\xi}}_\alpha) \quad .$$

In this case, the second stability condition (31) gives the stability threshold

$$2 \int_V dV \left\{ \sum_\alpha \left[\eta \frac{e_\alpha^2}{c} n_\alpha^2 \dot{\underset{\sim}{\xi}}_\alpha \cdot \dot{\underset{\sim}{\xi}}_\alpha + \dot{\underset{\sim}{\xi}}_\alpha \cdot \nabla \cdot \underset{\approx}{\Pi}_\alpha^H (\dot{\underset{\sim}{\xi}}_\alpha) \right] - 2 \eta \frac{e_\alpha^2}{c} n_e n_i \dot{\underset{\sim}{\xi}}_e \cdot \dot{\underset{\sim}{\xi}}_i \right\}$$

$$\geq \sigma^2 \int_V dV \sum_\alpha \frac{e_\alpha^2 n_\alpha}{m_\alpha} (\underset{\sim}{B}_o \times \dot{\underset{\sim}{\xi}}_\alpha) \cdot (\underset{\sim}{B}_o \cdot \dot{\underset{\sim}{\xi}}_\alpha) \quad . \tag{35}$$

This condition shows that for small relative amplitude of fluctuations σ, the deterministic stability is preserved. Nevertheless the non-self-consistent description is valid only for a low level of random fluctuations of the magnetostatic field B_o.

Finally, the non-self-consistent stochastic disturbance of the density

$$n_\alpha = n_\alpha \left[1 + \sigma \varphi(t) \right]$$

leads to the threshold value of σ given by the relation

$$2\int_V dV \sum_\alpha \left\{ \eta \frac{e^2}{c} \left(n_\alpha^2 \dot{\underset{\sim}{\xi}}_\alpha \cdot \dot{\underset{\sim}{\xi}}_\alpha - n_\alpha n_\beta \dot{\underset{\sim}{\xi}}_\alpha \cdot \dot{\underset{\sim}{\xi}}_\beta \right) + \dot{\underset{\sim}{\xi}}_\alpha \cdot \nabla \cdot \underset{\approx}{\Pi}_\alpha^H \left(\dot{\underset{\sim}{\xi}}_\alpha \right) \right\}$$

$$\geq \sigma^2 \int_V dV \left\{ \sum_\alpha m_\alpha n_\alpha \left[\eta \frac{e^2}{c} \left(\frac{n_\alpha}{m_\alpha} \dot{\underset{\sim}{\xi}}_\alpha - \frac{n_\beta}{m_\alpha} \dot{\underset{\sim}{\xi}}_\beta \right) - \frac{1}{m_\alpha n_\alpha} \nabla \cdot \underset{\approx}{\Pi}_\alpha \left(\dot{\underset{\sim}{\xi}}_\alpha \right) \right]^2 + \left[\sum_\alpha e_\alpha n_\alpha \dot{\underset{\sim}{\xi}}_\alpha \right]^2 \right\} . \tag{36}$$

In deriving this condition we have supposed that for the equilibrium $\underset{\sim}{v}_\alpha = 0$, $E_o = 0$ and $\underline{\Psi} = 0$.

It is interesting to note that for small amplitudes σ the stochastic non-self-consistent perturbation of the magnetostatic field B_o and the density n_α may be stabilized by the resistivity and viscosity of the fluid.

CONCLUSION

In the present study we describe the influence of random fluctuations of one or, more precisely, several plasma parameters on the stability of the equilibrium of the magnetohydrodynamic fluid. Such fluctuations are due to the presence of MHD or kinetic turbulence in the plasma, random variations of the confining magnetic field or density generated from external sources. Discharges in many plasma devices, such as tokamaks, are characterized by a large proportion of random magnetic field components.

We established the general stability criteria for a stochastically perturbed equilibrium of dissipative MHD based on earlier results obtained for deterministic dissipative systems [19]. We examined the influence of random variation of plasma parameters on the MHD stability of a static plasma equilibrium in several cases - ideal MHD, and one-component and two-fluid dissipative MHD.

