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GARCHING BEI MÜNCHEN

Hain-Lüst Equation with Toroidal Corrections

C. Copenhaver

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Abstract

Variational theory is used to derive a generalized Euler equation
and a new energy functional which are convenient for analytical
studies of ideal MHD stability in tokamaks. This generalized Euler
equation, which is an explicit function of magnetic surface
coordinate ψ only, represents an infinite set of equations coupled
together by poloidal m mode coupling. In the infinite aspect ratio
limit, the toroidal curvature and mode coupling terms disappear
and an infinite set of uncoupled Euler equations for the diffuse
linear pinch (Hain-Lüst equation) for each m value result. The
continous spectrum is discussed for the circular toroidal case.
In this case, the equations are specialized further to three
modes m, m−1, m+1 and in the marginal stability limit reduce to known
results. Analytically eliminating the m−1 and m+1 modes for
arbitrary current profiles provides results on limiting ß poloidal
for tokamaks.

CONTENTS

I. INTRODUCTION

There has been renewed interest in analytical studies of mode coupling
in tokamaks. Recently, Zakharov[1] has shown, in a circular cross section
toroidal plasma having a fixed boundary, that the stability of the
plasma deteriorates with rise in pressure gradient as a result of
poloidal mode coupling of the fundamental mode m, where m is the
poloidal wave number, with the associated modes m + 1 and m − 1.
Zakharov's work is an extension of the earlier work of Ware and Haas[2],
in that the more recent work allows for non-uniform longitudinal
current and for the associated modes to be nonlocal in nature. Both
these works employ the same basic method of derivation. They start
with a minimized form of the potential energy functional and the
assumption of a circular tokamak. The perturbation is expanded in a
Fourier series in m, with the azimuthal wave number n being fixed
because the plasma is axisymmetric. The method of successive
approximations is then applied to the system, and then finally,
after integrating over poloidal angle, a minimized potential energy
functional is obtained. Minimizing the potential energy functional
results in three Euler equations.

The approach used here is different. Basically, variational theory
is used to first derive the Euler equations. Specifically, a ψ, χ, ϕ
orthogonal coordinate system is considered where the constant ψ surfaces
correspond to the magnetic flux surfaces, and the constant χ
surfaces radiate out from the magnetic axis and are perpendicular to
the ψ surfaces.

Using variational theory, the explicit χ dependence is eliminated by an infinite expansion in m and an integral over χ. This elimination results in a generalized Euler equation which explicitly depends only on ψ and is the principal derivation. From this equation, the energy functional is obtained.

Our main result is a generalized Euler equation, which has the following properties:

(1) The generalized Euler equation is an explicit function of the ψ coordinate only, and represents an infinite set of coupled m equations.

(2) In the infinite aspect ratio limit, the mode coupling terms and noncoupling curvature terms disappear and the infinite coupled set of Euler equations now uncouple. The infinite set of uncoupled Euler equations is identical to the infinite set of ideal MHD equations of motion for a diffuse linear pinch, or the Hain-Lüst[3] equation, one equation for each different mode number m.

(3) In the circular tokamak case, the continua are obtained as an infinite set of Alfven continua, ω_{1m}^2, and slow continua, ω_{2m}^2, corresponding to each mode number m. The cluster point at infinity is also present.

A secondary result is the derivation of an energy functional which follows from the generalized Euler equation. This energy functional has the following properties:

(1) When minimized it yields an infinite set of Euler equations for each m : the generalized Euler equation.

(2) In the marginal stability limit, $\omega^2 = 0$, and considering only the m, m −1, m +1 modes for the case of a circular tokamak, the potential energy functional of Ware and Haas and that of Zakharov are obtained.

The analysis proceeds as follows: In Sec. II, variational theory is used to derive an infinite set of Euler equations for each poloidal mode number m, and they are explicitly a function of the ψ coordinate only. The theory includes the use of a force operator in matrix form as developed by Goedbloed[4]. The derivation also includes the development and reduction of an infinite set of differential equations to a generalized Euler equation corresponding to each m value. Section III contains the generalized Euler equation corresponding to each m, which includes the coupling of the associated modes, m + 1 and m − 1. The derivation of the energy functional is given. Discussion of a general method for obtaining the continuous spectrum is presented. In Section IV, we specialize the generalized Euler equation to the circular tokamak. For the $\omega^2 \neq 0$ case, we first explicitly obtain the Alfven and slow continua, which are then discussed. Second, in the infinite aspect ratio limit it is shown that an infinite set of Hain-Lüst equations are obtained, one for each m value. Then still considering the circular tokamak case and using the energy functional for the marginal stability case, $\omega^2 = 0$, we

specialize further and consider only three modes m, m+1 and m-1. The resulting potential energy functional reduces to that of Zakharov, or correspondingly, when the gradient of the longitudinal current equals zero, to that of Ware and Haas. Section V contains a development to obtain ß-poloidal limitations on stability. The approach here is to analytically eliminate the m+1 and m-1 mode conponents resulting in a potential energy functional and Euler equation which are a function of the m mode only. This approach is more general than the previous approach[1] in that it allows for arbitrary current profiles and explicitly includes j_ϕ in the solution. The resulting limitations on β_p are less restrictive than those obtained previously and are discussed in Sec. VI where a summary of the principal results are given.

II BASIC EQUATIONS

A. Variational Theory

The variation of the Lagrangian is written as

$$\delta L = \delta K - \delta W = 0 ,$$

(1)

where δW = variation in potential energy,

$\delta K = \omega^2 \delta I$ = variation in kinetic energy,

since $\delta \omega^2 = 0$, and we are assuming an $e^{-i\omega t}$ time dependence.
Thus, we have

$$\omega^2 = \frac{\delta W}{\delta I} = \frac{-\pi \iint \sum_{jk} X_j^* F_{jk} X_k \, J \, d\psi \, dx}{\pi \iint \rho \sum_j X_j^* \alpha_j X_j \, J \, d\psi \, dx} .$$

(2)

Here, an orthogonal ψ, χ, ϕ coordinate system
is used, where ϕ is the toroidal angle. Since we are assuming
axisymmetric toroids, ϕ is an ignorable coordinate.
The X_j's are components of the perturbed displacement. The
components are obtained by projecting the ψ, χ, ϕ components
of the perturbed displacement in directions normal to the magnetic
surface, X_1, tangential to the magnetic surface and perpendicular
to field lines, X_2, and parallel to the fields lines, X_3.
In a similar projection, the components of the force operator F_{ij} are

obtained. The F_{ij}'s and α_j's are taken directly from Goedbloed[4] and are given in detail in Eqs.(10) and (12). Here J is the Jacobian and ρ is the density.

The components of the perturbed displacement are now Fourier expanded in the form

$$X_k(\psi, \chi, \phi) = \sum_{m=-\infty}^{m=\infty} X_{km}(\psi) \, e^{im\chi + in\phi} \,, \tag{3}$$

where, since ϕ is an ignorable coordinate, n is fixed. Then, the potential energy and virial variations can be expressed as

$$\delta W = -\pi \int d\psi \sum_{jm} \sum_{km'} X_{jm}^* \langle F_{jm,km'} \rangle X_{km'} \,, \tag{4a}$$

$$\delta I = \pi \int d\psi \sum_{jm} \sum_{m'} X_{jm}^* \langle \rho X_{jm,m'} \rangle X_{jm'} \,, \tag{4b}$$

where $m \neq m'$ in general, and where

$$\langle F_{jm,km'} \rangle = \int_0^{2\pi} \frac{d\chi}{2\pi} \, J(\psi,\chi) \, e^{-im\chi} \, F_{jk}(\psi,\chi) \, e^{im'\chi} \,. \tag{5}$$

In obtaining the $\langle F \rangle$'s, we have integrated over all the χ dependence. It is easy to show using the force operator elements, F_{jk}'s, that the $\langle F \rangle$'s are Hermitian, or

$$\langle F_{jm,km'} \rangle^{\dagger} = \langle F_{km',jm} \rangle^{*} . \tag{6}$$

Thus, since δL is variational with respect to the $X's$, it follows that

$$\frac{\delta L_o}{\delta X^*} = \omega^2 \frac{\delta I_o}{\delta X^*} - \frac{\delta W_o}{\delta X^*} = 0 \tag{7}$$

with $\delta \omega_o^2 = 0$, which gives

$$\sum_{km'} \langle F_{jm,km'}(\psi) \rangle X_{km}(\psi) = - \omega^2 \sum_{m'} \langle \rho \alpha_j(\psi) \rangle X_{jm'}(\psi). \tag{8}$$

Equation (8) is the principal equation used in the derivation and it is instructive to compare it with the orignial form,

$$\sum_{k} F_{jk}(\psi,\chi) X_k(\psi,\chi) = - \omega^2 \rho(\psi) \alpha_j(\psi,\chi) X_j(\psi,\chi). \tag{9}$$

Thus, the explicit X dependence has been eliminated by an expansion in m and an integral over X.

B. Force Operator

The force operator, taken from Goedbloed, and several other relations are repeated here, because several comments need to be made about them, and also our notation is slightly different.

The force operator is

$$F_{jk} =$$

$D(\Gamma_p + B^2)D - \dfrac{1}{J}F\dfrac{1}{R^2 B_x^2}F\dfrac{1}{J}$ $- \dfrac{2}{J^2}\left(\dfrac{\partial \tau}{\partial \psi} + q\dfrac{\partial \lambda}{\partial \psi}\right)$	$DG(\Gamma_p + B^2)$ $- \dfrac{2}{IJ}\left(\tau n + \lambda i\dfrac{\partial}{\partial x}\right)B^2$	$D\Gamma_p F$
$-(\Gamma_p + B^2)GD$ $- \dfrac{2B^2}{IJ}\left(n\tau + i\dfrac{\partial}{\partial x}\lambda\right)$	$-G\Gamma_p G - B^2 G\dfrac{1}{B^2}GB^2$ $-B^2 F\dfrac{B_x^2}{B_\phi^2 B^2}F B^2$	$-G\Gamma_p F$
$-F\Gamma_p D$	$-F\Gamma_p G$	$-F\Gamma_p F$

$$(10)$$

The first thing that is observed is the symmetric form of F_{jk}, where we note that the gradient operators D, F, G change sign upon taking the Hermitian conjugate. These operators are:

$$D = \frac{1}{J}\frac{\partial}{\partial \psi} , \tag{11a}$$

$$F = -\frac{i}{J}\frac{\partial}{\partial x} + \frac{nq}{J} \longrightarrow \vec{k}\cdot\vec{B} , \tag{11b}$$

$$G = -\frac{i}{J}\frac{\partial}{\partial x} - \frac{n B_x^2}{I} \longrightarrow \frac{B_x}{B_\phi}(\vec{k}\times\vec{B}) . \tag{11c}$$

The curvature terms λ and τ are defined in the Appendix.

