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Application of the Two-fluid Energy Principle to Large Aspect Ratio Tokamaks

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IPP 6/172 May 1978

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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Abstract

An "energy principle" derived in /3/ is applied to the stability of shaped large-aspect-ratio tokamaks. Most of the previous results on MHD, resistive and universal-type instabilities can be recovered in a simple way that demonstrates the power of this formalism. This allows 3 D tearing modes for general cross-sections to be investigated. An exact proof of the dissipative universal instability is given at least within the assumption of adiabatic motion. The relation of this to current work on collisionless drift waves is discussed.

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Partly supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNP_q).

I. Introduction

For any stability equation of the form

$$N\ddot{Y} + (P+M)\dot{Y} + QY = 0 \tag{1}$$

where N and M are symmetric and positive definite, Q is symmetric and P is antisymmetric, the necessary and sufficient condition for stability can be written [1], without looking for eigenmodes, in the form

$$(Y, QY) > 0 \tag{2}$$

The linearized equations of motion for conservative systems can be put, by using a Lagrangian representation, in the form (1) with M = 0 [2]. Introduction of dissipation in the Lagrangian system leads to a symmetric positive definite M, but also to an additional term $Q_D Y$ which in general has no symmetry [3].

A proper representation Y for which the linearized dissipative equations of motion can be put into form (1) was found for a one-fluid incompressible plasma [1, 4], a gravitating two-fluid plasma at rest [5] and a two-fluid plasma with long wavelength along the flow [3].

As an application of the "Energy Principle" derived in /3/, we show that it contains, in a shaped tokamak with large aspect ratio, MHD and resistive modes and "universal-type instabilities" caused by density and temperature gradients.

II. Basic Equations

The two-fluid theory can be represented by the following equations:

$$n_k m_k \frac{d\underline{v}_k}{dt} = n_k q_k (\underline{E} + \underline{v}_k \times \underline{B}) - \nabla P_k - \nabla\cdot\underline{\underline{\Pi}}_k - \frac{q_k^2}{c}\eta n_k (n_k \underline{v}_k - n_j \underline{v}_j) \,, \tag{3}$$

$$\frac{\partial n_k}{\partial t} + \nabla\cdot n_k \underline{v}_k = 0 \,, \tag{4}$$

$$P_k = P_k(n_k) \,, \tag{5}$$

$$\varepsilon_0 \nabla\cdot\underline{E} = \sum_k n_k q_k \,, \tag{6}$$

$$\nabla\times\underline{B} = \mu_0 \sum_k n_k q_k \underline{v}_k + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \,, \tag{7}$$

$$\nabla\times\underline{E} = -\frac{\partial \underline{B}}{\partial t} \,,$$

$$\nabla\cdot\underline{B} = 0 \,.$$

The index k denotes one of the fluids, ions or electrons, n is the density, m is the mass, $\underline{v}$ is the velocity field, q is the charge, $\underline{E}$ and $\underline{B}$ are the electric and magnetic fields and P is the scalar part of the pressure and $\underline{\underline{\Pi}}$ is the pressure tensor [8]. The equation of state (5) will assume different forms, depending on the nature of the fluid and the perturbations considered.

The perturbed quantities can be written in terms of Lagrangian displacements $\underline{\xi}_k$, which may be expressed as functions of the Euler coordinates of the equilibrium [2, 3] , and of the perturbed potential vector $\underline{A}$. The gauge is the one for which the electrostatic potential vanishes.

Assuming either that the perturbations decay at infinity for an infinite plasma or that they vanish on a surface surrounding the plasma, and considering two--dimensional equilibria, the ions at rest and the velocity of electrons $\underline{v}_e = v_e(x,y)\underline{e}$, Tasso [3] put the perturbed equations for an adiabatic equation of state in the form (1), provided the term

$$Q_D \underline{\xi}_e = -\frac{e}{c} n \eta \, \underline{j} \cdot \nabla \underline{\xi}_e \tag{10}$$

can be neglected (z -independent or long wavelength perturbations). This allows us to find instabilities simply by looking for test functions which do not satisfy the condition (2) [1] .

