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Abstract

The nonlinear behavior of the drift-tearing instability is investigated within the two fluid theory. It is shown analytically and verified in numerical computations that the diamagnetic effects and ion gyroviscosity which determine the frequency of the linear mode, are quenched at magnetic island size of the order of the resistive layer width. For finite island width growth and saturation of the mode depend only on resistivity and current profile, while its frequency is determined by the rotation of the plasma.

I. Introduction

Tearing modes appear to play an important role in tokamak plasmas. While the linear theory of the tearing instability has been treated intensely taking into account various nonideal effects [1)-4)] as well as toroidal geometry [5)], nonlinear theory is still mainly limited to the purely resistive case [6),7),8)]. The main result of the latter is that the (exponential) tearing instability is quenched at very low amplitude, below the observation limit in most applications, and that the further development of magnetic islands proceeds on the slow diffusive time scale given by the resistivity instead of some fractional power thereof (An exception is the m = 1 mode, which continues growing on the linear time scale to large amplitudes). In fact it can easily be shown for the resistive [6)] as well as the collisionless [9)] tearing instability that the driving mechanism is switched off as soon as the island size exceeds the thin resistive or inertial (in the collisionless case) layer width. One may therefore argue that the tearing instability is an artifact due to the assumption of an idealized equilibrium [7)].

In tokamaks tearing modes (we are restricting ourselves to mode numbers $m \geq 2$) are observed as Mirnov oscillations, i.e. magnetic perturbations of a certain amplitude rotating with a certain frequency. Prior to a major disruption the magnetic signals may grow considerably and the frequency is often reduced. A nonlinear theory of tearing modes should therefore predict the saturation amplitude of the mode (i.e. island size) for a particular current profile and also the rotation frequency of the island. In a previous paper [3] an appropriate set of nonlinear equations has been derived including effects of diamagnetic drifts and viscosity, which give rise to mode rotation in the linear tearing instability. In the present paper these equations, in a slightly modified form, are investigated to treat the nonlinear behavior of tearing modes. The main results are: 1) The nonlinear growth and saturation size of the magnetic islands depend only on resistivity and current profile and are independent of the additional nonideal effects like diamagnetic drifts and viscosity. 2) The island rotation frequency does not depend on the diamagnetic frequencies, which determine the frequency of the linear mode, but only on plasma rotation.

In section II we first discuss the model equations and then briefly review linear theory. In section III an estimate is given of the critical island size where exponential growth stops and the mode frequency is strongly changed. For finite island size a set of equations is obtained for the change of the magnetic configuration and the mode frequency, section IV. Section V gives some results of numerical solutions of the nonlinear equations verifying the results of the previous sections.

II. Discussion of the model

The model used in the present paper is based on the two-fluid equations in cylindrical geometry using the tokamak ordering $\varepsilon = a/R \ll 1$ and $q(a) \sim 1$, where a is the plasma radius, $2\pi R$ the axial periodicity length (R = major radius of equivalent torus), and q the safety factor. This implies that the poloidal component of the magnetic field is small $B_p/B_z \sim \varepsilon$, while the variation of the axial field B_z is small $\delta B_z/\delta B_p \sim \varepsilon$. Taking into account only lowest order terms in ε we have $B_z = B_o$ = const. We further assume helical symmetry $f = f(r, m\theta - kz)$, $k = n/R$; r, θ, z cylindrical coordinates, restricting consideration to the nonlinear evolution of a particular tearing mode.

Since the problem is two-dimensional, the canonical momentum in the third direction is conserved and the component of the vector potential in this direction ψ defines magnetic surfaces In our case ψ is called the helical flux function satisfying the equations

$$\nabla z \times \nabla \psi = \underline{B_p} - \frac{kr}{m} B_o \nabla z$$

(1)

$$\nabla^2 \psi = j - c$$

where $B_p = (B_r, B_\theta)$, $j = j_z$ and $c = 2kB_o/m$. Neglecting viscosity and inertia in the electron equation of motion, but taking into account plasma compressibility, parallel plasma flow and the ion stress tensor yields the following equations for ψ, the plasma density n, the vorticity function A and the parallel plasma velocity $v_\parallel$ (for details see Ref. 3)