The results obtained are pessimistic. In all cases (with the exception of the case of non-self-consistent fluctuation of the magnetic field and density in the two--fluid model, when the perturbations are proportional to the Lagrangian velocity $\dot{\underset{\sim}{\xi}}$ only) the random modulation of the fluid parameters has a destabilizing influence proportional to σ^2, σ being the relative amplitude of stochastic disturbances. Thus, the stochastic fluctuations destroy the equilibrium (if such equilibrium exists in the deterministic case) with growth for which we have defined a general estimate.

Nevertheless, the approach used has several basic limitations. First, the considered fluctuations should have small relative amplitudes σ such that σ can be considered as a small parameter. In several practical situations (strong stochastic magnetic fields in the tokamak discharge in the regime close to disruptions) our theory does not apply. In these cases it would be desirable to derive a new hierarchy of transport equations including the random fields. Some attempts in this direction have already been made [24]. From the same standpoint all calculations, based on the assumption of non-self-consistent variation of one plasma parameter, apply only if the initial stage of evolution of the random fluctuations where σ is small enough.

The second objection may be raised against the assumption of the white-noise character of the stochastic process, which represents the real random fluctuations. Although the white-noise process is a mathematical artifice, it approximates quite well the behaviour of a number of real stochastic processes as verified on numerous examples in technical applications. Experience also shows [25] that the use of the white-noise model leads to good results even if we deal with a small portion of the frequency spectrum. However, the replacing of a Gaussian process by the Gaussian white noise is a non-trivial matter. A fundamental discussion of this point may be found in [26].

An alternative approach which would allow the mathematical model to be related closely to the real spectrum of fluctuations will be the use of the non-linear Kalman-Bucy filtering [27, 28] and consequently the solution of the evolutional equation based on the method of moments. This approach represents a considerable mathematical task and does not afford the transparency of the Liapunov technique used in this paper. The resulting conditions for stochastic stability of the evolutional equation (1), although more precise, will not be qualitatively different from our results.

Acknowledgements

The author gratefully acknowledges a fruitful discussion with Prof. D. Pfirsch and the hospitality of Max-Planck-Institut für Plasmaphysik, Garching.

Appendix A

We consider a viscous, resistive MHD fluid. Taking Ohm's law in the form

$$\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} - \eta \underset{\sim}{j} = 0$$

where $\eta(\underset{\sim}{r})$ is the resistivity, we have for the equilibrium mass velocity $\underset{\sim}{v} = 0$ the following perturbed first-order equations for Lagrangian displacements $\underset{\sim}{\xi}$, $\delta\underset{\sim}{B}$:

$$\rho \ddot{\underset{\sim}{\xi}} - \frac{1}{\mu_0}(\nabla \times \underset{\sim}{B}) \times (\delta \underset{\sim}{B}) - \frac{1}{\mu_0}[\nabla \times (\delta \underset{\sim}{B})] \times \underset{\sim}{B} - \nabla(\underset{\sim}{\xi} \cdot \nabla p)$$
$$- \gamma \nabla [p(\nabla \cdot \underset{\sim}{\xi})] - [\nabla \cdot (\rho \underset{\sim}{\xi})] \nabla \Psi + \nabla \cdot \underset{\approx}{\Pi}(\dot{\underset{\sim}{\xi}}) = 0 \ ,$$

$$\varepsilon_0 \eta (\delta B)^{\cdot\cdot} + (\delta B)^{\cdot} + \frac{1}{\mu_0} \nabla \times [\eta \nabla \times (\delta \underset{\sim}{B})] - \nabla \times (\dot{\underset{\sim}{\xi}} \times \underset{\sim}{B})$$
$$+ \frac{1}{\mu_0} \nabla [\eta (\nabla \cdot \underset{\sim}{\xi}) - (\underset{\sim}{\xi} \cdot \nabla) \eta] \times (\nabla \times \underset{\sim}{B}) = 0 \ . \tag{A.1}$$

