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# NONLINEAR THRESHOLD FOR TEARING MODE GROWTH


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Abstract

It is shown that the threshold for growth of tearing modes at island size exceeding the resistive layer width is $\Delta' = 0$, independent of effects such as parallel sound wave propagation and radial equilibrium plasma flow.

Tearing modes, or resistive kink modes as they are sometimes called, are the predominant internal macroscopic modes in low-ß current carrying torii, since ideal MHD modes are either stable or saturate at low amplitudes. Because of their potential importance, in particular in causing current disruptions, these modes have attracted much interest. Most theoretical work was devoted to the linear tearing instability taking into account various refinements of the original theory [1], e.g. diamagnetic effects [2)3)] and a kinetic description of the electrons [4]. These additional nonideal effects not only lead to a modification of the (complex) eigenfrequencies but may also change the thresholds for instability [5) 6)].

It has recently been pointed out, however, that the characteristics of the linear tearing instability, i.e. exponential growth with growth rates proportional to a fractional power of the resistivity $\eta$ and with frequencies given by the diamagnetic frequencies, are radically changed at small amplitude corresponding to a magnetic island size $\Delta_I$ of the order of the resistive layer width $\delta$ [7)8)] (an exception is the m = 1 mode). Further island growth proceeds on the resistive diffusion time scale, while mode rotation is determined by plasma flows and not by diamagnetic drifts. Since experimentally only magnetic perturbations of a certain amplitude are observed usually exceeding that corresponding to islands width $\Delta_I \sim \delta$, the linear properties are of little relevance.

In this letter it is pointed out that also the modifications of the threshold for instability, as discussed in Refs. 5), 6), vanish for

$\Delta_I > \delta$ and that the threshold for nonlinear growth is $\Delta' = 0$.

Let us briefly review the essentials of linear and nonlinear theory of the tearing instability. Resistivity and other nonideal effects play a role only within a narrow layer $\delta$ around the mode rational surface where $q = m/n$. We restrict ourselves to the most commonly treated case of $\psi$ = const across $\delta$, thus excluding $m = 1$ for instance. Eigenfrequencies are obtained by matching the inside solution $\psi_i$ to the ideal outside solution $\psi_a$ and are characterized by the jump in slope of $\psi_a$ across the resistive layer, $\Delta' = (\psi'_{a+} - \psi'_{a-})/\psi_a$ .

To be definite we consider the following set of equations derived from two-fluid theory in the limit of large aspect ratio ("cylindrical torus", for details see Ref. 9), neglecting ion temperature and viscosity for simplicity:

(1) $$\frac{\partial \psi}{\partial t} + u \cdot \nabla \psi = \eta j_{\parallel} - E_o - \alpha \frac{T_e}{n} \nabla_{\parallel} n$$

(2) $$\frac{\partial n}{\partial t} + u \cdot \nabla n = \alpha \nabla_{\parallel} j_{\parallel} - \nabla_{\parallel} \Gamma_{\parallel}$$

(3) $$\frac{\partial \Gamma_{\parallel}}{\partial t} = - T_e \nabla_{\parallel} n$$

(4) $$\frac{\partial A}{\partial t} + u \cdot \nabla A - \frac{1}{2} (\nabla n \times \nabla u^2)_{\parallel} = B \cdot \nabla j_{\parallel}$$

Units are the plasma radius a, a typical poloidal magnetic field $B_{\theta o}$, a typical density $n_o$. Thus the normalized resistivity is $\eta = \tau_A/\tau_S = S^{-1}$ [1)] and $\alpha = c/\omega_{pi} a$, $\varepsilon = B_{\theta o}/B_z$, $\psi$ = helical flux function $\nabla_\perp^2\psi = j_\parallel - 2$, $A = \nabla_\perp^2 \phi$, $\underline{b} \times \nabla\phi = \underline{u}$, $\underline{b} = \underline{B}/|B|$ . In the linearized limit eqs. (1) - (4) yield the dispersion relation for drift tearing modes, where the real part of the frequency is proportional to the diamagnetic frequency $\omega^* = -\frac{m}{r}\ \varepsilon\ \alpha\ T_e\ n_o'/n_o$. The threshold for instability, which for pure tearing modes in the absence of equilibrium flow is $\Delta' = 0$, may be different from zero, if effects like parallel plasma motion (sound waves) [5)] or a zero order radial plasma[6)] are included.