$$(2) \qquad \frac{\partial \psi}{\partial t} + \underline{u}\cdot\nabla\psi = \eta j - E_o - \alpha\frac{T_e+\gamma_i T_i}{n}\nabla z\cdot\nabla\psi\times\nabla n$$

$$(3) \qquad \frac{\partial n}{\partial t} + \underline{u}\cdot\nabla n = \alpha\nabla z\cdot\nabla\psi\times\nabla j - \nabla_\parallel n v_\parallel$$

$$(4) \qquad \frac{\partial A}{\partial t} + \nabla\cdot(\underline{u}-\underline{v}_i^*)A - \nabla z\cdot\nabla n\times\nabla\frac{u^2}{2}$$

$$= \nabla z\cdot\nabla\psi\times\nabla j + \nabla\cdot\mu\nabla A$$

$$(5) \qquad \frac{\partial n v_\parallel}{\partial t} = -(T_e+\gamma_i T_i)\nabla_\parallel n + \mu_o n\nabla_\parallel^2 v_\parallel$$

with

$$\nabla_\parallel = \frac{1}{B_o}\nabla z\times\nabla\psi\cdot\nabla \quad ; \quad v_i^* = \alpha\gamma_i T_i\frac{\nabla z\times\nabla n}{n} .$$

In these equations $\underline{u}$ is the incompressible part of the perpendicular plasma velocity $\underline{u} = \nabla z\times\nabla\phi$, $A = \nabla z\cdot\nabla\times n\underline{u} = \nabla\cdot n\nabla\phi$; we assume $T_{e,i}(n)$, $\nabla_\parallel T_e = 0$, $\nabla_\perp n T_i = \gamma_i T_i \nabla_\perp n$; v_i^* is the ion diamagnetic drift, the term $\nabla\cdot\underline{v}_i^* A$ in (4) representing the effect of nondissipative gyroviscosity; μ is the perpendicular magnetic viscosity, μ_o the parallel viscosity. The equations are written in dimensionless form with plasma radius a, typical poloidal magnetic field B_{p_o} and typical plasma density n_o as units. Temperatures are in units of $B_{po}^2/4\pi n_o$ and hence characterize β_p, the poloidal β. Normalization of η is such that $\eta \sim \tau_A/\tau_s$, which is the inverse of the parameter S introduced in Ref. 1. The parameter $\alpha = (c/\omega_{pi} a)\, B_{po}/B_o$ together with $T_{e,i}$ give the magnitude of the diamagnetic effects. The α-term in (3) represents perpendicular plasma compression, connected with the ion polarization drift [3]. Equation (4) has an obvious constant of motion, the total vorticity $W = \int A\, r dr d\theta$. Integrating (4) over the plasma cross section yields

$$\frac{dW}{dt} = 2\pi\mu \left.\frac{\partial A}{\partial r}\right|_{r=1} . \tag{6}$$

Hence if μ vanishes at the boundary, W is conserved. In addition to the vorticity we expect the total angular momentum to be constant in time, $\dot{P} = \frac{d}{dt}\int r n \frac{\partial \phi}{\partial r}\, r dr d\theta = 0$ *). To investigate conservation of P in eq. (4), the following identity is useful

$$\begin{aligned} P &= -\frac{1}{2}\int r^2 A\, r dr d\theta + \frac{1}{2} n \left.\frac{\partial \phi}{\partial r}\right|_{r=1} \\ &= -\frac{1}{2}\int r^2 A\, r dr d\theta + \frac{1}{2} W . \end{aligned} \tag{7}$$

Multiplying (4) by r^2 and integrating, however, yields that P is not strictly conserved even for $\mu = 0$, the nonconservation arising through the gyro-viscosity term $\nabla \cdot \underline{v}_i^* A$. This is inconsistent with the original equation of motion, which conserves angular momentum for any symmetric stress tensor $\Pi_{ik} = \Pi_{ki}$ (apart from boundary effects) :

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*) The compressible part of the plasma motion does not contribute to P.

$$\int d\underline{F} \cdot (\underline{r} \times \nabla \cdot \underline{\underline{\Pi}}) = 0$$