For completeness, the α_j's are

$$\alpha_1 = \frac{1}{J^2 R^2 B_\chi^2}, \quad \alpha_2 = \frac{B_\chi^2 B^2}{B_\phi^2}, \quad \alpha_3 = B^2. \tag{12}$$

While the analysis here is generalized to axisymmetric plasmas in the ψ, χ, ϕ coordinate system, we later specialize to the circular tokamak case, where $\psi \to r$ and $\chi \to \theta$, so that we then use an r, θ, ϕ coordinate system. The terms to the right of the arrows are the forms we expect then. We also use the notation in the circular tokamak case of

$$F_m = \vec{k}_m \cdot \vec{B}, \tag{13}$$

where $k_m = \frac{m}{r}\hat{\theta} + \frac{n}{R}\hat{\phi}$.

Correspondingly, $F_{m+1} = \vec{k}_{m+1} \cdot \vec{B}$, where m in $\vec{k}_m$ is replaced by m+1 and so forth. Also, $F_{m+1} = 0$ at the m+1 mode rational surface defined by r_{m+1}. The circular coordinate notation is introduced early so that steps in the derivation process can be clarified by additionally discussing the circular tokamak case.

It is important to note that the D operators, which involve ψ derivatives, occur only in the first row and column of F_{jk}. This property is used in generating the differential equation solution.

Another important property of the F_{jk} matrix to be observed is the occurence of the toroidal curvature related terms λ_s in the F_{11}, F_{12} and F_{21} matrix elements only. The general expressions for the poloidal curvature K_p and toroidal curvature K_t terms are

$$\frac{B_x K_p}{R} = \frac{B_x}{J}(JB_x)' \equiv \tau , \tag{14a}$$

$$\frac{B_\phi K_t}{JB_x} = \frac{B_\phi}{J} R' \equiv \lambda . \tag{14b}$$

Specializing to the circular tokamak case for clarity, the component of the radius of curvature, R_c, normal to the magnetic surface can be written as

$$\frac{1}{R_c} = -\left[\frac{B_\theta^2}{B^2}\left(\frac{1}{r}\right) + \frac{B_\phi^2}{B^2}\left(\frac{\cos\theta}{R}\right)\right] . \tag{15}$$

For this case it is directly seen that

$$K_p = \frac{1}{r} , \quad \text{and} \quad K_t = \frac{\cos\theta}{R} .$$

Now, referring back to Eq (5), it is see that when m and m' differ by one, a $\cos^2\theta$ term results wherever a λ occurs. Thus, it is essential to consider mode coupling terms associated with the

F_{11}, F_{12}, F_{21} elements for the circular case.

In the more general case, however, the noncircular equilibriums will depend not only on $\cos \chi$ terms but also on $\cos 2\chi$, $\cos 3\chi$ and so forth. Then, mode coupling terms associated with all elements of F_{jk} need to be considered. Ic is necessary to emphasize here that the second order generalized Euler equation involving ψ of the form obtained in Eq (19) will still result. Now, however, in the noncircular case, instead of the dominant coupling of fundamental mode m to the two nearest associated modes m-1, m+1 that results in the circular case, direct coupling to all modes would be possible. It should be noted that the χ dependence also occurs in R and the B'_s and so forth, which will contribute toroidal terms and that all of these terms were included initially. Further, no restrictions have been made either to high or low ß at this point.

By using a different theoretical route and a simple mathematical treatment, which does not obscure the phyiscs, we are able to obtain some new results that reduce to known results in appropiate limits.

C. Set of Equations

Now going back to Eq.(8), for the purpose of clarity, we shall momentarily only consider the three modes m, m = 1 and m + 1. We would then have 9 differential equations and 9 unknown $X's$. Considering the second equation corresponding to 1m and the fifth equation corresponding to 2m we have:

$$
\left.
\begin{aligned}
&\langle F_{1m,1m}\rangle X_{1m-1} + \langle F_{1m,1m}\rangle X_{1m} + \langle F_{1m,1m+1}\rangle X_{1m+1} \\
&+ \langle F_{1m,2m-1}\rangle X_{2m-1} + \langle F_{1m,2m}\rangle X_{2m} + \langle F_{1m,2m+1}\rangle X_{2m+1} \\
&+ \langle F_{1m,3m-1}\rangle X_{3m-1} + \langle F_{1m,3m}\rangle X_{3m} + \langle F_{1m,3m+1}\rangle X_{3m+1}
\end{aligned}
\right\}
=
\left\{
\begin{aligned}
&-\omega^2\langle\rho\kappa_{1m,1m-1}\rangle X_{1m-1} \\
&-\omega^2\langle\rho\kappa_{1m,1m}\rangle X_{1m} \\
&-\omega^2\langle\rho\kappa_{1m,1m+1}\rangle X_{1m+1}
\end{aligned}
\right.
$$

$$(16)$$

$$
\left.
\begin{aligned}
&\langle F_{2m,1m}\rangle X_{1m} + \langle F_{2m,1m}\rangle X_{1m} + \langle F_{2m,1m+1}\rangle X_{1m+1} \\
&+ \langle F_{2m,2m-1}\rangle X_{2m-1} + \langle F_{2m,2m}\rangle X_{2m} + \langle F_{2m,2m+1}\rangle X_{2m+1} \\
&+ \langle F_{2m,3m-1}\rangle X_{3m-1} + \langle F_{2m,3m}\rangle X_{3m} + \langle F_{2m,3m+1}\rangle X_{3m+1}
\end{aligned}
\right\}
=
\left\{
\begin{aligned}
&-\omega^2\langle\rho\kappa_{2m,2m-1}\rangle X_{2m-1} \\
&-\omega^2\langle\rho\kappa_{2m,2m}\rangle X_{2m} \\
&-\omega^2\langle\rho\kappa_{2m,2m+1}\rangle X_{2m+1}
\end{aligned}
\right.
$$

The notation here follows that introduced in Eq.(8), e.g., for $\langle F_{jm'km'}\rangle = \langle F_{1m'1m-1}\rangle$ then j=1, k=1, m' = m-1, where we do not use additional commas in order to keep the notation compact.

The top equation in (16) is seen to be the Euler equation for the fundamental harmonic X_{1m}. This follows because only terms up to second order in curvature are considered, so that of the $\langle F_{1m,1m'} \rangle$ terms only the $\langle F_{1m,1m} \rangle$ terms enter in. Also, if we added terms to this equation, such as $\langle F_{jm,km-2} \rangle$, they would not contribute since m and m' differ by two. Thus, considering a circular tokamak the resulting Euler equation for X_{1m} will be complete. Those for X_{1m-1} and X_{1m+1} will not be complete, since terms such as $\langle F_{jm-1,km-2} \rangle$ and $\langle F_{jm+1,km+2} \rangle$ will enter in.

The last six differential equations, represented by the subset 2m-1 through 3m+1, are used to eliminate the six perturbed displacements X_{2m-1} through X_{3m+1} in the usual mathematical way. For example, the perturbed displacement X_{2m-1} can be expressed as the ratio of two determinants

$$X_{2m-1} = \frac{D_{x2m-1}}{D_{xm}} = \frac{D_{x2m-1}}{D_m D_{m+1} D_{m-1}} \, , \qquad (17)$$

where the D_{xm} determinant is an algebraic expression of the $X's$ and that can be written as a product of the three m modes separately. In particular, D_{xm} is the determinant of the matrix given next.

$$D_{xm} = \text{Determinant of} \qquad\qquad \text{Eq. (18)}$$

$\langle F_{2m-1,2m-1}\rangle$ $+w^2\langle\rho\alpha_{2m-1,2m-1}\rangle$			$\langle F_{2m-1,3m-1}\rangle$		
	$\langle F_{2m,2m}\rangle$ $+w^2\langle\rho\alpha_{2m,2m}\rangle$			$\langle F_{2m,3m}\rangle$	
		$\langle F_{2m+1,2m+1}\rangle$ $+w^2\langle\rho\alpha_{2m+1,2m+1}\rangle$			$\langle F_{2m+1,3m+1}\rangle$
$\langle F_{3m-1,2m-1}\rangle$			$\langle F_{3m-1,3m-1}\rangle$ $+w^2\langle\rho\alpha_{3m-1,3m-1}\rangle$		
	$\langle F_{3m,2m}\rangle$			$\langle F_{3m,3m}\rangle$ $+w^2\langle\rho\alpha_{3m,3m}\rangle$	
		$\langle F_{3m+1,2m+1}\rangle$			$\langle F_{3m+1,3m+1}\rangle$ $+w^2\langle\rho\alpha_{3m+1,3m+1}\rangle$

Correspondingly, the D_{x2m-1} determinant follows from the determinant of the matrix formed by replacing the first column of Eq. (18) by the appropriate X_{1m}, X_{1m-1}, X_{1m+1} factors from the last six differential equations.

The sparseness of the matrix of Eq. (18) is what makes the problem solution relatively easy. However, this sparseness is artifical in that only the lowest order mode coupling terms were considered,

since it was anticipated that the result would be specialized to circular geometry, Mathematically, it is possible to include all higher order mode coupling elements and eliminate variables so that only the m series components of the X_1 perturbation remains. The cancellation of terms that occurs, particularly common expressions in the ratios of determinants, would be expected to continue and the complete expressions obtained should be similar to those in the next Section but of expanded form.

