Kinetic Theory of Alfvén Wave Propagation+)

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IPP 4/173 Oktober 1978

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+) Vortrag gehalten anläßlich der APS-Tagung in Colorado Springs, USA (30.10.-4.11.1978)

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Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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Abstract

Assuming the validity of the local dielectric tensor approximation, a description of Alfvén wave propagation in a quasi-homogeneous hot plasma is obtained. The Alfvén velocity surface is found to be a linear (cut-off) rather than a singular (resonance) turning point with no evidence of wave conversion. The kinetic (finite temperature) effects occur predominantly through electron inertia rather than due to the finite ion-gyroradius. Boundary value solutions in a slab geometry and using idealized antennas are obtained. While the compressional Alfvén wave (TTMP) is found to possess extremely weak damping, the shear Alfvén wave may encounter difficulty in penetrating to the interior of a thermonuclear plasma.

KINETIC THEORY OF ALFVÉN WAVE PROPAGATION

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I. Introduction

Both compressional[1-7] (TTMP) and shear[8-11] Alfvén waves have been considered as possible candidates for radio-frequency heating of thermonuclear plasmas. The different theoretical treatments use either the MHD[8,9] or the kinetic[1-7, 10-11] description of the plasma. In the kinetic treatment of the shear Alfvén wave, Hasegawa and Chen[10] identify the kinetic Alfvén wave propagating in the interior of the plasma which causes heating through absorption by Landau daming and collisional dissipation.

In this paper we extend the work of Refs. 10-11 and derive the relevant differential equations from the Maxwell's equations in which the local value of the dielectric tensor is assumed to be valid. This approximation is valid provided the elements of the dielectric tensor $\underline{\underline{\epsilon}}$ have a small fractional variation over a wavelength or if the wave length is large compared to the ion-gyroradius. The compressional (c) and the shear (s) modes decouple if we assume $\omega \ll \omega_{ci}$ and that $k_y \equiv o$, $\partial/\partial z = ik_z$, the static magnetic field B_o in the z-direction is constant and the density variation is along the x-direction. The effects due to gradient drifts is neglected and the plasma is assumed to possess an isotropic Maxwellian velocity distribution. The dissipation due to the electron-ion momentum transfer collisions is included by replacing the electron mass by $m_e(1+ i\nu ei/\omega)$.

II. The differential equation

The Maxwell's equations assuming the validity of the local approximation may be written as

$$-i k_z E_y = i\omega\mu_0 H_x \tag{1}$$

$$i k_z E_x - \frac{\partial E_z}{\partial x} = i\omega\mu_0 H_y \tag{2}$$

$$\frac{\partial E_y}{\partial x} = i\omega\mu_0 H_z \tag{3}$$

$$-i k_z H_y = -i\omega\epsilon_0\epsilon_x E_x \tag{4}$$

$$i k_z H_x - \frac{\partial H_z}{\partial x} = -i\omega\epsilon_0\epsilon_x E_y \tag{5}$$

$$\frac{\partial H_y}{\partial x} = -i\omega\epsilon_0\epsilon_z E_z \tag{6}$$

where ϵ_y has been dropped being negligibly small for the frequency range $\omega \ll \omega_{ci}$, while ϵ_x and ϵ_z are given by

$$\epsilon_x = 1 + \sum_j \frac{\omega_p^2}{\omega\omega_c} \xi_0 \frac{e^{-\Lambda}}{\Lambda} \sum_{-\infty}^{\infty} p\, I_p(\Lambda) Z_p \tag{7}$$

and $$\epsilon_z = 1 - \sum_j \frac{\omega_p^2}{\omega^2} \xi_0^2\, e^{-\Lambda} \sum_{-\infty}^{\infty} I_p(\Lambda) Z_p' \,. \tag{8}$$

Here j represents the particle species, I_p is the modified Bessel function, $Z_p = Z(\xi_p)$ is the plasma dispersion function approximated for small and large real values of the parameter ξ_p by

$$Z(\xi_p) \sim -i\,\pi^{1/2} e^{-\xi_p^2} - 2\xi_p \,, \qquad \xi_p \ll 1 \tag{9}$$

$$Z(\xi_p) \sim -i\,\pi^{1/2} e^{-\xi_p^2} - \xi_p^{-1} \,, \qquad \xi_p \gg 1 \tag{10}$$

$$\xi_{pj} = \frac{c}{n_z v_{zj}} \left(1 - \frac{p\,\omega_{cj}}{\omega}\right) \tag{11}$$

$$\Lambda_j = \frac{1}{2} k_x^2 r_{cj}^2 \tag{12}$$

$Z_p' = \partial Z_p/\partial \xi_p$, $I_p' = \partial I_p/\partial \Lambda$, v_{zj} is the particle thermal speed. From (1), (3) and (5), one obtains the differential equation for the compressional wave as

$$\frac{\partial^2 E_y}{\partial x^2} + k_0^2 (\epsilon_x - n_z^2) E_y = 0 . \tag{13}$$