The solution of this discrepancy is that the term $\nabla \cdot \underline{v}_i^* A$ represents the gyroviscous effect only to first order in the ion pressure variation. The exact form is

$$\nabla z \cdot \nabla \times \nabla \cdot \underline{\underline{\Pi}}_g = - \alpha \Big[\nabla z \times \nabla p_i \cdot \nabla \nabla^2 \phi$$

(8)

$$+ \frac{\partial^2 \phi}{\partial r\, r \partial \theta} \left(r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial p_i}{\partial r} - \frac{1}{r^2} \frac{\partial^2 p_i}{\partial \theta^2} \right)$$

$$- \frac{\partial^2 p_i}{\partial r\, r \partial \theta} \left(r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) \Big] ,$$

where $\underline{\underline{\Pi}}_g$ is the part of the stress tensor $\underline{\underline{\Pi}}$ [10] not depending on the collision time τ_i , and $p_i = nT_i$. In the linear limit the first term on the l.h.s. of (8) is dominating, since $n_o'' \phi'$, $\phi_o'' n' \ll n_o' \phi''$. Only in this limit (8) reduces to $- \nabla \cdot \underline{v}_i^* A$. For finite amplitude the terms containing second order derivatives of p_i have to be taken into account to conserve angular momentum. Because of the approximate nature of the fluid stress tensor when applied to a hot plasma, it does not appear worthwhile to retain the complex expression (8). Instead we assume that the coefficient αp_i multiplying the strain tensor $\partial u_i / \partial x_k + \partial u_k / \partial x_i$ in $\underline{\underline{\Pi}}$, do not contain a fluctuating (i.e. θ-dependent) part but only the average quantity $p_o(r,t)$. In this approximation v_i^* has only an azimuthal component.

This is a reasonable model conserving angular momentum, and reproducing the linear properties briefly reviewed in the remainder of this section, as well as the nonlinear quenching of the diamagnetic effects discussed in section IV. It will be shown in section IV, that the effect of gyroviscosity vanishes for finite amplitude also in the exact form (8).

Let us briefly summarize the results of linear theory using the linearized form of eqs. (2) - (5), see for instance Ref. 3. We need consider only the resistive layer, since outside this layer the nonideal effects are in general negligible. For a tearing mode of mode number m, m $\gtrsim$ 2 (the axial mode number, i.e. the helicity of the mode, is contained in the helical flux function ψ_0) we obtain after elimination of the density perturbation and neglecting the parallel flow term and the perpendicular viscosity

(9a) $$(\Omega - \omega^*)(\tilde{\psi} + F_0 \frac{\tilde{\phi}}{\Omega}) = i\left(\eta + i\,\frac{\alpha^2 F_0^2 (T_e + \gamma_i T_i)}{n_0 \Omega}\right)\tilde{\psi}''$$

(9b) $$n_0(\Omega - \omega_i^*)\tilde{\phi}'' = - F_0 \tilde{\psi}''$$

$$\omega^* = \omega_e^* + \omega_i^* \; ,$$

$$\omega_e^* = -\frac{m}{r}\alpha T_e \frac{n_0'}{n_0} \; , \quad \omega_i^* = -\frac{m}{r}\alpha \gamma_i T_i \frac{n_0'}{n_0}$$

$$\Omega = \omega - \frac{m}{r}\phi_0' \; , \quad F_0 = \frac{m}{r}\psi_0' \; .$$

Here ϕ_o' is the zero order azimuthal plasma flow which was taken zero in Ref. 3. The dispersion relation obtained from (9) is

(10) $$\Omega\,(\Omega - \omega_i^*)(\Omega - \omega_e^*)^3 = i\gamma_T^5 \quad ,$$

where γ_T is the purely resistive tearing mode growth rate. Equation (10) yields for $\omega^* \ll \gamma_T$

(11) $$\Omega = i\gamma_T \quad ,$$

while for $\omega^* \gg \gamma_T$, typical for a hot plasma, the growth rate is reduced ,

(12) $$\Omega = \omega^* + \frac{1}{2}\, i\gamma_T \left(\frac{\gamma_T^2}{\omega^* \omega_e^*} \right)^{1/3} .$$

Note that the frequency in (12) is determined by the diamagnetic frequency ω^* and the plasma rotation ϕ_o' taken at the singular surface r_s.