Let me now discuss the modifications of the linear instability that occur at finite but very low (in most applications) amplitude. There are two effects that radically change the properties of linear drift tearing modes:

i) Equation (1) gives rise to a nonlinear current contribution $\delta j_o \simeq \frac{1}{\eta}\frac{m}{r}\frac{\partial}{\partial r}(\tilde\phi\ \tilde\psi)$, which inserted into the equation of motion (4) yields that the inertia terms are negligible for magnetic island size $\Delta_I = 4\sqrt{\tilde\psi / \psi_o''}$ exceeding the resistive layer width $\delta$ . Hence $B \cdot \nabla j = 0$ for $\Delta_I > \delta$ , the further development proceeding on the resistive diffusion time scale as a sequence of MHD equilibrium states. When averaged over flux surfaces V(V(r,t) being the volume enclosed by a surface $\psi$ = const), the set (1) - (4) reduces to the equations first discussed in Ref. 10 ,

$$(5) \qquad \frac{\partial\psi(V,t)}{\partial t} = \eta \frac{\partial}{\partial V} K \frac{\partial\psi}{\partial V} - E_o , \quad K = < |\nabla V|^2 >$$

$$(6) \qquad \nabla^2_\psi = j(V,t) - 2 \ .$$

Evidently the development of $\psi$ decouples from that of the density.

ii) According to (2) the density will become a flux function for $\Delta_I > \delta$ since $\nabla_{\parallel} j_{\parallel} = 0$ from i) . Because of the high parallel thermal conduction $T_e$ is also a flux function. Hence there can be no net azimuthal diamagnetic current across the islands, $n_o'(r_s) = 0$, i.e. $\omega^* = 0$. Consequently a rotation of the mode cannot be due to diamagnetic drifts but only to plasma rotation. It is shown in Ref. 8, that $\omega^* \simeq 0$ for $\Delta_I \gtrsim \delta$.

Equations (5), (6) have been treated in the limit of small island size [7), 11)], where one obtains

$$(7) \qquad \frac{\partial\Delta_I}{\partial t} = \frac{\pi}{2} \eta \Delta' + O(\Delta_I)$$

Thus the threshold for island growth is $\Delta' = 0$. Equation (7) can easily be verified qualitatively. For small island size $\Delta_I \simeq \delta$ we have in the vicinity of the singular surface $\psi = \psi_o(r) + \psi_1(t) \cos m\theta$, since $\delta\psi_o/\psi_1 \sim \eta^{2/5}$. Considering (1) at the O-point where the convective term disappears, one obtains

$$\frac{\partial\psi_1}{\partial t} = \eta_o j_1 + \eta_1 j_o \tag{8}$$

with $\eta_1 = (\delta\eta/\delta\psi)\psi_1$, where $\eta(\psi)$ is determined by the electron energy balance. The average value of the perturbed current $j_1$ which is concentrated within the islands is

$$j_1 = \psi_1'' \simeq \psi_1 \frac{\Delta'}{\Delta_I} \simeq \psi_o'' \; \Delta_I \, \Delta' \quad . \tag{9}$$

Insertion into (10) yields

$$\frac{\partial\Delta_I}{\partial t} \simeq \eta_o \, \Delta' + j_o \frac{\delta\eta}{\delta\psi} \, \Delta_I \quad , \tag{10}$$

where $\Delta'$ is a function of $\Delta_I$, too, $\Delta' = \Delta'_o + O(\Delta_I)$. It is worthwhile to note that in (7) the growth rate for $\Delta_I \sim \delta$ essentially equals the linear growth rate $\gamma_T$, since

$$\gamma = \frac{1}{\psi_1} \; \frac{\partial\psi_1}{\partial t} = 2 \, \frac{1}{\Delta_I} \frac{\partial\Delta_I}{\partial t} \sim \eta_o \; \frac{\Delta'}{\Delta_I} \sim \gamma_T \, \frac{\delta}{\Delta_I}$$

Hence in constrast to the diamagnetic frequency $\omega^*$, which vanishes for $\Delta_I \geq \delta$, the growth rate is only gradually reduced as the islands grow to finite size $\Delta_I >> \delta$ .