III GENERALIZED EULER EQUATION

A. Coupled Euler Equations

From the previous section, it follows directly that if an infinite expansion in m had been used, the resulting second order Euler equation for the fundamental mode m would be unchanged, while those for the m+1 and m−1 modes would be changed. Now, the Euler equations for the different m' modes would be the same as that of the m mode, with m replaced by m'. When writting the generalized Euler equation, we want to display the ψ derivatives. In order to do this, our χ surfaced averaged quantities are redefined. For example,

$$
\langle F_{1m-1,\,1m-1} \rangle + \omega^2 \langle \rho\,\alpha_{m-1,\,m-1} \rangle \equiv \frac{\partial}{\partial\psi}\, a_{11}{}_{m-1,\,m-1} \frac{\partial}{\partial\psi} + b_{11}{}_{m-1,\,m-1} \,,
$$

where the a's associated with the ψ derivatives and the b's are the left-over terms.

Thus, we can write the generalized Euler equation as

$$\frac{\partial}{\partial\psi}\left[\underset{m}{a_{11}} - \frac{\underset{m}{a_{12}}\,\underset{m}{a_{21}}\,\underset{m}{b_{33}}}{D_m} + \frac{\underset{m}{a_{13}}\,\underset{m}{b_{32}}\,\underset{m}{a_{21}}}{D_m} + \frac{\underset{m}{a_{12}}\,\underset{m}{b_{23}}\,\underset{m}{a_{31}}}{D_m} - \frac{\underset{m}{a_{13}}\,\underset{m}{b_{22}}\,\underset{m}{a_{31}}}{D_m}\right]\frac{\partial X_{1m}}{\partial\psi}$$

$$+\left[\frac{-\underset{m}{a_{12}}\,\underset{m}{b_{33}}\,\underset{m}{b_{21}} + \underset{m}{b_{12}}\,\underset{m}{b_{23}}\,\underset{m}{a_{31}} + \underset{m}{a_{13}}\,\underset{m}{b_{32}}\,\underset{m}{b_{21}} - \underset{m}{b_{12}}\,\underset{m}{b_{33}}\,\underset{m}{a_{21}}}{D_m}\right]\frac{\partial X_{1m}}{\partial\psi}$$

$$+\left[\underset{m}{b_{11}} - \frac{\underset{m}{b_{12}}\,\underset{m}{b_{33}}\,\underset{m}{b_{21}}}{D_m} + \frac{\partial}{\partial\psi}\left(\frac{-\underset{m}{a_{12}}\,\underset{m}{b_{33}}\,\underset{m}{b_{21}} + \underset{m}{a_{13}}\,\underset{m}{b_{32}}\,\underset{m}{b_{21}}}{D_m}\right)\right]X_{1m} =$$

$$(19)$$

$$-\frac{\partial}{\partial\psi}\left[\left(\frac{-\underset{m}{a_{12}}\,\underset{m}{b_{33}}\,\underset{m,m-1}{b_{21}} + \underset{m}{a_{13}}\,\underset{m}{b_{32}}\,\underset{m,m-1}{b_{21}}}{D_m}\right)X_{1m-1}\right] - \frac{\partial}{\partial\psi}\left[\left(\frac{-\underset{m}{a_{12}}\,\underset{m}{b_{33}}\,\underset{m,m+1}{b_{21}} + \underset{m}{a_{13}}\,\underset{m}{b_{22}}\,\underset{m,m+1}{b_{21}}}{D_m}\right)X_{1m+1}\right]$$

$$-\left[\underset{m,m-1}{b_{11}} - \frac{\underset{m,m-1}{b_{12}}\,\underset{m-1}{b_{33}}\,\underset{m-1,m-1}{b_{21}}}{D_{m-1}} - \frac{\underset{m,m}{b_{12}}\,\underset{m}{b_{33}}\,\underset{m,m-1}{b_{21}}}{D_m}\right]X_{1m-1} - \left[\underset{m,m+1}{b_{11}} - \frac{\underset{m,m+1}{b_{12}}\,\underset{m+1}{b_{33}}\,\underset{m+1,m+1}{b_{21}}}{D_{m+1}} - \frac{\underset{m,m}{b_{12}}\,\underset{m}{b_{33}}\,\underset{m,m+1}{b_{21}}}{D_m}\right]X_{1m+1}$$

$$-\left[\frac{-\underset{m,m-1}{b_{12}}\,\underset{m-1}{b_{33}}\,\underset{m-1}{a_{21}} + \underset{m,m-1}{b_{12}}\,\underset{m-1}{b_{23}}\,\underset{m-1}{a_{31}}}{D_{m-1}}\right]\frac{\partial X_{1m-1}}{\partial\psi} - \left[\frac{-\underset{m,m+1}{b_{12}}\,\underset{m+1}{b_{33}}\,\underset{m+1}{a_{21}} + \underset{m,m+1}{b_{12}}\,\underset{m+1}{b_{23}}\,\underset{m+1}{a_{31}}}{D_{m+1}}\right]\frac{\partial X_{1m+1}}{\partial\psi}$$

$$-\left[-\frac{\underset{m,m-1}{b_{12}}\,\underset{m-1}{b_{33}}\,\underset{m-1,m}{b_{21}}}{D_{m-1}} - \frac{\underset{m,m+1}{b_{12}}\,\underset{m+1}{b_{33}}\,\underset{m+1,m}{b_{21}}}{P_{m+1}}\right]X_{1m}$$

The symmetrical nature of the generalized Euler equation, Eq.(19), is immediately apparent. For each m there is an equation of this form. For example, the m+1 equation is obtained by replacing m by m+1 throughout. The boundary conditions for each equation are

$$X_{im}(\psi_0) = 0, \qquad X_{im}(\psi_a) = 0 . \tag{20}$$

Here the subscript o refers to the magnetic axis and the subscript a to the conducting wall; in the circular tokamak a is the minor radius. When the m subscript are not doubled on the a's and b's it is understood that they are the same, e.g. $b_{32} = b_{32}$.
$$\quad\quad\quad\quad m \quad\quad m,m$$

The right-hand side of Eq.(19) contains the mode coupling terms. The last term is interesting in that we see the m mode in effect being coupled to itself.

Writting Eq.(19) in more compact form we have

$$\frac{\partial}{\partial \psi}\left[f_m \frac{\partial X_m}{\partial \psi} \right] - \sum_{\ell=m-1}^{\ell=m+1} h_{m\ell} \frac{\partial X_\ell}{\partial \psi} - \sum_{\ell=m-1}^{\ell=m+1} g_{m\ell} X_\ell = 0 , \tag{21}$$

where

$$f_m = \underset{m}{a_{11}} - \frac{\underset{m}{a_{12}}\,\underset{m}{a_{21}}\,\underset{m}{b_{33}}}{D_m} + \frac{\underset{m}{a_{13}}\,\underset{m}{b_{32}}\,\underset{m}{a_{21}}}{D_m} + \frac{\underset{m}{a_{12}}\,\underset{m}{b_{23}}\,\underset{m}{a_{31}}}{D_m} - \frac{\underset{m}{a_{13}}\,\underset{m}{b_{22}}\,\underset{m}{a_{31}}}{D_m},$$

$$h_{m\ell} = -\frac{\underset{m,m}{a_{13}}\,\underset{m,m}{b_{32}}\,\underset{m,\ell}{b_{21}} - \underset{m,m}{a_{12}}\,\underset{m,m}{b_{33}}\,\underset{m,\ell}{b_{21}}}{D_m} - \frac{\underset{m,\ell}{b_{12}}\,\underset{\ell,\ell}{b_{23}}\,\underset{\ell,\ell}{a_{31}} - \underset{m,\ell}{b_{12}}\,\underset{\ell,\ell}{b_{33}}\,\underset{\ell,\ell}{a_{21}}}{D_\ell},$$

$$g_{m\ell} = -\underset{m,\ell}{b_{11}} + \sum_{\substack{j=m-1 \\ j=\ell\pm1}}^{j=m+1} \frac{\underset{m,j}{b_{12}}\,\underset{j,j}{b_{33}}\,\underset{j,\ell}{b_{21}}}{D_j} - \frac{\partial}{\partial\psi}\left(\frac{\underset{m,m}{a_{13}}\,\underset{m,m}{b_{32}}\,\underset{m,\ell}{b_{21}} - \underset{m,m}{a_{12}}\,\underset{m,m}{b_{33}}\,\underset{m,\ell}{b_{21}}}{D_m} \right).$$

The description of Eq. (21) as a generalized Euler equation now becomes clear, since this equation repeats itself for each m value.

The continuous spectrum are associated with the first term of Eq. (21), or more precisely associated with f_m. In examining Eq. (17) we observe that it is mathematically improper to divide by zero in eliminating variables in the process of going to a higher order differential equation. Thus, in general for any m, setting

D_m equal to zero will yield only apparent continua. The form of D_{xm}, and correspondingly the associated D products and apparent continua, depends on which variables are eliminated. The true continua can be calculated from f_m by eliminating the apparent continua associated with D_m. This is shown explicitly in the circular toroidal case in Eqs. (25), (26) and (28).

B. Energy Functional

The Euler equation, Eq. (21), is the m_{th} Euler equation corresponding the functional

$$
J = \int_{\psi_0}^{\psi_a} d\psi \sum_{m=-\infty}^{m=+\infty} \left\{ f_m(\omega^2) X_m'^2 + \sum_{\ell=m-1}^{\ell=m+1} h_{m\ell}(\omega^2) X_m X_\ell' + \sum_{\ell=m-1}^{\ell=m+1} g_{m\ell}(\omega^2) X_m X_\ell \right\}.
$$

(22)

In obtaining this functional we have used the relations

$$
h_{\ell m} = - h_{m\ell} , \quad \text{so that} \quad h_{mm} = 0 ,
$$

and

$$
h_{\ell m}' = g_{\ell m} - g_{m\ell} .
$$

(23)

These properties follow from the fact that the $\langle F \rangle$'s are

Hermitian, Eq.(6), and then examining properties of the

various force operators, Eq.(10), where it should be noted that

the differential operators change sign upon taking the Hermetian

conjugate so that in particular $a_{31} = -a_{13}^*$ and $a_{21} = -a_{12}^*$.

Since the a's and b's are real the above relations for h and g

follow. The fact that $h_{mm} = 0$ means that the second bracket on

the left hand side of Eq.(19) identically goes to zero. This can

also be seen directly noting the above properties for the a's and

that $b_{jk} = b_{kj}^* = b_{kj}$.

Summarizing, the minimization of J gives an infinite number of

Euler equations, where the m_{th} Euler equation is given by Eq.(21)

and the m+1 Euler equation is the same except that m is replaced

by m+1. There are two types of toroidal terms. Those that result

from mode coupling when m ≠ m' in the $\langle F \rangle$'s, and those

geometrical curvature terms that result when m = m'.