III. Saturation of the exponential drift tearing instability

It is not difficult to estimate the amplitude $\tilde{\psi}_c$, or island size Δ_{Ic}, $\Delta_I = 4\sqrt{\tilde{\psi}/\psi_0''}$, at which the linear eigenfrequency is strongly changed (For the purely resistive case this has been done in Ref. 6). The main stabilizing term arises from the second order current contribution δj_0 on the r.h.s. of (4)

$$\frac{\partial \tilde{A}}{\partial t} + i\omega_i^* \tilde{A} + i\frac{m}{r}\delta j_0' \tilde{\psi} = i\frac{m}{r}\psi_0' \tilde{\psi}'' \tag{13}$$

Here δj_0 is determined by eq. (2):

$$\frac{\partial \delta\psi_0}{\partial t} \cong \eta \delta j_0 - \frac{2m}{r}\left(1-\frac{\omega^*}{\omega}\right)\frac{\partial}{\partial r}\left[\psi_R \phi_I - \psi_I \phi_R\right] \cong 0 , \tag{14}$$

where the right hand side equality arises, since the relaxation of the current δj_0 across the resistive layer δ occurs in a time short compared to the growth time, $\eta/\delta^2 \gg \gamma$. In (14) the notation $\tilde{\psi} = \psi_R + i\psi_I$ etc. is used. Since the variation of $\tilde{\phi}$ across the resistive layer is stronger than that of $\tilde{\psi}$, one has

$$\delta j_0' \simeq \frac{1}{n_0 \eta} \frac{2m}{r}\left(\psi_R A_I - \psi_I A_R\right)\left(1 - \frac{\omega^*}{\omega}\right) \tag{15}$$

Inserting (15) into (13) the critical island size Δ_{Ic} is obtained by the condition that the nonlinear term equals the inertia term. For $\omega^* < \gamma_T$ the result is known to be $\Delta_{Ic} \sim \delta$ [6]. Hence we limit ourselves to the opposite case $\omega^* > \gamma_T$, where ω, γ are given by (12). Here we find

(16) $$\Delta_{Ic}^{4} \sim \frac{\omega^{*}\omega_{e}^{*}}{\gamma F_{o}^{\prime 2}}\eta n_{o} \sim \left(\frac{\omega^{*}\omega_{e}^{*} n_{o}}{\Delta' F_{o}^{\prime 2}}\right)^{4/3}$$

since

(17) $$\gamma \sim \gamma_{T}^{5/3}/(\omega^{*}\omega_{e}^{*})^{1/3} \quad , \quad \gamma_{T}^{5} \sim \eta^{3}\Delta^{\prime 4} F_{o}^{\prime 2}/n_{o}$$
$$F_{o}' = \frac{m}{r}\psi_{o}'' .$$

The resistive layer width δ is estimated using eq. (9a),

$$(\omega-\omega^{*})\tilde{\psi} \sim i\eta\tilde{\psi}''^{\,*)} \; , \; \tilde{\psi}'' \sim \frac{\tilde{\psi}}{\delta}\Delta' \; , \; \Delta' \equiv \frac{\tilde{\psi}_{+}' - \tilde{\psi}_{-}'}{\tilde{\psi}}$$

valid within δ, which yields the general relation independent of the ratio ω^{*}/γ_{T} ,

(18) $$\delta \sim \frac{\eta\Delta'}{\gamma} .$$

Inserting (17) into (18), comparison with (16) gives $\Delta_{Ic} \sim \delta$ also for $\omega^{*} > \gamma_{T}$, which is thus independent of ω^{*}/γ_{T} (the magnitude of δ, of course, depends on ω^{*}/γ_{T}).