Similarly from (2), (4) and (6) the differential equation for the shear-wave amplitude becomes

$$\frac{\partial^2 H_y}{\partial x^2} - \frac{1}{\epsilon_z} \frac{\partial \epsilon_z}{\partial x} \frac{\partial H_y}{\partial x} + k_0^2 (\epsilon_x - n_z^2) \frac{\epsilon_z}{\epsilon_x} H_y = 0 . \tag{14}$$

In the next two sections we treat the propagation and coupling properties for the two waves separately.

III. The compressional (TTMP) wave

Using ik_x for $\partial/\partial x$ in (13) one obtains the dispersion relation for the compressional wave in a uniform plasma,

$$n_x^2 = \epsilon_x - n_z^2 \tag{15}$$

Expanding in powers of Λ one obtains from (7),

$$\epsilon_x \simeq 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\left(1 + i\frac{\nu_{ei}}{\omega}\right) + \frac{\omega_{pi}^2}{\omega_{ci}^2}\left(1 + \frac{\Lambda}{4}\right) + 2i\pi^{1/2}\frac{\omega_{pi}^2}{\omega_{ci}^2}\frac{c}{n_z V_z}\, e^{-\frac{\omega_{ci}^2}{\omega^2}\frac{c^2}{n_z^2 V_{zi}^2}} \tag{16}$$

Because of the presence of p in the summation product in (7) the term p = 0 vanishes identically and there can be no Landau damping of the compressional (TTMP) wave. The second damping term in (16) is from the first cyclotron-harmonic (p=1) contribution and is negligibly small compared to the first damping term arising due to the collisional dissipation. Since for the compressional wave $\Lambda \ll 1$ (16) becomes

$$\epsilon_x \simeq 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} + \frac{\omega_{pe}^2}{\omega_{ce}^2}\left(1 + i\frac{\nu_{ei}}{\omega}\right) \simeq \frac{c^2}{V_A^2} \; . \tag{17}$$

From (15) and (17) it may be seen that the compressional wave has a cutoff ($k_x = 0$) when the phase velocity ω/k_z of the wave along the static magnetic field direction equals the local Alfvén speed. We shall refer to this linear turning point as the Alfvén cutoff. An electromagnetic wave launched into the plasma of monotonically increasing density is evanescent ($k_x^2 < 0$) outside the Alfvén cutoff and propagating within.

The wave propagates as a TE (finite E_y, H_x and H_z) electromagnetic wave with the electric field transverse to the direction of propagation. Such a wave may be launched by imposing a y-directed current antenna carrying a current $I_y = J \exp(i\, k_z z)$. We assume symmetric excitation of a slab plasma model with density varying linearly from the plasma edges towards the center as shown in Fig. 1. Writing ϵ_x as

$$\epsilon_x = 1 + \gamma^2(x + a) , \qquad x \leq 0 \tag{18}$$

where $$\gamma^2 = \frac{d}{dx}\left\{\frac{\omega_{pi}^2(x)}{\omega_{ci}^2} + \frac{\omega_{pe}^2(x)}{\omega_{ce}^2}\left(1 + i\frac{\langle\nu_{ei}\rangle}{\omega}\right)\right\} \quad , \tag{19}$$

equation (13) takes the form of an Airy equation

$$\frac{\partial^2 E_y}{\partial x^2} + k_o^2\left\{\gamma^2 x + (1 + \gamma^2 a - n_z^2)\right\} E_y = 0 , \tag{20}$$

with the solution

$$E_y^I(\xi) = p^I Ai(X) + q^I Bi(X) \tag{21}$$

where $\xi = (x + a) + (1-n_z^2)/\gamma^2$ and $X = (\gamma k_o)^{2/3}\ \xi$. Writing down expressions for the electric field in regions II and III as

$$E_y^{II}(x) = p^{II} e^{\kappa_x x} + q^{II} e^{-\kappa_x x} , \tag{22}$$

and $$E_y^{III}(x) = p^{III} e^{\kappa_x x} , \tag{23}$$

where $\kappa_x = (k_z^2 - k_o^2)^{1/2}$ and upon applying the relevant boundary conditions for Ey and H_z at $x = -a$ and $x = -b$ one finally obtains the electric field everywhere within and without the plasma from Eqs. (21) - (23) where $p^I = \Delta 1/\Delta$, $q^I = \Delta 2/\Delta$, $p^{II} = \Delta_3/\Delta$, $q^{II} = \Delta_4/\Delta$, $p^{III} = \Delta_5/\Delta$