These latter terms are referred to as noncoupling curvature

terms throughout the paper. In the infinite aspect ratio limit,

both the mode coupling terms and noncoupling curvature terms go

to zero, and only reduced forms of $f_m(\omega^2)$ and $g_{mm}(\omega^2)$ remain.

The equation for J reduces to $J(r)$ for a diffuse linear pinch

given by Goedbloed and Sakanaka[5], except that here there is an infinite

sum of m. Since the Euler equations are now all uncoupled, this

is an understandable difference.

IV   CIRCULAR TOKAMAK

A. $\omega^2 \neq 0$.

Using the generalized Euler equation, Eq. (19), we now specialize to circular geometry, and diagrammatically show the two coordinate systems: the original orthogonal coordinate system ψ, χ, ϕ in Fig. 1a and the circular coordinate system r, θ, ϕ in Fig. 1b.

The metric is given in the Appendix. In lowest order ψ goes over to r, χ goes over to θ, and ϕ of course is unchanged. One difference here, from the usual circular toroid coordinate system, is that γ instead of δ is used to express the displacement of the centers of the magnetic flux surfaces (see Fig. 1b). An expression for γ', derived in the Appendix as Eq. (A10), is

$$\gamma' = -\frac{1}{r B_\theta^2} \int_0^r B_\theta^2 r \left(1 - \frac{2 p' r}{B_\theta^2} \right) dr = -\delta', \qquad (24)$$

where all primes now denote derivatives with respect to r. This expression for δ' was obtained by Shafranov[6] in 1964. That Eq. (24) is reasonable follows directly, since $\gamma + \delta$ = constant for a given equilibrium.

To reemphasize a statement made previously; in the noncircular case, R in Fig. 1b would be a sum over cos m'θ, and then there would be direct coupling of the m mode to all other m' modes. Although the form of Eq. (19) would be unchanged, there would then be an extended array of coupling terms on the right hand side of the equation.
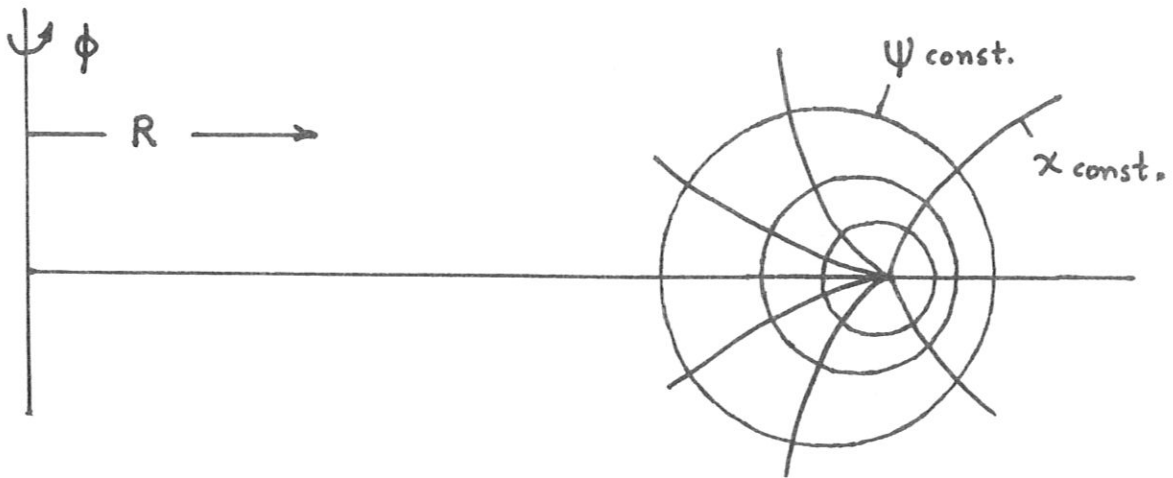
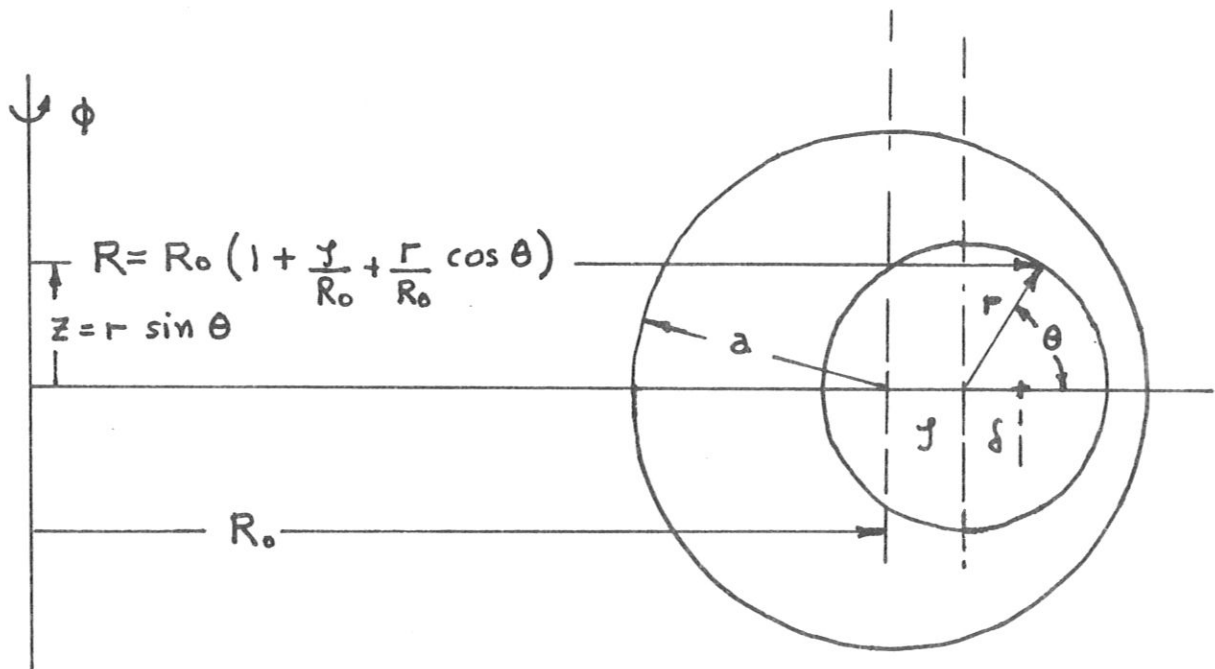Fig. 1a. Orthogonal coordinate system.

Fig. 1b. Circular coordinate system.

Choose $\theta(r,\chi)$ to make χ = constant surfaces

orthogonal to r = constant surfaces.

However, restricting ourselves to the circular toroidal case, we obtain using Eq.(11).

$$\frac{d}{dr}\left[\frac{(\Gamma p + B^2)}{r}\frac{(\omega^2 - \omega_{1m}^2)(\omega^2 - \omega_{2m}^2)}{(\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)}\frac{d}{dr}(r\xi_m)\right] + \frac{(\rho\omega^2 - F_m^2)}{r}\, r\xi_m$$

$$+ \left[-\frac{2B_\theta}{r}\frac{d}{dr}\left(\frac{B_\theta}{r}\right) - \frac{2B_\phi^2}{rR^2}\right] r\xi_m + \frac{4B_\theta^2}{\rho r^3 R}\frac{(n+mq\sigma)^2\left[\omega_{2m}(\Gamma p + B^2) - \omega^2 B^2\right]}{(\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)} r\xi_m$$

$$+ \frac{B_\theta^2}{\rho r^3}\left\{\frac{q^2(m+1)^2}{r^2}\frac{\left[\omega_{2m+1}^2(\Gamma p + B^2) - \omega^2 B^2\right]}{(\omega^2 - \omega_{3m+1}^2)(\omega^2 - \omega_{4m+1}^2)} + \frac{q^2(m-1)^2}{r^2}\frac{\left[\omega_{2m-1}^2(\Gamma p + B^2) - \omega^2 B^2\right]}{(\omega^2 - \omega_{3m-1}^2)(\omega^2 - \omega_{4m-1}^2)}\right\} r\xi_m$$

$$- \frac{1}{R}\left\{\frac{d}{dr}\left[\frac{2B_\theta}{\rho r^2}(n+mq\sigma)(\vec{k}_m \times \vec{B})\frac{(\Gamma p + B^2)(\omega_{2m}^2 - \omega^2)}{(\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)}\right]\right\} r\xi_m =$$

$$\tag{25}$$

$$- \frac{1}{R}\frac{d}{dr}\left\{\left[\frac{B_\phi m(\vec{k}_m \times \vec{B})(\Gamma p + B^2)(\omega_{2m}^2 - \omega^2)}{\rho r^2(\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)}\right](r\xi_{m-1} + r\xi_{m+1})\right\}$$

$$- \left\{\frac{B_\phi^2}{Rr^2} - \frac{2B_\theta B_\phi m\left[\omega_{2m}^2(\Gamma p + B^2) - \omega^2 B^2\right](n+mq\sigma)}{R^2\rho r^3(\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)}\right\}(r\xi_{m-1} + r\xi_{m+1})$$

$$+ \left\{\frac{2B_\theta B_\phi(m-1)\left[\omega_{2m-1}^2(\Gamma p + B^2) - \omega^2 B^2\right]\left[n + (m-1)q\sigma\right]}{R^2\rho r^3(\omega^2 - \omega_{3m-1}^2)(\omega^2 - \omega_{4m-1}^2)}\right\} r\xi_{m-1}$$

$$+ \left\{\frac{2B_\theta B_\phi(m+1)\left[\omega_{2m+1}^2(\Gamma p + B^2) - \omega^2 B^2\right]\left[n + (m+1)q\sigma\right]}{R^2\rho r^3(\omega^2 - \omega_{3m+1}^2)(\omega^2 - \omega_{4m+1}^2)}\right\} r\xi_{m+1}$$

$$+ \left[\frac{B_\phi(m-1)(\vec{k}_{m-1} \times \vec{B})(\Gamma p + B^2)(\omega_{2m-1}^2 - \omega^2)}{R\rho r^2(\omega^2 - \omega_{3m-1}^2)(\omega^2 - \omega_{4m-1}^2)}\right]\frac{d}{dr}(r\xi_{m-1})$$

$$+ \left[\frac{B_\phi(m+1)(\vec{k}_{m+1} \times \vec{B})(\Gamma p + B^2)(\omega_{2m+1}^2 - \omega^2)}{R\rho r^2(\omega^2 - \omega_{3m+1}^2)(\omega^2 - \omega_{4m+1}^2)}\right]\frac{d}{dr}(r\xi_{m+1})$$