Here p is the scalar pressure, $\delta \underset{\sim}{B}$ the perturbation of the magnetic induction, $\underset{\approx}{\Pi}$ the viscosity tensor, and Ψ the gravitational potential. The operator $\nabla \cdot \underset{\approx}{\Pi}(\dot{\underset{\sim}{\xi}})$ may be split into a Hermitian $\nabla \cdot \underset{\approx}{\Pi}^{H}$ and a anti-Hermitian component [10] $\nabla \cdot \underset{\approx}{\Pi}^{AH}$. Writing eq. (A.1) in the operator form and expressing all involved operators as a sum of Hermitian and anti-Hermitian components, eq. (A.1) takes the form of eq. (1). In order to obtain simpler equation suitable for analysis of the stochastic stability, we sacrifice the generality of eq. (A.1), assuming for the equilibrium $\nabla \times B = 0$ and neglecting the Eulerian perturbation of resistivity ($\delta\eta$). Then using the method proposed by Barston [13], systematically neglecting the contribution of the displacement current in the equation for the momentum balance, we obtain

$$A\ddot{\hat{\xi}} + (B+C)\dot{\hat{\xi}} + D\hat{\xi} \doteq 0 \tag{A.2}$$

where $\hat{\xi} = (\underset{\sim}{\xi}, \int dt\,(\delta \underset{\sim}{B}))^T$. The operators in eq. (A.2) have the following non-zero elements (ρ is the density):

$$A_{11} = \rho \ , \quad A_{22} = \varepsilon_0 \eta B_{22} \ , \quad B_{11}\dot{\underset{\sim}{\xi}} = \nabla \cdot \underset{\approx}{\Pi}^{H}(\dot{\underset{\sim}{\xi}})$$

$$C_{11}\dot{\underset{\sim}{\xi}} = \nabla \cdot \underset{\approx}{\Pi}^{AH}(\dot{\underset{\sim}{\xi}}) \ , \quad B_{22}\underset{\sim}{\xi} = \frac{1}{\mu_0}\nabla \times (\eta \nabla \times \underset{\sim}{\xi}) \ ,$$

$$D_{11}\underset{\sim}{\xi} = -\nabla(\underset{\sim}{\xi}\cdot\nabla p) - \gamma\nabla[p(\nabla\cdot\underset{\sim}{\xi})] - (\nabla\cdot\rho\underset{\sim}{\xi})\nabla\Psi - \frac{1}{\mu_0}[\nabla\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B})]\times\underset{\sim}{B} \ ,$$

$$D_{12}(\delta\underset{\sim}{B}) = \frac{1}{\mu_0^2}\{\nabla\times[\nabla\times(\eta\nabla\times(\delta\underset{\sim}{B})]\}\times\underset{\sim}{B} \ ,$$

$$D_{21}\underset{\sim}{\xi} = -\frac{1}{\mu_0^2}\nabla\times[\eta\nabla\times(\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}))] \ ,$$

$$D_{22}(\delta\underset{\sim}{B}) = \frac{1}{\mu_0^3}\nabla\times[\eta\nabla\times(\nabla\times(\eta\nabla\times(\delta\underset{\sim}{B}))] \ .$$

In the limit $\eta \to 0$, eq. (A.2) gives the form of the corresponding equation of ideal MHD, with the constraint $\nabla \times B = 0$. Equation (A.2) differs from the model previously discussed by Barston [13] in that we keep the displacement current in the field equation, take the full viscosity tensor and do not require incompressibility of the fluid.

In this as well as in other examples we assume a perfectly conducting wall at the fluid boundary.