$$\Delta_1 = 2J\beta\ Bi(X)_o\ e^{-\kappa_x b}$$

$$\Delta_2 = -2J\beta\ Bi(X)_o\ e^{-\kappa_x b}$$

$$\Delta_3 = -Je^{\kappa_x(a-b)}\left[Ai(X)_o\left\{\beta\, Bi(X)_{-a} + Bi'(X)_{-a}\right\}\right.$$
$$\left. - Bi(X)_o\left\{\beta\, Ai(X)_{-a} + Ai'(X)_{-a}\right\}\right]$$

$$\Delta_4 = J e^{-K_x(a+b)} \left[Ai(X)_o \left\{ -\beta\, Bi(X)_{-a} + Bi'(X)_{-a} \right\} \right.$$
$$\left. - Bi(X)_o \left\{ -\beta\, Ai(X)_{-a} + Ai'(X)_{-a} \right\} \right]$$

$$\Delta_5 = J\, Ai(X)_o \left[Bi(X)_{-a} \left\{ -\beta e^{-K_x(a-b)} - \beta e^{K_x(a-b)} \right\} \right.$$
$$\left. - Bi'(X)_{-a} \left\{ - e^{-K_x(a-b)} + e^{K_x(a-b)} \right\} \right]$$
$$- J\, Bi(X)_o \left[Ai(X)_{-a} \left\{ -\beta e^{-K_x(a-b)} - \beta e^{K_x(a-b)} \right\} \right.$$
$$\left. - Ai'(X)_{-a} \left\{ - e^{-K_x(a-b)} + e^{K_x(a-b)} \right\} \right]$$

and

$$\Delta = 2\, Ai(X)_o \left[\beta e^{-K_x a} Bi(X)_{-a} - e^{-K_x a} Bi'(X)_{-a} \right]$$
$$- 2\, Bi(X)_o \left[\beta e^{-K_x a} Ai(X)_{-a} - e^{-K_x a} Ai'(X)_{-a} \right]$$

$\beta = K_X/(\gamma k_0)^{2/3}$ and $(X)_o = [X]_{X=0}$ etc. The energy flux P_{in} into the plasma is given by

$$P_{in} = \frac{1}{2} J \, Re[E_y]_{x=-b} = \frac{1}{2} J\, p^{III} e^{-K_x b} \qquad (24)$$

and in order to obtain the field quantities normalized to the unit energy flux, they should all be multiplied by P_{in}^{-1}. The power dissipated per unit area in the antenna is given by

$$P_{diss} = \frac{1}{2} J^2 \frac{\rho}{\delta} \qquad (25)$$

where ρ and δ are the specific resistence and skin depth of the antenna material. The heating efficiency η is given by

$$\eta = \frac{P_{in}}{P_{in} + P_{diss}} \quad . \tag{26}$$

The heating efficiency η in (26) is typically less than one percent for a plasma of thermonuclear parameters. Nor is the situation much improved by including ion-ion collisions in the treatment of a three component plasma.

IV. The shear wave

From (14), the dispersion relation for the shear wave in a uniform plasma becomes

$$n_x^2 = \frac{\epsilon_z}{\epsilon_x} \left(\epsilon_x - n_z^2 \right) . \tag{27}$$

In case the wave travels exactly along the ambient static magnetic field ($k_x^2 = 0$) one obtains the well known form of the shear Alfvén wave dispersion relation $n_z^2 = \epsilon_x$ which we now recognize as the Alfvén cutoff. For this wave Λ is no longer small compared to unity and attains values as high as 0.5 as found through computations using the full hot-plasma dispersion relations without simplifying approximations. From (16) we observe that ϵ_x is still given within 10 % accuracy by the approximate expression (17). The actual accuracy is better by a factor of 2 due to the diminishing values of Z_p for higher values of p. This process, in fact, dominates in the summation (8) so that ϵ_z is given within close approximation by

$$\epsilon_z \simeq 1 - \frac{\omega_{pi}^2}{\omega^2} \xi_{oi}^2 Z_{oi}' - \frac{\omega_{pe}^2}{\omega^2} \left(1 + i \frac{\nu_{ei}}{\omega}\right) \xi_{oe}^2 Z_{oe}' \quad . \tag{28}$$