Here $\quad \omega_{1m}^2 = \dfrac{F_m^2}{\rho}$, $\qquad \omega_{2m}^2 = \dfrac{\Gamma p}{\Gamma p + B^2}\, \dfrac{F_m^2}{\rho}$, $\qquad$ (26)

$$\omega_{\substack{3m \\ 4m}}^2 = \frac{1}{2\rho}\left(\frac{m^2}{r^2}+\frac{n^2}{R^2}\right)(\Gamma p + B^2)\left[\,1 \mp \left(1 - \frac{4\Gamma p\, F_m^2}{\left(\frac{m^2}{r^2}+\frac{n^2}{R^2}\right)(\Gamma p + B^2)}\right)^{\!\!1/2}\,\right],$$

and

$$F_m = \vec{k}_m \cdot \vec{B} \;, \qquad \vec{k}_m = \frac{m}{r}\hat{\theta} + \frac{n}{R}\hat{\phi} \;, \qquad \vec{B} = B_\theta\,\hat{\theta} + B_\phi\,\hat{\phi} \;,$$

$$\sigma \equiv 2 + \frac{R}{r}\int_0^r p' \frac{dr}{r} \;. \tag{27}$$

In the circular case X_{1m} reduces to $r\,\xi_m$, where ξ_m is the perturbed radial displacement. Here, $q = \dfrac{r\,B_\phi}{R\,B_\theta}$ is the safety factor and the other terms have been defined previously.

It is now easy to explicitly obtain the continuous spectrum. The most direct way of showing that ω_{3m}^2 and ω_{4m}^2 are apparent continua, and thus by elimination that ω_{1m}^2 and ω_{2m}^2 are true continua, all given by Eq. (26), is by examining the D_{xm} determinant of Eq. (18) in the circular toroidal case. We obtain

$$D_{xm} = \rho^6 \frac{r^6 B'^2}{B_\phi^6}\left[\,(\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)(\omega^2 - \omega_{3m+1}^2)(\omega^2 - \omega_{4m+1}^2)\right.$$
$$\left. (\omega^2 - \omega_{3m-1}^2)(\omega^2 - \omega_{4m-1}^2)\,\right]. \tag{28}$$

From the argument advanced previously, we know that setting D_{xm} equal to zero results in apparent continua, and in particular ω_{3m}^2, ω_{4m}^2, ω_{3m+1}^2, ω_{4m+1}^2, ω_{3m-1}^2 and ω_{4m-1}^2 are all apparent continua. In the more general infinite expansion over m, where m can take on any integer value, ω_{3m}^2 and ω_{4m}^2 are apparent continua, while ω_{1m}^2 and ω_{2m}^2 are true continua. There is also a cluster point at $\omega^2 = \infty$. Thus, we have two infinite sets of continua resulting from an expansion in m: ω_{1m}^2, an Alfvén continua; and ω_{2m}^2, a slow continua; and the cluster point at infinity.

In the large aspect ratio limit ($n \to \infty$, $q \to 0$, nq fixed) all mode coupling terms and other toroidal correction terms disappear and Eq.(25) becomes

$$\frac{d}{dr}\left[f_m \frac{d}{dr}(r\,\xi_m)\right] - g_m\, r\, \xi_m = 0 \quad , \tag{29}$$

$$f_m = \frac{\Gamma p + B^2}{r}\,\frac{(\omega^2 - \omega_{1m}^2)(\omega^2 - \omega_{2m}^2)}{(\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)} \quad ,$$

$$g_m = \frac{F_m^2 - \rho\omega^2}{r} + \frac{2B_\theta}{r}\frac{\partial}{\partial r}\left(\frac{B_\theta}{r}\right)$$

$$- \frac{B_\theta^2}{\rho r^3}\frac{4n^2}{R^2}\frac{\left[\omega_{2m}^2(\Gamma p + B^2) - \omega^2 B^2\right]}{(\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)}$$

$$+ \frac{d}{dr}\left[\frac{2B_\theta(\Gamma p + B^2)(\vec{k}_m \times \vec{B})(\omega_{2m}^2 - \omega^2)}{\rho r^2 (\omega^2 - \omega_{3m}^2)(\omega^2 - \omega_{4m}^2)}\frac{n}{R}\right] \quad .$$

This is the ideal MHD equation of motion for a diffuse linear pinch and was first derived by Hain and Lüst[3]. Thus, in the large aspect ratio limit, the infinite set of coupled differential equations for each m represented by Eq.(25) uncouple, and reduce to an infinite set of uncoupled Hain-Lüst equations for each m represented by Eq.(29).

B. $\omega^2 = 0$

In this Section it is shown that the energy functional, J of Eq.(27), reduces to the potential energy functionals of Zakharov[1] and that of Ware and Haas[2], if the limits employed by these authors are used. These appropriate limits are a circular tokamak with $\omega^2 = 0$ (marginal stability case) and considering the m-1, m, and m+1 modes only. Employing these limits we obtain

$$
J_0 = \int_0^a \frac{dr}{R} \left\{ \frac{r F_m^2}{(m^2 + k^2 r^2)} \left[\frac{d}{dr}(r \xi_m) \right]^2 + \frac{r F_{m+1}^2}{[(m+1)^2 + k^2 r^2]} \left[\frac{d(r \xi_{m+1})}{dr} \right]^2 + \frac{r F_{m-1}^2}{[(m-1)^2 + k^2 r^2]} \left[\frac{d}{dr}(r \xi_{m-1}) \right]^2 \right.
$$

$$
+ R^2 B_\theta \left[g_0_{mm} r^2 \xi_m^2 + g_0_{m+1,m+1} r^2 \xi_{m+1}^2 + g_0_{m-1,m-1} r^2 \xi_{m-1}^2 + 2 g_0_{m,m-1} r^2 \xi_m \xi_{m-1} + 2 g_0_{m,m+1} r^2 \xi_m \xi_{m+1} \right]
$$

$$
\left. + R \left[2 h_0_{m,m-1} r \xi_m (r \xi_{m-1})' + 2 h_0_{m,m+1} r \xi_m (r \xi_{m+1})' \right] \right\} ,
$$

(30)

where integration by parts has been employed. The subscript - o denotes that $\omega^2 = 0$.

At this point, approximations are used with respect to low β tokamak ordering, $kr \ll 1$, etc., and the approximations employed are principally valid in the vicinity of the m-mode rational surface. The approximations are also valid for $m \geq 2$, where m is the fundamental mode. Thus, we obtain

$$
\begin{aligned}
R J_0 = \int_0^a dr \Bigg\{ & \frac{r^3 F_m^2 \xi_m'^2}{m^2} + \left[r F_m^2 \left(\frac{m^2-1}{m^2} \right) + \frac{4 B_\theta^2 \beta r}{R^2} \left(2\beta + 1 - \frac{n^2}{m^2} \right) \right] \xi_m^2 \\
& + r \left[\frac{r F_{m+1} \xi_{m+1}}{m+1} \right]'^2 + r \left[\frac{r F_{m-1} \xi_{m-1}}{m-1} \right]'^2 \\
& + \left[1 + \frac{r j_\phi'}{F_{m+1}(m+1)} \right] r F_{m+1}^2 \xi_{m+1}^2 + \left[1 + \frac{r j_\phi'}{F_{m-1}(m-1)} \right] r F_{m-1}^2 \xi_{m-1}^2 \\
& + \frac{4 \beta r B_\theta \xi_m}{R} \left[\frac{r F_{m+1} \xi_{m+1}}{m+1} \right]' + \frac{4 \beta B_\theta \xi_m r F_{m+1} \xi_{m+1}}{R} \\
& + \frac{4 \beta r B_\theta \xi_m}{R} \left[\frac{r F_{m-1} \xi_{m-1}}{m-1} \right]' - \frac{4 \beta B_\theta \xi_m r F_{m-1} \xi_{m-1}}{R} \Bigg\} .
\end{aligned}
\tag{31}
$$

Comparing with Zakharov, $R J_0$ here identically equals his $\frac{\delta W}{(2\pi R)^2}$, where δW is given by his Eq. (4), with the normalization factors being somewhat different in the two cases. The β here is local beta poloidal, or $\beta = -\frac{rp'}{2 B_\theta^2}$. The only difference in the potential energy functional here and that of Ware and Haas (given here in Eq. (34)) is that they did not include the gradient of the longitudinal current, j_ϕ'.

The argument advanced by both previous works for the neglect
of higher associated modes is that the m ± 2 modes only
contribute square terms to the potential energy functional
and thus its sign is unchanged. An additional argument is
that in circular tokamaks having large aspect ratios, the
dominant contribution to the m mode coupling should come
from the m−1 and m+1 modes, and the m+2 and m−2 contributions
would be lower order in inverse aspect ratio expansion.