III. Explicit Criteria and Application

The "Energy Principle" (2) will be applied, in the sense of perturbation theory, to long wavelength perturbations along the electron flow. This applies to tokamak plasmas for which the perturbation wavelength in the toroidal direction is much larger than in the poloidal one.

Using the definitions of $\mathcal{Q}$ and Y [3] the condition (2) can be written in the form

$$(Y, \mathcal{Q} Y) = \int d\tau (\nabla \times \underline{A})^2 + \sum_k \int d\tau \Big[\nabla \cdot \underline{\xi}_k (\underline{\xi}_k \cdot \nabla P_k) + \gamma P_k (\nabla \cdot \underline{\xi}_k)^2 +$$

$$(\nabla P_k \cdot \nabla \ln n)\left(\underline{\xi}_k \cdot \frac{\nabla n}{|\nabla n|}\right)^2 \Big] - \int d\tau (\nabla \ln n \cdot \nabla P)\left(\underline{\xi}_e \cdot \frac{\nabla n}{|\nabla n|}\right)^2 +$$

$$2 \int d\tau\, \underline{A} \cdot (\underline{\xi}_e \cdot \nabla \underline{j} - \underline{j} \cdot \nabla \underline{\xi}_e + \underline{j}\, \nabla \cdot \underline{\xi}_e) -$$

$$\int d\tau (\underline{B}_0 \times \underline{\xi}_e) \cdot (\underline{\xi}_e \cdot \nabla \underline{j} - \underline{j} \cdot \nabla \underline{\xi}_e) > 0 \qquad (14)$$

where

$$P = P_i + P_e \quad , \quad n_i = n_e = n$$

and $\underline{j}$ is the electric current given by the equilibrium conditions.

We consider next different classes of test functions and so obtain a classification and a generalization of known plasma instabilities such as ideal MHD, resistive and "universal-type instabilities", expression (14) being the guideline of the investigation.

a) Ideal MHD instabilities

Taking the limit

$$\underline{A} \sim \underline{\xi}_k \times \underline{B}_0 \quad ,$$
$$\underline{\xi}_i \sim \underline{\xi}_e \quad , \tag{15}$$

the necessary and sufficient condition for stability of the fluids described by the ideal MHD equations [9] can be obtained from the condition (14):

$$(Y, QY) \simeq \int d\tau \left\{ \frac{1}{\mu_0} (\nabla \times \underline{A})^2 + \frac{1}{\mu_0} \nabla \times \underline{B}_0 \cdot \left[\underline{\xi} \times (\nabla \times \underline{A}) \right] + (\nabla \cdot \underline{\xi})(\underline{\xi} \cdot \nabla P) + \gamma P (\nabla \cdot \underline{\xi})^2 \right\} > 0 \quad . \tag{16}$$

b) Two-dimensional resistive instabilities

For perturbations satisfying

$$\underline{A} \sim A(x, y) \underline{e}_z \quad , \tag{17}$$

$$\underline{\xi}_k \sim \underline{e}_z \times \nabla \mu (x, y) \quad , \tag{18}$$

with the same μ for ions and electrons, and supposing that

$$\underline{j} = j(\Psi) \underline{e}_z \quad ,$$

where Ψ (x,y) is the meridional magnetic flux, the condition (14) can be reduced to

$$(Y, QY) \simeq -\int d\tau \left[\frac{dj}{d\psi} (\underline{e}_z \times \nabla\psi \cdot \nabla\mu)^2 + 2 \frac{dj}{d\psi} A (\underline{e}_z \times \nabla\psi \cdot \nabla\mu) \right]$$

$$+ \int d\tau \, |\nabla A|^2 > 0 \, . \tag{19}$$

This condition was obtained by Tasso [1, 4] in the case of a two-dimensional incompressible fluid and with "Ohm's Law" in the form

$$\underline{E} + \underline{v} \times \underline{B} = \eta \underline{j} \tag{20}$$

A simple application to tokamaks is the instability of skin currents, present if $\frac{dj}{d\psi} > 0$ [1, 4], which is similar to the rippling mode in one-dimensional geometry [10].