In a cold plasma $\xi_o = c/n_z v_z \to \infty$, $Z_0' \to \xi_0^{-2}$ and ϵ_z has the familiar form

$$\epsilon_z = 1 - \frac{\omega_p^2}{\omega^2} \quad . \tag{29}$$

In addition to the Alfvén cutoff, the shear wave in a cold plasma has a plasma cutoff (ϵ_z = 0) very near the plasma edge. In practice a tenuous plasma is likely to extend upto the antenna surface and the plasma cutoff may not play any significant role. Figure 2 shows a plot of k_x^2 versus density in a cold, collisionless plasma both for the compressional and the shear Alfvén waves where

$$n_z \sim \frac{\omega_{pi}}{\omega_{ci}} \tag{30}$$

was chosen for the Alfvén cutoff to lie midway between the plasma edge (n_e =0) and the maximum plasma density ($n_e = 10^{15}$ cm^{-3}). With increasing plasma temperature v_z, the particle thermal speed increases and

$$\xi_{oe} = \frac{c}{v_{ze}} \frac{\omega_{ci}}{\omega_{pi}} \tag{31}$$

decreases and as it passes through a value near unity, the sign of the real part of Z'_{oe} and hence ϵ_z reverses and the propagating and evanescent regions for the shear wave in the plasma are interchanged. Since the finite gyroradius has only a nominal quantitative effect on the plasma dispersion relation, one may identify the finite electron inertia as the dominating kinetic process causing modifications of the shear Alfvén wave propagation. Figure 3 is a plot of the compressional and shear wave dispersion characteristics as a function of density in a hot Maxwellian plasma. The approximate dispersion relation for the case ξ_e << 1 may be obtained from (17), (27), and (28) as

$$\omega^2 = k_z^2 v_A^2 \left\{ 1 + \frac{1}{2} \frac{T_e}{T_i} \rho_i^2 k_x^2 \right\} \tag{32}$$

which is similar but not identical to Eq. (8) of Ref. 10. Writing (32) in the form

$$k_x^2 = \frac{2}{k_z^2 v_A^2 \rho_i^2} \frac{T_i}{T_e} \left(\omega^2 - k_z^2 v_A^2 \right) \tag{33}$$

one can also see that the kinetic Alfvén wave (named after Ref. 10) propagates on the high density side of the Alfvén cutoff but is evanescent on the low density side. Using the same approximation namely ϵ_x, $\epsilon_z >> 1$, ϵ_z/ϵ_x = constant and ϵ_x varies linearly with x one may reduce (14) to an Airy equation and calculate the field components proceeding in a manner identical to the one used for the compressional wave. Since the shear Alfvén wave is a TM (finite E_x, E_z and H_y) wave, one must resort to using a TM antenna which excites the relevant field components. Note that since the electric field is finite along the direction of propagation, this wave has a mixed electrostatic and electromagnetic character[10].

Unlike its compressional counterpart, the shear Alfvén wave is readily absorbed by collisions as well as electron Landau damping both occuring due to the presence of ϵ_z in the dispersion relation. The problem in this case arises due to the rapid attenuation of the wave near the plasma boundary either due to wave evanescence or absorption. A measure of this attenuation is given by the integral

$$\Gamma = \int_0^x k_{xi}\, dx \,, \tag{34}$$

where

$$k_{xi} = \mathrm{Im}\left[\frac{\epsilon_z}{\epsilon_x}\left(k_0^2\epsilon_x - k_z^2\right)\right]^{1/2}$$

which using (17) and (28) may be approximated as

$$k_{xi} = \mathrm{Im}\left[-\frac{T_i}{T_e}\frac{Z_{0e}'}{\rho_i^2}\left(1 + i\,\frac{\nu_{ei}(x)}{\omega}\right)\left(\frac{1}{n_z^2}\,\frac{\omega_{pi}^2(x)}{\omega_{ci}^2} - 1\right)\right]^{1/2} . \tag{35}$$

In (35) both $\nu_{ei}(x)$ and $\omega_{pi}^2(x)$ are linear functions of density and of x for a linear density profile. All other quantities are constants if no gradients of magnetic field or temperature exist. Equation (34) in this case can be exactly integrated to give

$$\Gamma = \mathrm{Im}\left[\left(i\,\frac{T_i}{T_e}\,\frac{Z_{oe}'}{\rho_i^2}\right)^{1/2}\left\{\frac{2cx+b}{4c}\sqrt{R}+\frac{4ac-b^2}{8c}\,\ln\left(2\sqrt{cR}+2cx+b\right)\right\}\right]_0^x$$

where $a = -i$, $b = -\gamma_1 + i\,\gamma_2$, $c = \gamma_1\gamma_2$, $\gamma_1 = d\nu_{ei}/\omega dx$, $\gamma_2 = \partial\omega_{pi}^2/n_z^2\omega_{ci}^2\partial x$, $R = a + bx + cx^2$ and the results are shown in Fig. 5 where the distance from the plasma edge at which the wave amplitude falls by 90 % of its value at the antenna is plotted versus n_z. The effect of altering n_z is to displace the Alfvén cutoff along x. For $n_z = 1$ the Alfvén cutoff occurs at the plasma edge while for $n_z = 100$, this occurs very near the plasma center. The increased penetration at T = 100 eV is due to ζ_e becoming greater than unity and the wave propagates between the plasma edge and the Alfvén cutoff as for the cold plasma (Fig. 2). For $T \leq 100$ eV, electron-ion collisions contribute significantly to wave attenuation.