At this point one might argue that because of the nonlinear reduction of tearing mode growth just discussed, the system always reaches a state where $\omega \gg \gamma$ even if initially $\omega \lesssim \gamma$, so that modifications due to a nonlocal behaviour of drift modes in the tearing instability for $\omega \gg \gamma$ (see Ref. 3) would be very important nonlinearily. This

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*) This is a very rough estimate, though it gives the correct δ, eq. (18). For a more elaborate analysis see Ref. 4

is, however, not true, since there is a rapid nonlinear quenching of the diamagnetic frequencies, due to a flattening of the density profile $n_o(r)$ around r_s. The island size necessary for quenching of ω^* can be estimated by assuming that the gradient of n_o is compensated by the gradient of the density perturbation which has a much shorter scale length, typically δ,

$$\frac{\tilde{n}}{\delta} \sim n_o' \quad . \tag{19}$$

Equation (3) yields $\tilde{n} \sim (m/r) n_o' \tilde{\phi}/\omega$, while from eq. (2) $\tilde{\psi} \sim (m/r)(\psi_o'' \delta)\tilde{\phi}/\omega$. We thus recover

$$\Delta_I \sim \delta \sim \Delta_{Ic} \quad .$$

Hence the diamagnetic frequencies vanish for island size $\Delta_I > \Delta_{Ic}$. The flattening of the density profile around the singular radius r_s can also be seen from the quasi-linear diffusion equation which is easily derived to describe the change of $n_o(r,t)$:

$$\frac{\partial n_o}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r D \frac{\partial n_o}{\partial r} \quad , \quad D = \frac{2\gamma}{\omega^2}\frac{m^2}{r^2}|\tilde{\phi}|^2 . \tag{20}$$

This equation gives the same estimate of the island size required to flatten the density profile around r_s.

The rapid nonlinear quenching of the diamagnetic frequencies leads to a peculiar behavior of the growth rate. For a drift tearing mode with $\omega^* >> \gamma_T$, where the linear growth rate is strongly reduced

$\gamma \ll \gamma_T$, see eq. (12), the quenching of ω^* leads to a nonlinear increase of γ , until at island size $\Delta_I \sim \delta$ one has $\gamma \sim \gamma_T$. The initially nonlocal mode for $\omega^* \gg \gamma_T$ (due to drift wave propagation [3]) becomes localized for $\Delta_I \gtrsim \delta$.

Since $\omega^* = 0$ nonlinearly, the mode frequency is given by $\omega = \frac{m}{r_s} \phi_o(r_s)$. In contrast to n_o' the rotation velocity ϕ_o' is not strongly changed by the nonlinear development of the mode for $\Delta_I \lesssim \delta$. First we note that the value of A_o is not strongly changed, since it satisfies an equation of the type

$$\frac{\partial A_o}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} F = 0$$

where F is bilinear in the perturbed quantities and thus strongly localized around r_s. Now ϕ_o' is given in terms of A_o by

$$\frac{1}{r}\frac{\partial}{\partial r} r n_o \frac{\partial \phi_o}{\partial r} = A_o - \frac{2}{r}\frac{\partial}{\partial r} r \left[n_R \frac{\partial \phi_R}{\partial r} + n_I \frac{\partial \phi_I}{\partial r} \right]$$

hence

$$(21) \qquad n_o \phi_o' = \frac{1}{r} \int^r r A_o dr - 2\left(n_R \phi_R' + n_I \phi_I' \right) .$$

Let us estimate the last term on the r.h.s. of (21). Using $\tilde{n} \sim \frac{m}{r} n_o' \tilde{\phi} / \omega$, $\tilde{\phi} \sim \omega \tilde{\psi} / \frac{m}{r} \psi_o'' \delta$ and $\Delta_I \sim \delta$ we find

$$\frac{n \phi'}{n_o \phi_o'} \sim \frac{\omega}{\frac{m}{r}\phi_o'} \frac{\delta}{r} \ll 1$$

Thus ϕ_o' remains essentially unchanged for $\Delta_I \sim \delta$ and ω is determined by the original plasma rotation at $r = r_s$. The case $\Delta_I \gg \delta$ will be discussed in the following section.

IV. Behavior of tearing modes at finite island size

The equations (2) - (5) yield several general properties for finite island size, $\Delta_I \gg \delta$. It has been shown in section III that inertia effects are negligible for the evolution of the magnetic configuration, in particular the island growth, hence from eq. (4) $\nabla\psi \times \nabla\dot{j} = 0$, i.e. $\dot{j} = \dot{j}(\psi)$. Thus the α-term in the n-equation (3), which is due to the compressible part of the perpendicular plasma motion, is negligible, so that the plasma is incompressible perpendicular to $\underline{B}$. Equation (3) now implies that n moves essentially along with ψ, where slight variations of n on a magnetic surface are smoothed out by the last term, since they would give rise to sound waves rapidly propagating along $\underline{B}$, eq. (5) (the parallel flow term is important within the islands). For $n = n(\psi)$ the diamagnetic term in (2) vanishes, as predicted in section III. Taking averages over flux surfaces characterized by the volume $V(r,\theta,t)$ enclosed by a flux surface $\psi = \text{const}$, $V(\psi,t)$, we obtain the following equations, using the notation $\psi_V \equiv \partial/\partial V$ etc,