Minimizing J_0 , we obtain the three Euler equations

$$\frac{d}{dr}\left[r\,\frac{d}{dr}(\Psi_{m-1})\right] - \left[(m-1)^2 + \frac{r(m-1)}{F_{m-1}}\,\frac{dj_\phi}{dr}\right]\frac{\Psi_{m-1}}{r}$$

$$= -\frac{d}{dr}\left[\frac{2\beta r B_\theta \xi_m}{R}\right] - \frac{2\beta B_\theta \xi_m (m-1)}{R}\quad , \tag{32}$$

$$\frac{d}{dr}\left[\frac{r^3 F_m^2}{m^2}\,\frac{d\xi_m}{dr}\right] - r F_m^2\left(\frac{m^2-1}{m^2}\right)\xi_m - \frac{4r B_\theta^2 \beta}{R^2}\left(2\beta+1-\frac{n^2}{m^2}\right)\xi_m$$

$$= \frac{2\beta r B_\theta}{R}\,\frac{d\Psi_{m-1}}{dr} + \frac{2\beta r B_\theta}{R}\,\frac{d\Psi_{m+1}}{dr} \tag{33}$$

$$- \frac{2\beta B_\theta (m-1)}{R}\,\Psi_{m-1} + \frac{2\beta B_\theta (m+1)}{R}\,\Psi_{m+1}\quad ,$$

$$\frac{d}{dr}\left[\frac{r\,d}{dr}(\psi_{m+1})\right] - \left[(m+1)^2 + \frac{r(m+1)}{F_{m+1}}\frac{dj_\phi}{dr}\right]\frac{\psi_{m+1}}{r}$$

$$= -\frac{d}{dr}\left[\frac{2\beta r B_\theta \xi_m}{R}\right] + \frac{2\beta B_\theta \xi_m (m+1)}{R} \quad, \tag{34}$$

where we employ the functions

$$\psi_{m+1} = \frac{r\,F_{m+1}\,\xi_{m+1}}{m+1} \quad,$$

$$\psi_{m-1} = \frac{r\,F_{m-1}\,\xi_{m-1}}{m-1} \quad.$$

These Euler equations agree with those of Zakharov[1], or, again when $j_\phi' = 0$, with those of Ware and Haas[2].

It is revealing to examine the ψ_{m+1} and ψ_{m-1} Euler equations with their right hand sides set equal to zero. These equations then become those appropriate for the study of the m+1 and m-1 kink modes in cylindrical geometry. The m mode form of this kink instability equation was derived by Rutherford, Furth and Rosenbluth[7] and more recently used by Glasser et al[8] in studying resistive kink modes in tokamaks. This latter study suggested that the toroidal stabilization

indicated in another study[9] would result in even greater stabilization of resistive kink modes in tokamaks. However, since this second study[9] did not include the effects of poloidal mode coupling, as for example represented by the incomplete right hand terms in Eq.(32) and (34), the suggested greater stabilization may be optimistic.

An important point, that is more apparent when the marginal stability Euler equations are obtained directly from the generalized Euler equation, Eq.(25), is the cancellation of the lowest order toroidal terms when $\omega^2 = 0$. The lowest order mode coupling terms originally appeared as the X_{1m} term on the right hand side of Eq.(19) and then reappear as the m+1 and m-1 terms on the left hand side of Eq.(25). These mode coupling terms cancel in lowest order the $-\dfrac{2B_\phi^2}{rR^2}$ term in Eq.(25) which resulted originally from the geometrical curvature terms in $b_{\parallel_m}$ of Eq.(19). This cancellation ensures that at the magnetic axis that $r\,\xi_m$ goes as r^m in general. The noncancellation of toroidal terms in lowest order when $\omega^2 \neq 0$ has implications for the disruptive instability as discussed in the conclusions.

V BETA LIMITATIONS ON STABILITY

A Stability Criteria

Since the approach used here is parallel to but significantly different from that of Zakharov[1], it is necessary to discuss the model here in some depth. First we briefly discuss the analysis of Ware and Haas[2].

Setting j_ϕ' equal to zero, writting the associated mode terms in squared form, and using $B_{rm} = i \xi_m F_m$ in general which applies in ideal MHD, where B_r is the perturbed radial magnetic field, Eq. (31) for J_o becomes

(35)

$$
R J_o = \int_0^a r\, dr \left\{ \frac{r^2 F_m^2}{m^2} (\xi_m')^2 + \left[F_m^2 \left(\frac{m^2-1}{m^2} \right) + \frac{4 B_\theta^2 \beta}{R^2} \left(-2\beta + 1 - \frac{1}{q^2} \right) \right] \xi_m^2 \right.
$$

$$
+ \left[\frac{i\, (r B_{r\,m+1})'}{m+1} - \frac{2\beta B_\theta \xi_m}{R} \right]^2 + \left[\frac{i\, (r B_{r\,m-1})'}{m-1} - \frac{2\beta B_\theta \xi_m}{R} \right]^2
$$

$$
\left. + \left[i B_{r\,m+1} - \frac{2\beta B_\theta \xi_m}{R} \right]^2 + \left[i B_{r\,m-1} + \frac{2\beta B_\theta \xi_m}{R} \right]^2 \right\} \quad .
$$

This potential energy functional was obtained by Ware and Haas. They reasoned that the only nonsquared terms constituted a sufficient criterion for stability. That is,

$$\beta \left(-2\beta + 1 - \frac{1}{q^2} \right) > 0 \qquad (36)$$

is sufficient for ideal MHD stability of a circular tokamak with large aspect ratio. Zakharov[10] analytically eliminated the associated harmonics in Eq. (35) and showed that the resultant J_o is positive for $q^2 > 1$ and thus that the sufficient criterion of Eq. (36) is unjustifiably stringent.

Ware and Haas also obtained a necessary criterion for stability

$$\frac{1}{4} \left(\frac{r q'}{q} \right)^2 + \frac{2 r p'}{B_\phi^2} (1 - q^2) > 0 \; . \qquad (37)$$

This is a specialized form of Merciers[11] criterion for circular tokamaks. Ware and Haas obtained this criterion from Eq. (35) by noting that in the case of high shear the terms $B_{r_{m+1}}$ and $B_{r_{m-1}}$ are small near the m mode rational surface and then minimizing such that the other squared terms involving associated modes are zero. Correspondingly, they observed that the same result is obtained for the case of low shear and high m when the terms containing $(r B_{r_{m+1}})'$ and $(r B_{r_{m-1}})'$ are small and minimizing

such that the last two squared brackets in Eq.(35) are zero. The q'^2 term in Eq.(37) results from taking the minimum value of the $\xi_m'^2$ term in Eq.(31).

The question then arises as to what happens to stability when $\dot{j}_\phi'$ is also included with arbitrary shear and small m, and this is discussed next.

B. Stability Analysis

The Euler equations for ψ_{m+1} and ψ_{m-1} can be written in the form

$$\psi_\ell{}'' + \frac{\psi_\ell{}'}{r} - \left(\ell^2 + \frac{r j_\phi' \ell}{F_\ell}\right)\frac{\psi_\ell}{r^2} = G_\ell(r), \tag{38}$$

where ℓ takes on the values m+1 and m-1. If we express

$$\left(\frac{-j_\phi' \ell r}{F_\ell}\right) = \sum_{k=1}^{\infty} (a_k r^k)^2 , \tag{39}$$

that is as a polynomial series in r, then the solution of the left hand side of Eq.(38), or homogeneous equation, can be written in terms of the function

$$Z(r,s) = r^s \left[1 + \left\{\frac{-a_1^2}{[(s+2)^2 - \ell^2]}\right\}r^2 + \left\{\frac{a_1^4}{[(s+4)^2 - \ell^2][(s+2)^2 - \ell^2]} - \frac{a_2^2}{[(s+4)^2 - \ell^2]}\right\}r^4 \right. \tag{40}$$

$$\left. + \left\{\frac{-a_1^6 + a_2^2 a_1^2 [(s+2)^2 + (s+4)^2 - 2\ell^2]}{[(s+6)^2 - \ell^2][(s+4)^2 - \ell^2][(s+2)^2 - \ell^2]} - \frac{a_3^2}{[(s+6)^2 - \ell^2]}\right\}r^6 + \cdots \right]$$

which follows from

$$Z(r,s) = \frac{r^s}{C_0}\sum_{n=0}^{\infty} C_n r^n ,$$

and the recursion relation

$$\left[(s+j)^2 - \ell^2\right] C_j + a_1^2 C_{j-2} + a_2^2 C_{j-4} \cdots + a_{\frac{j}{2}}^2 C_0 = 0.$$

The indicial equation gives, $s^2 = \ell^2$. One independent solution Z_ℓ^1 follows directly from $Z(r,s)$ with $s = \ell$. However, for $s = -\ell$, it is clear from $Z(r,s)$ that all denominators beyond a certain point vanish. Recognizing this, the independent solutions can be written as

$$Z_\ell^1(r) = \frac{\left\{ Z(r,s) \right\}_{s=\ell}}{\Gamma(\ell+1)},$$

$$Z_\ell^2(r) = \left\{ \frac{\partial}{\partial s} \left[(s+\ell) Z(r,s) \right] \right\}_{s=-\ell} \Gamma(\ell).$$

(41)

The gamma functions, Γ's , are introduced so that the Wronksian of Z_ℓ^1 and Z_ℓ^2, $W(s)$, is independent of ℓ.

The polynomial series used in Eq. (39) is a convenient way to represent the current profiles, and the resultant Z_ℓ functions have the property that, for any $a_n \neq 0$ and all other $a_k = 0$ ($k \neq n$), they reduce to Bessel functions (J_ℓ, Y_ℓ, I_ℓ, K_ℓ, etc.). The case that all the a_k's are zero corresponds to a flat current profile or in the vicinity of the magnetic axis.

Although Taylor expansions of a general type of equation that embraces Bessel equation are not new[12], functions such as the Z_ℓ functions are not normally presented because they depend on the particular values of the a_k's chosen. With computers, however, such functions do not represent a problem. Not all a_k's will appear, since they depend on the current profile chosen, and further

a polynomial series with more than three k's would not be meaningful,

recognizing the approximate nature of the model to this point.

Once we know the two independent solutions of the homogeneous

equation, Z_ℓ^1 and Z_ℓ^2, we can write the general solution as

$$\Psi_\ell(r) = C_1 Z_\ell^1(r) + C_2 Z_\ell^2(r) + \int G_\ell(s) ds \left[\frac{Z_\ell^1(s) Z_\ell^2(r) - Z_\ell^2(s) Z_\ell^1(r)}{Z_\ell^{1'}(s) Z_\ell^{2'}(s) - Z_\ell^{2'}(s) Z_\ell^{1'}(s)} \right] ,$$

where the denominator of the particular soltuion is henceforth

written in terms of the Wronskian, $W(s)$. Although the Z_ℓ^1 and Z_ℓ^2

functions do not satisfy the Bessel function recursion relations

in general, it can be shown that for $a_k r^k \sim 1$, it is a good

approximation to use

$$\frac{Z_\ell^{1'}}{Z_\ell^1} \sim + \frac{\ell}{r} \quad , \qquad \frac{Z_\ell^{2'}}{Z_\ell^2} \sim - \frac{\ell}{r} \quad . \qquad (43)$$

However, for peaked current profiles the a_k's ~ 3, so that the

approximation is only fair at the conducting wall, but this is partially

compensated for by the fact that there Ψ_ℓ is small.