Appendix B

The Euler-Lagrange equation (1) is derived in the case of two-fluid magnetohydrodynamics on the assumption of the static equilibrium $\underset{\sim}{v}_\alpha = 0$ (α is the plasma species) and the gauge $\Phi = 0$, Φ being the electrostatic potential. The first-order equations read

$$m_\alpha n_\alpha \ddot{\underset{\sim}{\xi}}_\alpha + e_\alpha n_\alpha \underset{\sim}{B}_0 \times \dot{\underset{\sim}{\xi}}_\alpha + e_\alpha n_\alpha \dot{\underset{\sim}{A}} + e_\alpha (\nabla \cdot n_\alpha \underset{\sim}{\xi}_\alpha) \underset{\sim}{E}_0 + \nabla \cdot \underset{\approx}{\Pi}_\alpha (\dot{\underset{\sim}{\xi}}_\alpha)$$

$$- \nabla (\underset{\sim}{\xi}_\alpha \cdot \nabla p_\alpha) - \nabla [\gamma p_\alpha (\nabla \cdot \underset{\sim}{\xi}_\alpha)] - (\nabla \cdot n_\alpha \underset{\sim}{\xi}_\alpha) m_\alpha \nabla \Psi +$$

$$\eta \frac{e^2}{c} (n_\alpha^2 \dot{\underset{\sim}{\xi}}_\alpha - n_\alpha n_\beta \dot{\underset{\sim}{\xi}}_\alpha) = 0 \; ,$$

$$\frac{1}{\mu_0 c^2} \ddot{\underset{\sim}{A}} + \frac{1}{\mu_0} \nabla \times \nabla \times \underset{\sim}{A} - (e n_i \dot{\underset{\sim}{\xi}}_i - e n_e \dot{\underset{\sim}{\xi}}_e) = 0 \; .$$

After introducing the congruent mapping eq. (2), we can cast the linearized evolutional equation in the form

$$\ddot{\hat{\xi}} + (M + N) \dot{\hat{\xi}} + P \hat{\xi} = 0 \; , \qquad \text{(B.1)}$$

where the elements of the operators M, N and P are as follows:

$M_{11} = Mi, \; M_{22} = Me$,

$$M_\alpha \, \hat{\dot{\underset{\sim}{\xi}}}_\alpha = \eta \frac{e^2}{c} \frac{n_\alpha}{m_\alpha} \hat{\dot{\underset{\sim}{\xi}}}_\alpha + m_\alpha^{-1} n_\alpha^{-1/2} \nabla \cdot \underset{\approx}{\Pi}_\alpha^H \left(\hat{\dot{\underset{\sim}{\xi}}}_\alpha n_\alpha^{-1/2} \right),$$

$$M_{21} = M_{12} = -\eta \frac{e^2}{c} \left(\frac{n_e n_i}{m_e m_i} \right)^{1/2},$$

$N_{11} = Ni, \; N_{22} = Ne$,

$$N_\alpha \, \hat{\dot{\underset{\sim}{\xi}}}_\alpha = \frac{e_\alpha}{m_\alpha} \underset{\sim}{B}_0 \times \hat{\dot{\underset{\sim}{\xi}}}_\alpha + m_\alpha^{-1} n_\alpha^{-1} \nabla \cdot \underset{\approx}{\Pi}_\alpha^{AH} \left(\hat{\dot{\underset{\sim}{\xi}}}_\alpha n_\alpha^{-1/2} \right),$$

$$N_{13} = - N_{31} = c e \mu_0^{1/2} \left(\frac{n_i}{m_i} \right)^{1/2},$$

$$N_{23} = - N_{32} = - c e \mu_0^{1/2} \left(\frac{n_e}{m_e} \right)^{1/2}$$

$$P_{11} = Pi, \; P_{22} = Pe, \; P_{33} \underset{\sim}{A} = c^2 \nabla \times \nabla \times \hat{\underset{\sim}{A}} ,$$