Another application is the stability of configurations with stagnation points (such as Doublet or for islands in tokamaks) [1, 4, 11].

c) Generalized helical resistive instabilities

For complicated geometries it might be convenient to adopt general coordinates. The condition (14) can then be written in the following way:

$$(Y,QY)=\iiint dx^1dx^2dx^3\sqrt{g}\Big\{g_{il}\,\epsilon^{ijk}\,\epsilon^{lmn}\,\frac{\partial A_k}{\partial x^j}\frac{\partial A_n}{\partial x^m}+\sum_\alpha\Big[$$

$$\frac{1}{m}\frac{dP_\alpha}{dm}\Big(\xi^i_\alpha\frac{\partial m}{\partial x^i}\Big)\Big(\xi^j_\alpha\frac{\partial m}{\partial x^j}\Big)+\frac{2}{\sqrt{g}}\frac{\partial(\sqrt{g}\,\xi^i_\alpha)}{\partial x^i}\xi^j_\alpha\frac{\partial P}{\partial x^j}+$$

$$\gamma P_\alpha\Big(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i}\xi^i_\alpha\Big)\Big(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^j}\xi^j_\alpha\Big)-\frac{1}{m}\frac{dP}{dm}\Big(\xi^i_e\frac{\partial m}{\partial x^i}\Big)\Big(\xi^j_e\frac{\partial m}{\partial x^j}\Big)$$

$$-\Big(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i}\xi^i_e\Big)\xi^j_e\frac{\partial P}{\partial x^j}+\Big[\delta^{pq}_{jk}B^j_0\xi^k+2A_i\epsilon^{ipq}\Big]\frac{\partial}{\partial x^p}\Big(\epsilon_{qmn}\,j^m\,\xi^n_e\Big)\Big\}>0 \tag{21}$$

where the index α denotes one of the fluids, ions or electrons, and g_{ik} , g , ϵ^{klm} and δ^{pq}_{jk} are defined as in /14/.

For shaped cross-sections it is convenient to choose Hamada coordinates [12]

$$x^1=V\ ,\quad x^2=\theta\ ,\quad x^3=\zeta\ , \tag{22}$$

which are orthogonal for straight plasmas. In these coordinates the contravariant components of the magnetic field and of the current density are

$$\begin{aligned} &B^1=0\ ,\quad B^2=\dot{\chi}\ ,\quad B^3=\dot{\psi}\ ,\\ &j^1=0\ ,\quad j^2=0\ ,\quad j^3=\dot{I}\ , \end{aligned} \tag{23}$$

where the dots indicate differentiation with respect to V. The functions ψ and χ describe the longitudinal and azimuthal magnetic fluxes, and the function I the longitudinal current inside the magnetic surface S bounding the volume V.

The metric $g_{ik}\, dx^i dx^k$ is explicitly

$$ds^2 = \left(\frac{\dot{\chi}}{L B_m}\right)^2 dV^2 + \left(\frac{B_m}{\dot{\chi}}\right)^2 d\theta^2 + L^2 d\xi^2 \quad , \tag{24}$$

where B_m is the meridional magnetic field, L is the length of the cylinder, and since $\sqrt{g} = 1$ it follows that

$$\epsilon^{k\ell m} = \begin{cases} +1 & k,\ell,m \quad \text{cyclic} = 1,\ 2,\ 3 \\ -1 & k,\ell,m \quad \text{cyclic} = 1,\ 3,\ 2 \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