(22) $$\frac{\partial\psi(V,t)}{\partial t} = \eta\left[(K_1\psi_V)_V + c\right] - E_0$$

(23) $$\frac{\partial n(V,t)}{\partial t} = 0$$

(24) $$\nabla^2\psi = \dot{j}(V,t) - c$$

where K_1 is a geometrical factor

$$K_1 = \langle |\nabla V|^2 \rangle \tag{25}$$

Here the normalization $\oint ds/|\nabla V| = 1$ is used, hence $K_{1V} = \langle \nabla^2 V \rangle$. These equations are identical to those introduced and treated in Ref. 7. They describe a sequence of MHD equilibria developing on the resistive time scale. Since we do not consider plasma cross-field diffusion, the density on a magnetic surface $V =$ const remains constant. The change of geometry is described by eq. (24), which determines the perpendicular velocity in $\underline{u} \cdot \nabla \psi$ up to a rigid rotation of the magnetic configuration.

In the absence of perpendicular particle and momentum transport the rotation frequency of the magnetic islands is determined kinematically by the initial plasma rotation $\phi_0'(r, t=0)$ and the geometry of the magnetic configuration given by $V(r, \theta, t)$ (the actual definition of V is quite complicated in our case because of the existence of more than one magnetic axis). We consider the evolution of the magnetic islands on a quasi-infinitely slow time scale, i.e. $\eta \to 0$. In this limit (2) describes a rigid rotation of the magnetic configuration with rotation frequency ω_0

$$\psi = \psi(V) \quad , \qquad V = V(r, \theta - \omega_0 t) \tag{26}$$

and the streamfunction

$$\phi = \varphi(V) + \frac{\omega_0 r^2}{2} . \tag{27}$$

For finite η, ϕ contains a term $\phi_\eta(r,\theta,t)$ describing the change of geometry due to island growth

$$(28) \qquad \frac{\partial V(r,\theta',t)}{\partial t} + \underline{u}_\eta \cdot \nabla V = 0$$

$$\theta' = \theta - \omega_0 t \;, \quad \underline{u}_\eta = \nabla z \times \nabla \phi_\eta \;.$$

The resistive change of geometry which is described by (22) - (25) leads to quasi-static change of $\varphi = \varphi(V,t)$ and $\omega_0 = \omega_0(t)$. These quantities are determined by eq. (4). Averaging over flux surfaces $V=$ const yields

$$(29) \qquad \left\langle \frac{\partial A}{\partial t} + u \cdot \nabla A \right\rangle = 0 \;, \quad A = \nabla \cdot n(V) \nabla \phi$$

or, splitting off the rotational time dependence ,

$$(30) \qquad \left\langle \frac{\partial A(r,\theta',t)}{\partial t} + \underline{u}_\eta \cdot \nabla A \right\rangle = 0 \;.$$

It is interesting to note that in the averaged equation (29) the gyroviscosity term given by (8) vanishes exactly for $p_i = p_i(V)$ and ϕ as given in (27).

Because of (28) equation (30) implies

$$\frac{\partial \langle A \rangle}{\partial t} = 0$$

Using (27) we therefore obtain

$$(31) \qquad (K_1 n \varphi_V)_V + \omega_0 (2n + n_V K_2) = d(V) \,,$$

where $d(V)$ is determined by the initial conditions, and $K_2 = \langle r \frac{\partial V}{\partial r} \rangle$.

Given the magnetic configuration, i.e. the coefficients K_i, φ is determined as a function of ω_o. To fix the value of ω_o in terms of the initial conditions d, a further equation is needed. This is provided by the conservation of angular momentum (note that by eq. (29) the total vorticity is automatically conserved):

$$P = \int n\left[\langle r^2\rangle \omega_o + \varphi_V K_2\right] dV$$

yielding

(32)
$$\omega_o = \frac{P - \int n \varphi_V K_2 \, dV}{\int n \langle r^2 \rangle \, dV} .$$

Hence for finite island size the mode frequency $\omega = m\omega_o$ is determined by certain average values over φ and n.