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Proceeding, using approximation (43) and integrating by parts, we obtain

$$\psi_{m+1}^{i}(r) = \lambda_{m+1}^{i}\left(\frac{Z_{m+1}^{1}(r)}{Z_{m+1}^{1}(r_m)} + \sigma_{m+1}^{i}\frac{Z_{m+1}^{2}(r)}{Z_{m+1}^{2}(r_m)}\right)$$

$$+ Z_{m+1}^{2}(r)\int_{0}^{r}\frac{4\beta B_\theta \varsigma_m(m+1)Z_{m+1}^{1}(s)\,ds}{R\,s\,W(s)} \quad,$$

$$\psi_{m-1}^{i}(r) = \lambda_{m-1}^{i}\left(\frac{Z_{m-1}^{2}(r)}{Z_{m-1}^{2}(r_m)} + \sigma_{m-1}^{i}\frac{Z_{m-1}^{1}(r)}{Z_{m-1}^{1}(r_m)}\right)$$

$$+ Z_{m-1}^{1}(r)\int_{0}^{r}\frac{4\beta B_\theta \varsigma_m(m-1)Z_{m-1}^{2}(s)\,ds}{R\,s\,W(s)} \quad,$$

for $0 < r < r_m$, and

$$\psi_{m+1}^{e}(r) = \lambda_{m+1}^{e}\left(\frac{Z_{m+1}^{1}(r)}{Z_{m+1}^{1}(r_m)} + \sigma_{m+1}^{e}\frac{Z_{m+1}^{2}(r)}{Z_{m+1}^{2}(r_m)}\right) \qquad (44)$$

$$- Z_{m+1}^{2}(r)\int_{r}^{a}\frac{4\beta B_\theta \varsigma_m(m+1)Z_{m+1}^{1}(s)\,ds}{R\,s\,W(s)} \quad,$$

$$\psi_{m-1}^{e}(r) = \lambda_{m-1}^{e}\left(\frac{Z_{m-1}^{2}(r)}{Z_{m-1}^{2}(r_m)} + \sigma_{m-1}^{e}\frac{Z_{m-1}^{1}(r)}{Z_{m-1}^{1}(r_m)}\right)$$

$$- Z_{m-1}^{1}(r)\int_{r}^{a}\frac{4\beta B_\theta \varsigma_m(m-1)Z_{m-1}^{2}(s)\,ds}{R\,s\,W(s)} \quad,$$

for $r_m < r < a$.

The inversion of the independent solutions Z_ℓ^1 and Z_ℓ^2 for m+1 and m-1 in Eq. (44) arises from the fact that $G_\ell(r)$ of Eq. (38) differs in sign depending on the form of ℓ (last terms of Eqs. (32) and (31)). Setting all the a_k's equal to zero, which corresponds to $j_\phi' = 0$, or also in the vicinity of the magnetic axis, the Ψ_ℓ expressions (44) become independent of k and reduce to those of the previous paper[1]. The advantages of the approach here are that arbitrary current profiles can be treated, and that j_ϕ' is explicity included in the homogeneous solution and thus in the particular solution.

Equating the respective values of Ψ_ℓ and their derivatives at r_m, we obtain

$$\lambda_{m+1}^i = \lambda_{m+1}^e = \frac{Z_{m+1}^2(r_m)}{\sigma_{m+1}^e - \sigma_{m+1}^i} \int_0^a \frac{4(m+1)\beta B_\theta \, S_m \, Z_{m+1}^1(s) \, ds}{R s W(s)} \quad,$$

$$\lambda_{m-1}^i = \lambda_{m-1}^e = \frac{Z_{m-1}^1(r_m)}{\sigma_{m-1}^e - \sigma_{m-1}^i} \int_0^a \frac{4(m-1)\beta B_\theta \, S_m \, Z_{m-1}^2(s) \, ds}{R s W(s)} \quad. \tag{45}$$

It should be noted that when a step current model is considered with the step taken at r_m, it is not valid to equate the derivatives of Ψ_ℓ at r_m, since there is a discontinuity in Ψ_ℓ' in this case. Further, the treatment of Glasser et al.[8] has shown that a step current at r_m is precisely the worst thing to do with respect to stability. The model here avoids both these difficulties.

Using the m-Euler equation, Eq. (33), and then substituting the solutions for Ψ_{m+1} and Ψ_{m-1}, Eq. (44), employing integration by parts, we obtain a transformed m-Euler equation and associated potential energy functional, now both a function of $\mathcal{S}_m$ only.

$$\frac{d}{dr}\left[\frac{r^3 F_m^2}{m^2}\frac{d\mathcal{S}_m}{dr}\right] - \left[r F_m^2\left(\frac{m^2-1}{m^2}\right) + \frac{4\beta B_\theta^2 r}{R^2}\left(1-\frac{n^2}{m^2}\right)\right]\mathcal{S}_m$$

(46)

$$= \frac{4\beta B_\theta \lambda_{m+1}(m+1)}{R}\frac{Z_{m+1}'(r)}{Z_{m+1}'(r_m)} - \frac{4\beta B_\theta \lambda_{m-1}(m-1)}{R}\frac{Z_{m-1}^2(r)}{Z_{m-1}^2(r_m)}$$

The potential energy functional is

$$
R J_0 = \int_0^a dr \left\{ \frac{r^2 F_m^2 \xi_m'^2}{m^2} + \left[r F_m^2 \left(\frac{m^2-1}{m^2} \right) + \frac{4 B_\theta^2 \beta r}{R^2} \left(1 - \frac{n^2}{m^2} \right) \right] \right.
$$

$$
\left. + \sum C_\ell \lambda_\ell^2 (\xi_m) \right\} ,
$$

$$
\ell = \begin{cases} m+1 \\ m-1 \end{cases}
$$

(47)

where

$$
C_{m+1} = \frac{\beta B_\theta r (\sigma_{m+1}^e - \sigma_{m+1}^i) Z_{m+1}^1 (r)}{Z_{m+1}^1 (r_m) Z_{m+1}^2 (r_m) \int_0^a \frac{\beta B_\theta Z_{m+1}^1 (s)}{s W(s)}} ,
$$

$$
C_{m-1} = \frac{-\beta B_\theta r (\sigma_{m-1}^e - \sigma_{m-1}^i) Z_{m-1}^2 (r)}{Z_{m-1}^1 (r_m) Z_{m-1}^2 (r_m) \int_0^a \frac{\beta B_\theta Z_{m-1}^2 (s) ds}{s W(s)}} .
$$

The potential energy functional appears to be structurally the same as in the previous work, however, it differs, as can be seen by taking the small argument limit of the Z_ℓ functions, which are identical to the small argument limit of Bessel functions.

The criterion of Mercier can be directly obtained using either Eq.(46) or Eq.(47). The use of Suydam's trial function for ξ_m on the inside of r_m and its odd reflection on the outside of r_m

results in λ_ℓ being zero and directly yields Eq.(37). This result, previously obtained by Zakharov[1] in a more restricted development, assumes that the λ_ℓ dependence, other than ξ_m, varies slowly (esp. does not change sign) near the mode rational surface. This assumption is generally not valid, except at the magnetic axis; to be generally valid, there must be extreme radial localization of the modes.

In order to be more definitive, the boundary conditions for ψ_ℓ (r) need to be considered. For simplicity, we consider cases where the boundary conditions,

$$\psi_{m+1}(0) = 0 , \qquad \psi_{m+1}(a) = 0 , \tag{48}$$
$$\psi_{m-1}(0) = 0 , \qquad \psi_{m-1}(a) = 0 ,$$

apply. That is, only the m=2 mode rational surface is assumed to lie inside the plasma. The boundary conditions then give

$$\sigma_{m+1}^i = -\frac{Z_{m+1}^1(r) Z_{m+1}^2(r_m)}{Z_{m+1}^1(r_m) Z_{m+1}^2(r)} , \qquad \sigma_{m+1}^e = -\frac{Z_{m+1}^1(a) Z_{m+1}^2(r_m)}{Z_{m+1}^1(r_m) Z_{m+1}^2(a)} ,$$

$$\tag{49}$$

$$\sigma_{m-1}^i = -\frac{Z_{m-1}^2(r) Z_{m-1}^1(r_m)}{Z_{m-1}^2(r_m) Z_{m-1}^1(r)} , \qquad \sigma_{m-1}^e = -\frac{Z_{m-1}^2(a) Z_{m-1}^1(r_m)}{Z_{m-1}^2(r_m) Z_{m-1}^1(a)} .$$

In the small argument limit, the sign of the σ_ℓ terms determines the sign of the C_ℓ factors, since the remaining terms contribute positive factors. In this limit, the σ_ℓ terms are independent of k, positive and stabilizing. Thus, since the sufficient criterion (36) was obtained for precisely this case, $j'_\phi = 0$, it follows that it is too stringent.

In the general case, however, since the $Z_\ell's$ can change sign depending on the magnitude of the argument, which depends on j_ϕ and r, it does not follow a priori that the σ_ℓ^e and σ_ℓ^i terms are stabilizing, and one must calculate specific current profiles. To illustrate this point, we consider the m=2 mode and a peaked model[13], with the current distribution given by $j_\phi = j_{\phi_0}/(1+r^2)^2$. With the m=2 surface only in the plasma and an aspect ratio, R/a, of 4, the volume integrated limiting value of β_p for stability is determined to be $1.8\,R/a$. As the m-1=1 surface moves inside the plasma, $\sigma_{m-1}^i \neq \infty$, and the limiting value of β_p decreases. For the q = 1 surface at 0.1a and the previous parameters, the corresponding limiting value of β_p is determined to be 1.4 R/a. These numbers are indicative only and do not allow for the current profile inside the q=1 surface being flattened, nor the direct q=1 contribution to the potential energy. However, as expected, the preliminary numerical analysis here does indicate that limiting β_p depends sensitively on current profile and location of mode rational surfaces.

VI CONCLUSIONS

Linearized ideal MHD equations in variational form are used to derive a generalized Euler equation and associated energy functional. The generalized Euler equation, which is an explicit function of magnetic surface coordinate ψ only, represents an infinite set of equations coupled together by poloidal mode coupling. The form of the energy functional derived here is convenient for analytical studies of ideal MHD stability in tokamaks.