V. Discussion

Starting from Maxwell's equations and using the full hot-plasma dielectric tensor formulism, it was found that there is no collisionless damping term for the compressional (TTMP) Alfvén wave. Since this treatment is quite general, one may conclude that the so called magnetic Landau damping[2,3] is misleading in so far as it fails to reveal other obviously compensating mechanisms which exactly cancel this damping effect. We are therefore left with "gyrorelaxation"[1] as the sole mechanism leading to a weak absorption of the wave at elevated plasma temperatures occuring in fusion plasmas.

The more hopeful absorption processes occuring to damp the shear wave, on the other hand, remain ineffective in heating the plasma interior due to the strong attentuation of the wave occuring close to the plasma edge.

We are thus left in an uneviable position where the compressional wave is able to penetrate to the plasma interior but is unable to damp while the shear wave would have been able to heat the plasma, were it only able to gain access to the interior. If it were only possible for the two waves to couple efficiently in the plasma interior one might be able to transport the energy by the compressional wave and dissipate the energy converted into the shear wave.

This is strictly not possible for the present case where the two waves are decouplted for $n_y \equiv 0$. We have ignored the weak coupling which would occur if we had not assumed $|\epsilon_x| >> |\epsilon_y|$ at the outset. Finite n_y helps to further couple these two waves. The extent of this coupling and its effect on plasma heating are being presently investigated.

Parenthetically note that assuming the validity of the local dielectric tensor does not amount to the WKB or geometric optics assumption which, in addition, requires that k_x vary slowly as well. Local dielectric tensor assumption is a weaker assumption exactly valid in a cold plasma even though the WKB assumption breaks down near resonances or cutoffs. We were able to avoid the stronger WKB assumption because the Airy differential equations obtained for both the compressional and shear wave propagation in a plasma with linearly varying density profile are exactly integerable.