We shall choose divergence free test functions such that

$$\xi^1 = -\frac{\partial U}{\partial x^2} \quad , \quad \xi^2 = \frac{\partial U}{\partial x^1} \quad , \quad \frac{\partial \xi^3}{\partial x^3} = 0 \quad , \tag{26}$$

with $U = U\,(V, m\theta - n\xi)$ being the same for ions and electrons, and $2\pi m/n = 2\pi \dot{\chi}_0/\dot{\psi}_0 \equiv \iota_0$ where ι_0 is the rotational transform [12] and the subscripts denote evaluation on the surface S_0. Then condition (21) becomes

$$\iiint dv\,d\theta\,d\zeta \left(-\frac{d\dot{I}}{dv}\right) \frac{\partial U}{\partial \theta} \left[\frac{\partial U}{\partial \theta}\left(-\dot{\chi} + \frac{n}{m}\dot{\psi}\right) - 2 A_\zeta\right] +$$

$$\iiint dv\,d\theta\,d\zeta\; g_{il}\, \epsilon^{ijk} \epsilon^{lmn} \frac{\partial A_k}{\partial x^j} \frac{\partial A_n}{\partial x^m} > 0 \tag{27}$$

if B_0^3 is a constant and large enough to meet the tokamak scaling. The term $-\dot{\chi} + \frac{n}{m}\dot{\psi}$ vanishes on the magnetic surface S_0. Choosing $A_k \sim 0$ and concentrating $\left(\frac{\partial U}{\partial \theta}\right)^2$ at the negative side of the integrand, we see that the expression (27) can always be made negative. This is the rippling mode.

To study the stability of the tearing mode, the form (27) can be used as a starting point to derive the Euler equation, which may be useful for further numerical applications, in the same way as was done in /10/ for the plane case.

For circular cross-sections the expression (27) reduces to the "Energy Principle" obtained by Tasso [13] in the case of helical perturbations and incompressible one-fluid model.

d) Density and temperature gradient instabilities

In the equilibrium we have among other relations

$$E_{oz} = -\frac{e}{c} n \eta v_e = \text{constant} , \tag{28}$$

$$P_k = K n T_k , \tag{29}$$

$$\eta \sim T_e^{-3/2} , \tag{30}$$

where K is the Boltzmann constant.

Considering first a uniform electric current which excludes the previously treated resistive instabilities and using equations (28) to (30), we can write

$$\nabla \ln P_k = \nabla \ln n \qquad , P_i = P_e$$

and the condition (14) becomes

$$(Y, QY) = \int d\tau (\nabla \times \underline{A})^2 + \sum_k \int d\tau \left[\nabla P_k \cdot \nabla \ln n \left(\underline{\xi}_k \cdot \frac{\nabla n}{|\nabla n|} + \frac{\nabla \cdot \underline{\xi}_k}{|\nabla \ln n|} \right)^2 + (\gamma - 1) P_k (\nabla \cdot \underline{\xi}_k)^2 + \int d\tau \left[-\left(\underline{\xi}_e \cdot \frac{\nabla n}{|\nabla n|} \right)^2 \nabla \ln n \cdot \nabla P + 2 \underline{A} \cdot \underline{j} \, \nabla \cdot \underline{\xi}_e \right] > 0 . \tag{31}$$

If test functions are chosen such that

$$\nabla \cdot \underline{\xi}_k \sim 0 \quad \text{and} \quad A \sim 0 , \tag{32}$$

(Y, QY) can always be made negative because $\nabla n \neq 0$.

There are instabilities due to density gradients for uniform j.