In the infinite aspect ratio limit of a circular tokomak, the mode coupling terms and noncoupling curvature terms disappear in the generalized Euler equation and an infinite set of uncoupled Euler equations for each m results. The uncoupled Euler equation for m corresponds to the m_{th} Euler equation for the diffuse linear pinch, or Hain-Lüst equation.

For the case of a circular tokamak and marginal stability, $\omega^2 = 0$, and considering the m-1, m+1 and m modes only, the energy functional reduces to the form of previously derived potential energy functionals [1,2]. Although the potential energy functional is the same, the stability analysis here differs from the previous treatments. In particular, by expanding the term $\left(-\dfrac{j\phi' \ell}{F_\ell r}\right)$ in a polynomial series in r, it is possible to analytically eliminate the m-1 and m+1 perturbation components and obtain an Euler equation and

energy functional which now just depend on m. The advantage of the approach here over previous approaches is that arbitrary current profile j_ϕ can be treated and that j_ϕ' is explicitly included in the homogeneous solution and thus in the particular solution.

It is shown that the necessary criterion of Mercier, specialized to circular tokamaks, can be directly obtained using the transformed energy functional. This result, except near the magnetic axis, assumes extreme radial localization of the modes. In general, however, the nonlocal nature of the associated modes coupled with the fundamental mode results in limitations in β_p. For a peaked current profile with m=2 and R/a = 4, it is found that the volume integrated value of β_p is limited to less than 2R/a for stability. The fact that β_p limitation results are more in agreement with numerical computations[13] than previous analytical work, results from the use of a more realistic analytical model. The role of j_ϕ' becomes apparent and it is suggested that stability could be improved by optimized current profiles. In particular, stability should be improved by flattening the current profile near mode rational surfaces for m $\geqslant$ 2.

In order to obtain the stability results here, limited to circular tokamaks of moderate-to-large aspect ratio, a series of approximations had to be made, and it is generally more useful in design applications to use large computer codes[15], which numerically minimize the potential energy associated with perturbations of axisymmetric toroidal plasmas. However, the use of numerical codes generally

obscures the physics and often does not reveal direct ways to improve stability, so that it is useful to have both numerical and analytical results. Also, in limiting cases, such as ballooning modes with large toroidal mode number n, numerical codes cannot provide answers and then analytical results[16] are required to complete the analysis.

Since it is observed here that with finite growth rate there will be poloidal mode coupling in lowest order, as discussed at the end of Sec. IV, it follows that the linear growth rate will be strongly enhanced by mode coupling. This suggests that although many of the details of the nonlinear studies[17] of the disruptive instability in tokamaks are undoubtedly correct, the use of cylindrical geometry in which poloidal mode coupling effectively goes to zero is not valid. In cylindrical geometry the dominant effect in mode coupling is the closeness of the mode rational surfaces (assuming that either m or n of the coupled modes are the same), whereas, in toroidal geometry the strong poloidal mode coupling evidenced here should dominate over the nearness of the mode rational surfaces. This means that if toroidal geometry were used in the numerical computations, one would expect to see enhanced perturbations, growth rate and transport between the m=1, n=1 and m=2, n=1 surfaces relative to the cylindrical case. Thus, it is suggested that mode coupling in a tokamak between the m=1, n=1 and m=2, n=1 modes is more dominant in the disruptive instability than that between the m=3, n=2 and m=2, n=1 modes,

where the latter coupling is a manifestation of the numerical model used.

The analysis here indicates that poloidal mode coupling will strongly modify the Alfvén spectrum necessary for Alfvén wave heating in low aspect ratio tokamaks. However, although it is suggested that the continua can be calculated from f_m of Eq.(21), no results are given.

A simplified form of the model here has already been used to study the m=1, n=1 internal kink mode by Galvao et al.[17] A potential extension of the model here is the nonlinear analytical treatment of the effects of mode coupling in a circular tokamak. There are other extensions and applications of this model that could be studied.

The relatively simple model developed here demonstrates that poloidal mode coupling in tokamaks is a significant phenomenon and that analyses which regard mode coupling as a perturbation must be carefully questioned. The fact that known results are obtained in the infinite aspect ratio limit and also in the marginally stable circular tokamak case is a further confirmation of the use of variational theory, particularly in that a different approach is used here.

APPENDIX: CYLINDRICAL METRIC

In considering the special case of a circular tokomak we shall transform from the ψ , χ , ϕ coordinate system in Fig. 1a to that of r , θ , ϕ in Fig. 1b. We shall further choose a coordinate system based on $\mathcal{f}$ rather than the usual δ , where they are defined in Fig. 1b.

It is convenient to start with covariant bases vectors in the R , z , ϕ coordinate system,

$$\vec{e}_r = \frac{\partial R}{\partial r} \vec{e}_R + \frac{\partial z}{\partial r} \vec{e}_z + \frac{\partial \phi}{\partial r} \vec{e}_\phi \, ,$$

$$\vec{e}_r = \vec{e}_R \left(\mathcal{f}' + \cos\theta - r \frac{\partial \theta}{\partial r} \sin\theta \right) + \vec{e}_z \left(\sin\theta + r \frac{\partial \theta}{\partial r} \cos\theta \right) ,$$

$$(A1)$$

$$\vec{e}_\chi = \frac{\partial R}{\partial \chi} \vec{e}_R + \frac{\partial z}{\partial \chi} \vec{e}_z + \frac{\partial \phi}{\partial \chi} \vec{e}_\phi \, ,$$

$$\vec{e}_\chi = \vec{e}_R \left(-r \frac{\partial \theta}{\partial \chi} \sin\theta \right) + \vec{e}_z \left(r \frac{\partial \theta}{\partial \chi} \cos\theta \right) .$$

The primes here denote derivatives with respect to r,
we choose the metric tensor $g_{r\chi} = 0$ such that the χ surfaces are
orthogonal to the r surfaces and obtain

$$\frac{\partial \theta}{\partial r} = f' \frac{\sin \theta}{r} \, . \tag{A2}$$

Other pertinent relations derived in the circular tokamak, large
aspect ratio case are

$$F = \frac{m B_\theta}{r} + \frac{n B_\phi}{R} \, , \qquad G = \frac{B_\theta}{B_\phi} \left(\frac{m B_\phi}{r} - \frac{n B_\theta}{R} \right) ,$$

$$R = R_0 + f + r \cos \theta \, , \qquad I = R B_\phi = (R_0 + f) B_{\phi_0} \, ,$$

$$B_\phi = \frac{I}{R} \simeq B_{\phi_0}(r) \left(1 - \frac{r}{R_0} \cos \theta + \frac{r^2}{R_0^2} \cos^2 \theta \right) , \tag{A3}$$

$$B_\chi = B_\theta(r) \simeq B_{\theta_0}(r) \left[1 - \left(f' + \frac{r}{R_0} \right) \cos \theta \right] ,$$

$$J = \frac{r}{B_\theta} \frac{d\theta}{d\chi} \, , \qquad \frac{d\theta}{d\chi} = \frac{\sin \theta}{\sin \chi} \, , \qquad \theta = \chi + \sin \chi \int_0^r \frac{dr \, f'}{r} \, ,$$

$$\frac{d\psi}{dr} = B_{\theta_0}(R_0 + f) \, , \qquad \frac{\partial}{\partial \psi} = \frac{dr}{d\psi} \frac{\partial}{\partial r} \, .$$

The poloidal and toroidal curvature terms given in the axisymmetric toroidal case as

$$K_p = \frac{R}{J} \frac{d}{d\psi} (JB_\chi) , \qquad K_t = B_\chi \frac{dR}{d\psi} ,$$

now become

$$K_p = \frac{1}{r} , \qquad K_t = \frac{\cos\theta}{R} . \qquad (A4)$$

The transverse and longitudinal curvature terms given as

$$\tau = \frac{B_\chi}{h_3} K_p = \frac{B_\chi}{J} \frac{d}{d\psi}(JB_\chi), \qquad \lambda = \frac{B_\phi}{h_2} K_t = \frac{B_\phi}{J} \frac{dR}{d\psi} ,$$

now become

$$\tau = \frac{B_\theta}{Rr} , \qquad \lambda = \frac{B_\phi}{r \frac{\partial\theta}{\partial\chi}} \frac{\cos\theta}{R} . \quad (A5)$$

The equilibrium equation in the axisymmetric toroidal case

$$(A6)$$

$$\frac{dp}{d\psi} + \frac{1}{J} \frac{d}{d\psi} (JB_\chi) + \frac{I}{R^2} \frac{dI}{d\psi} = 0 ,$$

now becomes, in linearized form for the large aspect ratio circular tokamak

$$\frac{dp}{dr} + \left(1 - \frac{2r}{R}\cos\theta\right)\frac{1}{2(R_0+\jmath)^2}\frac{d}{dr}I^2(r) + (1 - 2\Lambda\cos\theta)\frac{d}{dr}\left(\frac{B_{\theta_0}^2}{2}\right)$$

$$\text{(A7)}$$

$$- B_{\theta_0}^2\cos\theta\frac{d\Lambda}{dr} + \frac{B_{\theta_0}^2}{r} - \frac{2B_{\theta_0}^2}{r}\Lambda\cos\theta + B_{\theta_0}^2\cos\theta\frac{\jmath'}{r} = 0 \quad,$$

where $\Lambda = \jmath' + \dfrac{r}{R_0}$. This results in the zero order relation

$$\frac{dp}{dr} + \frac{1}{2(R_0+\jmath)^2}\frac{d}{dr}I^2(r) + \frac{1}{2}\frac{dB_{\theta_0}^2}{dr} + \frac{B_{\theta_0}^2}{r} = 0, \qquad \text{(A8)}$$

and the first order relation

$$r\jmath'' + \left[2\frac{(rB_\theta)'}{B_\theta} - 1\right]\jmath' = -\frac{r}{R}\left(1 - \frac{2p'r}{B_\theta^2}\right) . \qquad \text{(A9)}$$

Integrating Eq. (A9) gives

$$\jmath' = -\frac{1}{rB_\theta^2}\int_0^r \frac{B_\theta^2 r}{R}\left(1 - \frac{2p'r}{B_\theta^2}\right)dr , \qquad \text{(A10)}$$

which is Eq. (24) of the text.

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