$$P_\alpha \hat{\underset{\sim}{\xi}}_\alpha = \frac{e_\alpha}{m_\alpha \sqrt{n_\alpha}} \left(\nabla \cdot \hat{\underset{\sim}{\xi}}_\alpha n_\alpha^{-1/2} \right) \underset{\sim}{E}_0 - m_\alpha^{-1} n_\alpha^{-1/2} \nabla \left(n_\alpha^{-1/2} \hat{\underset{\sim}{\xi}}_\alpha \cdot \nabla p_\alpha \right)$$
$$- m_\alpha^{-1} n_\alpha^{-1/2} \nabla \left[\gamma p_\alpha \left(\nabla \cdot \hat{\underset{\sim}{\xi}}_\alpha n_\alpha^{-1/2} \right) \right] - n_\alpha^{-1/2} \left(\nabla \cdot \hat{\underset{\sim}{\xi}}_\alpha n_\alpha^{-1/2} \right) \nabla \Psi .$$

The displacement vector $\hat{\underset{\sim}{\xi}}$ in eq. (B.1) is now

$$\xi = \left[(m_i n_i)^{-1/2} \hat{\underset{\sim}{\xi}}_i \, , \, (m_e n_e)^{-1/2} \hat{\underset{\sim}{\xi}}_e \, , \, c \mu_0^{1/2} \hat{\underset{\sim}{A}} \right]^T .$$

REFERENCES

1 F.E. Low, Proc. Roy. Soc. A 248, 283 (1958)

2 I. Bernstein, E.A. Tidman, M.D. Kruskal and R.M. Kulsrud, Proc. Roy. Soc. A 244, 17 (1958)

3 K. Hain, R. Lüst and A. Schlüter, Z.Naturforschung A 12, 833 (1957)

4 W.A. Newcomb, Nucl. Fusion Suppl. P.2. 451 (1962)

5 E. Frieman and M. Rotenberg, Rev.Mod.Phys. 32, 898 (1960)

6 H.P. Furth, in Propagation and Instabilities, Stanford Univ. Press, 87(1963)

7 H.P. Furth, J. Killen, M.N. Rosenbluth and B. Coppi, Culham IAEA Conf. 1, 103 (1965)

8 J.M. Green and B. Coppi, Phys.Fluids 8, 1745 (1965)

9 G. Laval, C. Mercier and R. Pellat, Nucl.Fusion 5, 156 (1965)

10 H. Tasso, P.P.J.M. Schram, Nucl.Fus. 6, 284 (1966)

11 J.M. Green and J.L. Johnson, Plasma Physics 10, 729 (1968)

12 E.M. Barston, Phys. Fluids 12, 2162 (1969)

13 E.M. Barston, J.Fluid.Mech. 42, 97 (1970)

14 H. Tasso, Plasma Phys. 17a, 1131 (1975)

15 H. Tasso, Plasma Phys. 19, 177 (1977)

16 E.M. Barston, J.Plasma Phys. 17, 409 (1977)

17 E.M. Barston, Int.J.Eng.Sci. 15, 71 (1977)

18 H. Tasso, Z. Naturforschung 33a, 257 (1978)

19 J. Teichmann, Phys. Rev. Letters, to be published

20 J. Teichmann, Phys. Letters, 72 A, 117, (1979)

21 K. Itô, "On stochastic differential equations", Memoirs, Amer.Math.Soc. No. 4 (1951)

22 H.J. Kushner,"Stochastic stability and control" Academic Press (1967)

23 F. Kozin, Automatica 5, 95 (1969)

24 Atsushi Inoue and Tadahisa Funaki, Comm.Math.Phys. 65, 83 (1979)

25 F. Kozin in: "Stability of Stochastic Dynamical Systems" 186, Springer-Verlag (1972)

26 E. Wong and M. Zakai, Int.J.Engng.Sci. 3, 213 (1965)

27 R.E. Kalman, Proc. First Symp. Engng. Appl. on Random Function Theory and Probability, N.Y. 270 (1963)

28 R.S. Bucy, IEEE Trans. Automatic Control 10, 198 (1965)