In the presence of viscosity μ the r.h.s. of (30) does not vanish, and a differential equation in time has to be solved. Since, however, the effect of collisional viscosity is usually weak, equation (31) is a good approximation to determine $\varphi(V,t)$ with changing geometry. Asymptotically the presence of the μ-term leads to a rigid rotor motion $A = 2\omega_o$ i.e. $\varphi = 0$, if no other effects driving a differential rotation are present.

V. Numerical computations

To verify the results of sections II - IV we have developed a simple but rather efficient numerical programme. From previous calculations using a two-dimensional Eulerian code, we find that for the $m \geqslant 2$ tearing mode the azimuthal variation is smooth (in contrast to the $m = 1$, where rather localized sheath currents are found close to the x-point). Hence Fourier analysing equations (2) - (5) and taking into account only the zeroth and first harmonics yields a reliable and particularly simple model, which is essentially one-dimensional in space. It should be noted that it would in general be of little use to include the second harmonics. If there were localized structures in θ, a Fourier approach would not be suitable at all.

The numerical model allows to use very small values of η, corresponding to $S \geq 10^6$, which makes a clear separation of the various time scales such as ω^*, γ_T, η possible. The results confirm the qualitative predictions of the previous sections:

i) Diamagnetic effects are rapidly quenched, so that $\omega^* = 0$ for $\Delta_I > \delta$. In the case of a drift tearing mode with $\omega^* \gg \gamma_T$ initially, the growth rate in fact increases nonlinearily up to $\gamma \sim \gamma_T$, the corresponding growth rate for $\omega^* < \gamma_T$.

ii) For finite island size $\Delta_I > \delta$ the further growth and saturation of the magnetic islands only depends on η and the current profile.

Plasma density and current density become flux functions, exhibiting the same island structure in the contour plots as the magnetic surfaces, with $n(r,\theta) = n_o(r) + \tilde{n} e^{im\theta} + \tilde{n}^* e^{-im\theta}$ etc.

iii) The rotation frequency of the mode is essentially given by the initial plasma rotation $\phi_o'(r_s)$. A change of ω_o as described by (31), (32) is not seen but at very large island size.

Details of the numerical model and results will be given elsewhere.

VI. Conclusions

In the present paper we have investigated the nonlinear properties of drift tearing modes within the framework of two fluid theory of a tokamak-like plasma in cylindrical approximation. We have discussed analytically and verified numerically that diamagnetic drifts and ion (collisionless) gyroviscosity, which strongly modify the linear properties of tearing modes, are quenched at small amplitude corresponding to magnetic island size of the order of the resistive layer width. The reason is that for finite amplitude n and T are functions of ψ, so that no net diamagnetic current can flow within the islands. The rotation of the magnetic configuration is due to plasma rotation, $\omega \simeq \frac{m}{r_s} \phi_o'(r_s)$. It thus appears that the interpretation of the experimentally observed frequency of the Mirnow oscillations in terms of the diamagnetic frequency is in general not correct. Instead these oscillations give insight into the rotational motion of the plasma. Of course conditions in a real tokamak are more complicated. Poloidal rotation should be damped by compressional ion heating due to the periodic variation of the toroidal field. On the other hand large toroidal plasma rotation is possible due to neoclassical effects. It appears experimentally that at large island size the magnetic configuration is rotating rigidly in the toroidal direction. A discussion of the theory of the various forms of plasma rotation in a toroidal plasma are, however, beyond the scope of the present paper.

We have demonstrated that the linear properties of the drift tearing mode, in particular $\omega \simeq \omega^*$ and $\gamma \ll \gamma_T$ for $\omega^* \gg \gamma_T$, are due to the assumption of an idealized equilibrium and are changed basically already at a small amplitude. For finite island size the behavior of tearing modes in a cylindrical low-ß plasma is well described by eqs. (22) - (25) as first given by Grad et al.[7] and (31), (32) to describe plasma and mode rotation.

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