References

1. A. Schlüter, Z. Naturforsch. 12A 822 (1957).

2. J.M. Berger, W.A. Newcomb, J.M. Dawson, E.A. Frieman, R.M. Kulsrud and A. Lenard, Phys. Fluids 1, 301 (1958).

3. T.H. Stix, The Theory of Plasma Waves, McGraw-Hill, New York (1962).

4. K.N. Stepanov, Soviet Phys. JETP 18 826 (1964).

5. J.M. Dawson and M.F. Uman, Nucl. Fusion 5 242 (1965).

6. C.N. Lashmore Davies and R.M. May, Phys. Fluids 15, 1616 (1972).

7. M. Brambilla, Plasma Phys. 16 482 (1974).

8. J.A. Tataronis and W. Grossman, Z. Phys. 261 217 (1973).

9. A. Hasegawa and L. Chen, Phys. Rev. Lett. 32 454 (1974); Phys. Fluids 17 1399 (1974).

10. A. Hasegawa and L. Chen, Phys. Rev. Lett. 35 370 (1975).

11. A. Hasegawa, Phys. Fluids 19 1924 (1976).

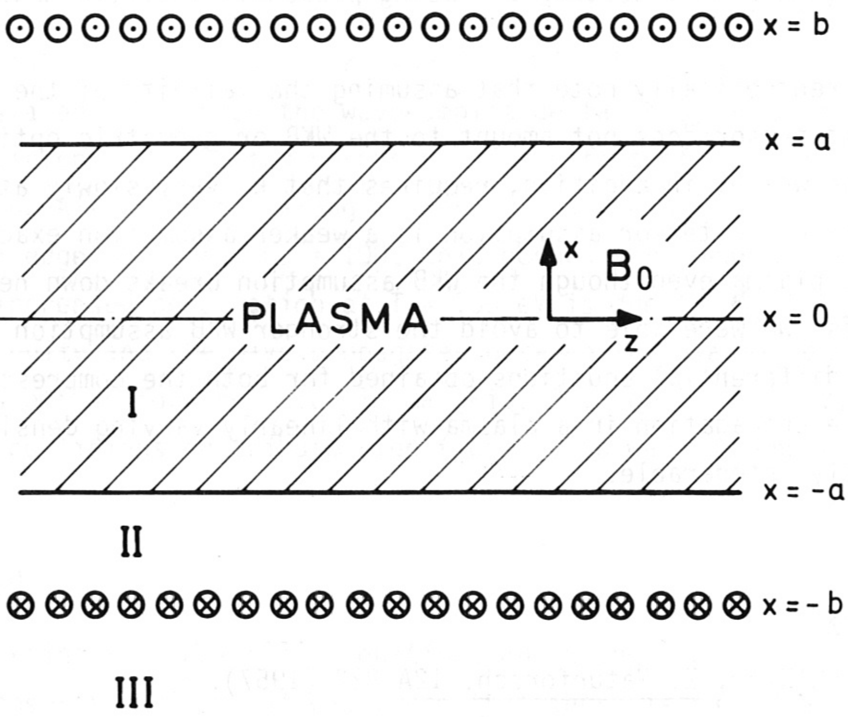

Fig. 1 Slab geometry of the boundary value problem for the compressional Alfvén wave excited by an idealized antenna carrying a surface current $I_y = \pm J \exp(ik_z z)$ on the two sides of the plasma. The plasma density is assumed to vary from $n_e = 0$ at the plasma edge to n_{max} at $x = 0$. In all the numerical results cited in this paper $n_{max} = 10^{15}$ cm^{-3} and $B_0 = 60$ kG. Gas used is deuterium and the parallel wavelength $\lambda_z = 10$ m, throughout.