For a uniform electron velocity we can obtain from the relations (28) to (30)

$$\nabla \ln P_k = \frac{5}{3} \nabla \ln n \ ; \quad j \neq \text{constant} \ ,$$

and the condition (14) becomes

$$(Y, \Theta Y) = \int d\tau \left[-2 \, \underline{\xi}_e \cdot \underline{j} \times (\nabla \times \underline{A}) \right] + \int d\tau \, (\nabla \times \underline{A})^2 +$$

$$\sum_k \int d\tau \, \frac{(\nabla \ln n)^2}{\gamma P_k} \left[\underline{\xi}_k \cdot \frac{\nabla n}{|\nabla n|} + \frac{\nabla \cdot \underline{\xi}_k}{|\nabla \ln n|} \right]^2 > 0 \ . \tag{33}$$

Choosing $\nabla \times \underline{A}$ small enough but not zero, it is always possible to find test functions $\underline{\xi}_k$ such that $(Y, \Theta Y) < 0$. For $\nabla \times \underline{A} = 0$ there are $\underline{\xi}_k$ for which the expression (33) can be made marginal.

IV. Isothermal Perturbations

Considering isothermal perturbations and a uniform as well as a uniform current density, it is possible to write

$$\delta\eta = 0 \tag{34}$$

and to show that

$$Q_D \underline{\xi}_e = -\frac{e}{c}\,\eta\, n\left(\underline{j}\cdot\nabla\underline{\xi}_e - \underline{j}\nabla\cdot\underline{\xi}_e\right) . \tag{35}$$

Now, even for z -independent $\underline{\xi}_e$, $Q_D\underline{\xi}_e$ does not vanish. However, when

$$\frac{\eta e n}{c B_0} = \frac{\nu_{ei}}{\Omega_e} << 1 , \tag{36}$$

where Ω_e is the electron Larmor frequency and ν_{ei} is the collision frequency between electrons and ions, we could write

$$\left| \underline{B}_0 \times (\underline{j}\cdot\nabla\underline{\xi}_e) \right| >> \left| \frac{e}{c}\,\eta\, n\, \underline{j}\,\nabla\cdot\underline{\xi}_e \right| \tag{37}$$

For typical tokamak plasmas

$$T \sim 1\ \text{Kev} \quad , \quad n \sim 5.10^{13}\ \text{cm}^{-3} \quad , \quad B_0 \sim 40\ \text{kG}$$

and

$$\frac{\eta e n}{c B_0} \sim 1.10^{-3} .$$

This approximation should at the same time keep the dissipative M operator dominant in order that statement (2) remain valid. This means that

$$\frac{eB_0}{m_e v_e k} = \frac{\Omega_e}{v_e k} >> 1 . \tag{38}$$

These remarks allow expression (31) to be used, as an approximation, for the isothermal case as well, in which heat conductivity and thermal effects are absent. This demonstrates the density gradient instability for rather high but constant temperature (T   1 keV). This may also suggest that the drift wave instability survives to large heat conductivities better than the rippling mode.

V. Discussion and Conclusion

The reader has probably noticed how easy it was to derive from the general energy principle (2) or in explicit form (14) a great deal of previous results, scattered throughout the literature in particular MHD, resistive, universal instabilities. This demonstrates the power of the formalism developed in /1/ and /3/, in which this general energy principle was derived. For a discussion of the validity of this derivation the reader is referred to the discussion and conclusion of /3/.

The progress achieved in this study is not only of a formal nature. In other words, progress in the physics can be achieved through the formalism. For example, the expression (27) can be taken as a basis for studying 3 D tearing modes in shaped tokamaks and shows how the code for 2 D modes developed in /11/ can be extended to 3 D modes in a straightforward way.

Another result is that it has been proved in the case of adiabatic motion that universal instabilities exist in a shaped but straight tokamak. This is not in contradiction with Ref. /15/, which is concerned with non-dissipative drift waves. In fact, the stability or marginality of the drift waves in the collisionless case found in /15/ was obtained for the drift approximation, i.e. small ion Larmor radii. As pointed out by Croci [16], the drift waves must be unstable if this approximation is improved by the solution of the full integro-differential equation [17].

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