The threshold $\Delta' = 0$ also follows from energy considerations. It has been shown [12] that $-\psi_1{}^2\Delta'$ is the magnetic energy of the tearing mode. Energy conservation implies that the mode may grow if $\Delta' > 0$. This argument remains valid for $\Delta_I \gtrsim \delta$ as long as $\Delta_I$ is small, since $\psi_a$ and therefore $\Delta'$ are not changed. $\Delta'$ thus characterizes the free magnetic energy of the global configuration, i.e. the current distribution, which is the only major energy source in a low-ß plasma.

It has been pointed out in Ref. 5, that in the linear theory parallel sound waves, eq.(3), give rise to a finite positive threshold value of $\Delta'$. The essential effect in the linearized equation is caused by a term $\nabla_{\parallel}^2 n = - k_{\parallel}^2 n$. Now according to ii) for finite island size $\nabla_{\parallel} n \simeq 0$, so that the contribution becomes negligible. It is easy to see directly how $k_{\parallel}^2 n$ is strongly affected for $\Delta_I \gtrsim \delta$:

$$\nabla_{\parallel}^2 n = - (k_{\parallel_o}^2 + (\delta k_{\parallel})^2)\ n$$

$$\simeq k_{\parallel_o}'^2\ (x^2 n - \Delta_I^4 n''),\ x = r - r_s\ .$$

As $n'' \sim n/\delta^2$, the nonlinear term is apparently important for $\Delta_I \simeq \delta$. Since the threshold for further island growth is $\Delta' = 0$, tearing modes may be linearily stable, but nonlinearily unstable, the necessary amplitude being small, however.

In ref. 6 it is found that a radial equilibrium plasma flow $v_o$, necessary to establish true resistive equilibrium if $(\eta_o j_o)' \neq 0$,

$$v_o \psi_o' = \eta_o j_o - E_o \ , \quad v_o = O(\eta_o) \quad ,$$

gives rise to a finite threshold $\Delta' \neq 0$ for the linear tearing instability. Assumption of a non-vanishing equilibrium flow leads to additional convective terms on the l.h.s. of the linearized equations. It can be seen that the term $v_o \partial A_1/\partial r$ in (4) is the most important one, while $v_o \ \partial\psi_1/\partial r$ in (1) is negligible because of the constant $\psi$ property. For $\Delta_I > \delta$, however, inertia effects in (4) including $v_o \partial A_1/\partial r$ are negligible. On the other hand the $v_o \partial\psi/\partial r$ term in (1) is still small as long as $\Delta_I << 1$. Therefore the threshold for island growth beyond the resistive layer width is $\Delta' = 0$ independent of the magnitude and direction of a radial equilibrium flow $v_o = O(\eta)$.

At finite island size $\Delta_I \sim 1$, the behavior of the resistivity $\delta\eta/\delta\psi$ in (12) and also a radial plasma flow become important effects. Obviously $\delta\eta/\delta\psi < 0$ within the islands is stabilizing, while $\delta\eta/\delta\psi > 0$ caused for instance by impurity radiation cooling in the island interior enhances island growth. It can also be seen that a radial outward flow $v_o > 0$ is destabilizing, while an inward flow has a stabilizing effect.

In this letter we have pointed out that the threshold for growth of tearing modes at island size exceeding the resistive layer width is determined by the free energy of the magnetic configuration characterized by the quantity $\Delta'$ and not by effects like parallel soundwaves or radial plasma flow that are important only within the resistive layer $\delta$ for $\Delta_I < \delta$. In particular the threshold for island growth at amplitudes $\Delta_I > \delta$ is $\Delta' = 0$, at least for sufficiently simple geometry as assumed in the present note.

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