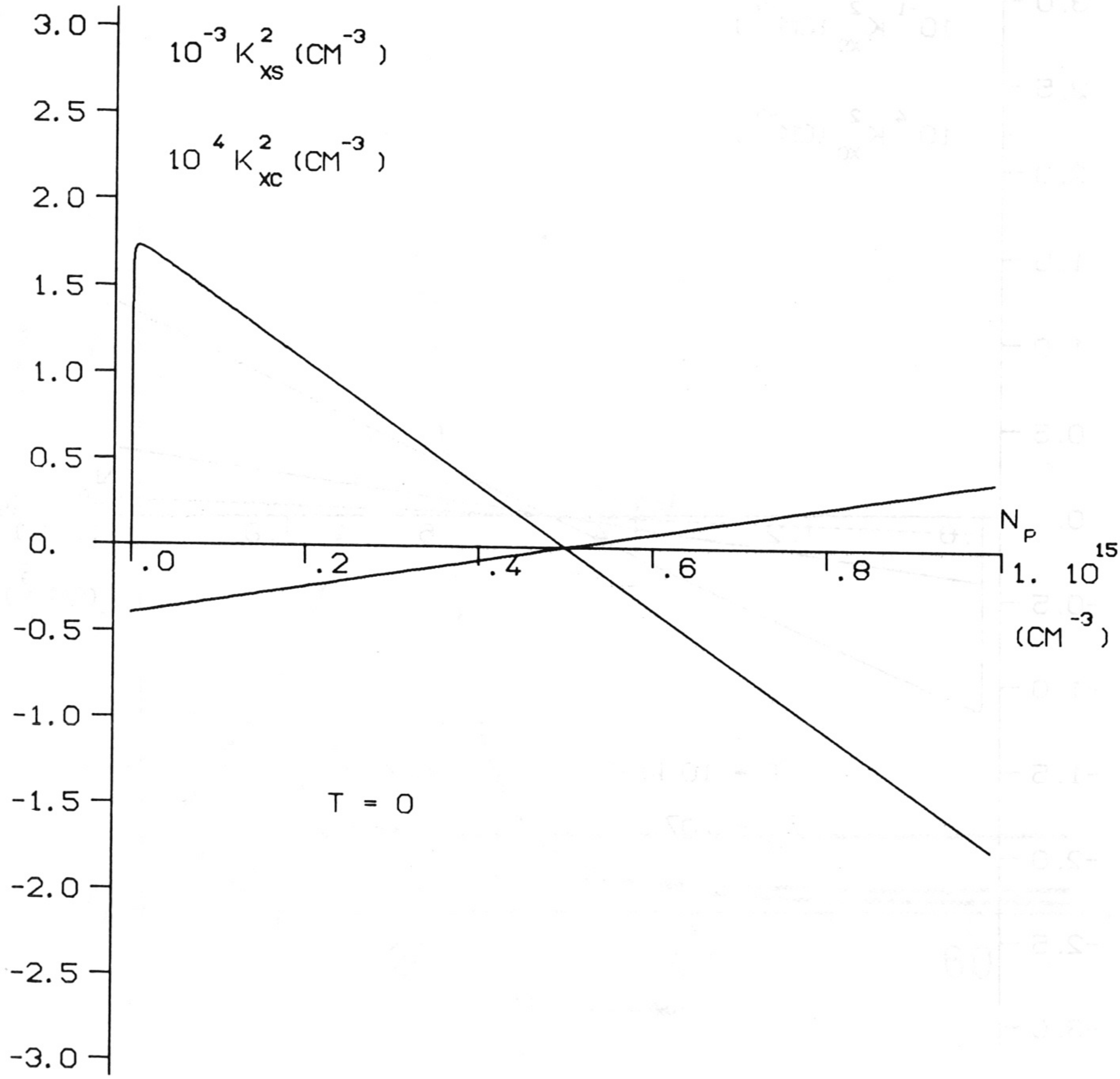

Fig. 2 The dispersion characteristics k_x^2 versus density for both the compressional and shear waves in a cold, collisionless plasma. The value of n_z was chosen for the Alfvén cutoff to lie midway between the plasma edge and center. The larger of the two roots corresponds to the shear wave.

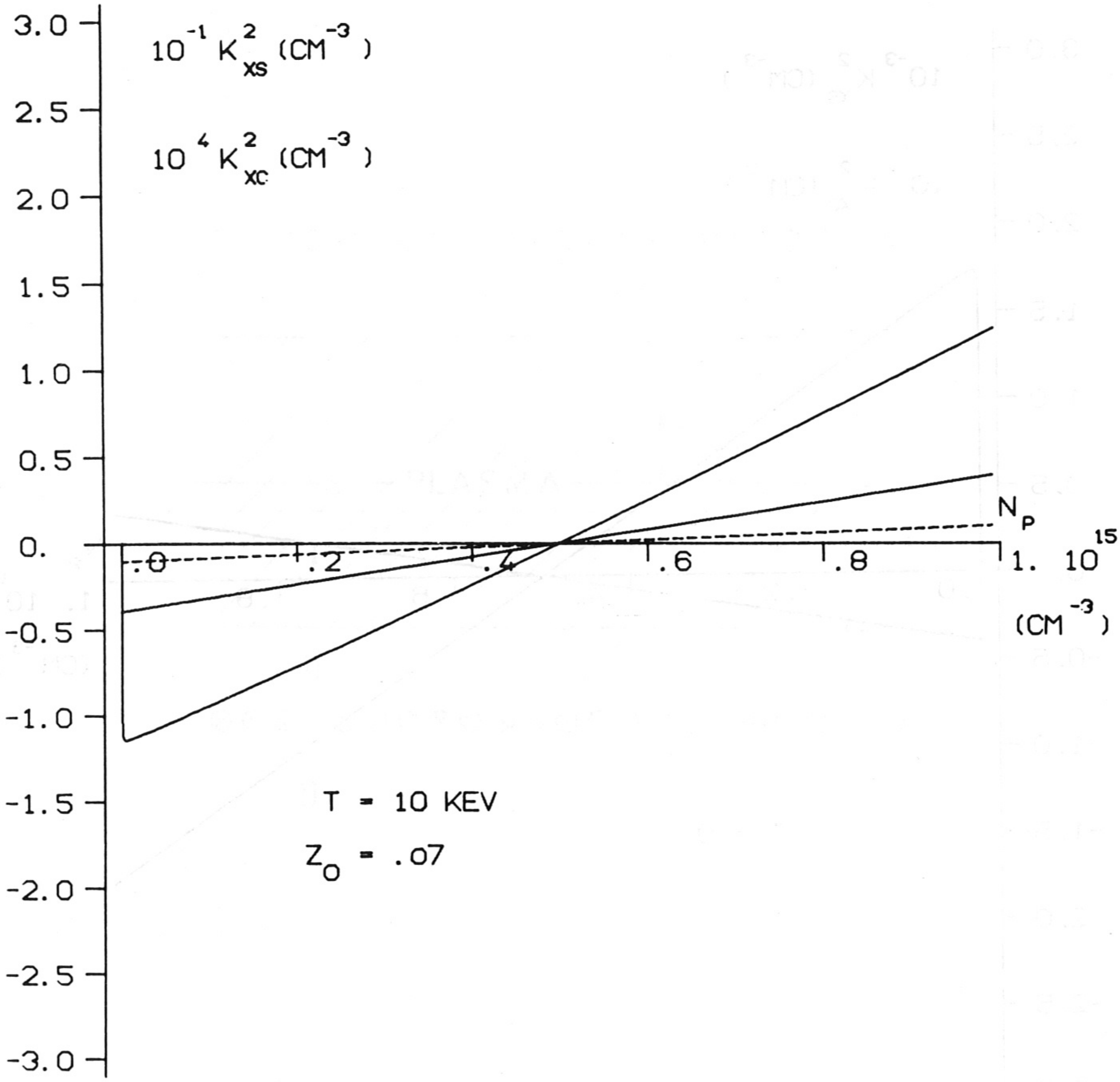

Fig. 3 Same as Fig. 2 but for a hot plasma with $T_e = T_i = 10^8\ {}^oK$. The dashed curve shows the $I_m(k_x^2)$ for the shear wave. These dispersion curves are the computed roots of the full hot-plasma dispersion relation in which all the six elements of the dielectric tensor[3] are included. Naturally, the two roots for the compressional and shear wave, in this case, are coupled. The very slight coupling, however, does not significantly alter the dispersion curves which would be obtained from the simplified relations (15) and (27) in which ϵ_y has been neglected. In particular, the compressional wave remains essentially undamped.

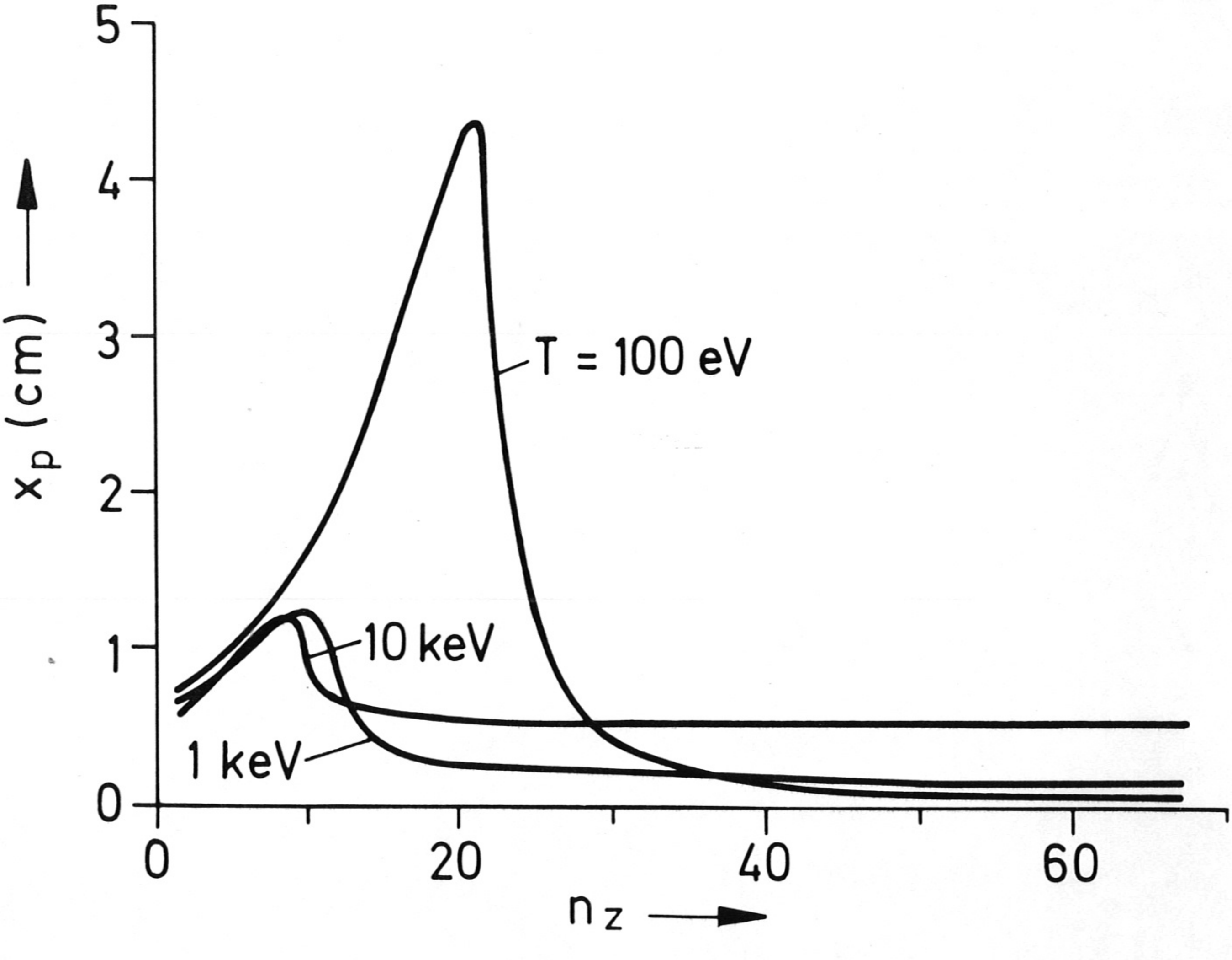

Fig. 4 Attenuation distance for which the shear wave amplitude drops to one-tenth of its value at the plasma surface versus n_z. The humps to the left occur because at the low n_z values ξ_{oe} becomes less than unity and the wave becomes propagating from the plasma edge till the Alfvén cutoff layer.