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Linear Theory of Drift-Tearing and Interchange Modes in a Screw Pinch

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Abstract

A drift dispersion relation, as applied to a resistive incompressible plasma in a screw pinch, is derived. This dispersion relation incorporates both drift-tearing and drift-interchange modes and is valid throughout the collisional regime by including kinetic theory factors. The dispersion relation reduces to the drift-tearing dispersion relation in the zero pressure gradient limit, and to the classical resistive dispersion relation in the zero drift limit. The electron temperature gradient instability is still present. Now, however, the introduction of the interchange-drift instability increases the growth rate further above the tearing-drift case.

I. INTRODUCTION

In past work [1-3], where the dispersion relation for drift instabilities has been derived, the resistive interchange mode has been overlooked. Here, the ions are modeled by the ion momentum balance equation as in two fluid theory, and the electrons are modeled by the kinetic theory version of Ohm's law as obtained by Hazeltine, Dobrott and Wang.[2] These two equations are used to derive a dispersion relation which includes both the resistive drift-interchange and drift-tearing modes. As contrasted to the Hazeltine et al.[2] treatment, where the principal effort was the derivation of the electron momentum balance equation, the principal effort here is the derivation of the ion momentum balance equation and its effect on the dispersion relation. All the analysis here is for an incompressible resistive plasma in a screw pinch, however, extensions of the theory to tokamaks are discussed.

Our main result is a dispersion relation which is valid throughout the collisional regime, and which has the following properties:

(i) Previously derived dispersion relations are obtained in appropriate limits. The Hazeltine et al.[2] tearing-drift dispersion relation is obtained in the zero pressure gradient limit. The Johnson, Greene and Coppi [4] resistive dispersion relation is obtained in the zero drift limit.

(ii) Although no new instabilities are produced, the electron temperature gradient instability of Hazeltine et al. still exists and the essential Johnson et al. interchange growth rate limits are also obtained.

A second result is the derivation of an exact momentum balance equation (one-fluid theory) in a form convenient for theoretical analysis. This equation, derived without invoking Ohm's law, involves only the perturbed radial components of the velocity and magnetic field. The portion of this equation, which does not exist in ideal magnetohydrodynamics, gives the interchange scaling in a reasonably direct fashion. In the appropiate limits, this equation reduces to the Hain Lüst [5] equation and to Newcomb's [6] equation.

The analysis proceeds as follows. In Sec. II, we derive, for the screw pinch, an exact momentum balance equation in the incompressible limit. Section III contains a heuristic derivation of the resistive interchange instability. In Sec. IV, we include diamagnetic frequencies in both the ion momentum balance equation and the electron momentum balance equation (Ohm's law). Section V contains the derivation of the drift dispersion relation. Previously derived dispersion relations are recovered by considering appropriate limits. In Sec. VI, we give a summary of our principal results.

II. MOMENTUM BALANCE EQUATION

In order to conveniently incorporate the resistive interchange mode into existing theory, it is useful to first derive a momentum balance equation in which no approximations are made. We start by considering the magnetohydrodynamic equations in the incompressible form:

$$\vec{F} = \rho \frac{d\vec{v}}{dt} = -\nabla p + \frac{\vec{J} \times \vec{B}}{c} \tag{1}$$

together with

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} \quad , \quad \nabla \cdot \vec{v} = 0 \quad , \quad \nabla \cdot \vec{B} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \quad , \qquad \frac{\partial p}{\partial t} + \nabla \cdot p \vec{v} = 0$$

Here $\vec{v}$ is the plasma velocity, $\vec{J}$ the current, p is the scalar pressure and ρ the density, $\vec{B}$ is the magnetic field, where B_θ and B_z are respectively, the poloidal and axial magnetic fields of a screw pinch. The pressure equation used follows from the adiabatic assumption.

We later treat Eq. (1) as the ion momentum balance equation and include the ion diamagnetic effects, which means including Π_i , the ion stress tensor. However, it is first useful to obtain an exact derivation of the classical momentum balance equation in a convenient form, since this gives the important pressure gradient terms. This is achieved by taking the radial component of the double curl of the linearized version of Eq. (1),

$$\left\{\nabla\times(\nabla\times\vec{F})\right\}_r = \left\{\frac{im}{r}(\nabla\times\vec{F})_z - ik(\nabla\times\vec{F})_\theta\right\}_r ,$$

and straightforward algebra. We choose the perturbed solutions of the form,

$$\vec{A}_1(r,t) = \vec{A}_1(r)\exp\left[-i\omega t + i(m\theta + kz)\right],$$

and also use $k = \frac{n}{R}$. Although the double curl expression given above does not include the vector component $(\nabla\times\vec{F})_r$, this component is needed to reduce the double curl expression to a convenient form.

After some algebra we obtain a momentum balance equation involving only the perturbed radial components V_{r_1} and B_{r_1}

$$\frac{\partial}{\partial r}\left[\frac{r\rho_0(-i\omega)}{m^2+k^2r^2}\frac{\partial}{\partial r}(rV_{r_1})\right] - \rho_0 V_{r_1}(-i\omega) - \frac{2B_{\theta_0}kr\rho_0(-i\omega)\frac{\partial}{\partial r}(rV_{r_1})}{(mB_{z_0}-krB_{\theta_0})(m^2+k^2r^2)}$$

$$+\frac{2B_{\theta_0}k\frac{\partial\rho_0}{\partial r}\left(\frac{V_{r_1}}{-i\omega}+\frac{iB_{r_1}}{F}\right)}{(mB_{z_0}-krB_{\theta_0})}$$

$$= irF\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r}{m^2+k^2r^2}\frac{\partial}{\partial r}(rB_{r_1})\right]\right. \qquad (2)$$

$$\left. -\frac{rB_{r_1}}{Fr^2}\left[\frac{G}{m^2+k^2r^2}+\frac{\partial}{\partial r}\left[\frac{r}{m^2+k^2r^2}\frac{\partial}{\partial r}(rF)\right]\right]\right\},$$

where
$$F \equiv \frac{mB_{\theta 0}}{r} + kB_{z0}$$

$$G \equiv F\,(m^2+k^2r^2-1) + \frac{2k^2r^2}{m^2+k^2r^2}\left(kB_{z0} - \frac{mB_{\theta 0}}{r}\right) + \frac{2k^2r}{F}\frac{\partial p}{\partial r}\ .$$

Ohm's law has not been invoked up to this point. We know that in the zero drift limit it will be of the form (see Eq. (17))

$$\frac{V_{r1}}{-i\omega} + \frac{iB_{r1}}{F} = \frac{\eta_0}{-i\omega F}\left(iB_{r_1}'' + \text{other perturbed magnetic field components}\right)$$

where η is the resistivity. Then, in the ideal MHD limit $(\eta \to 0)$, the $\frac{\partial p_0}{\partial r}$ term on the left side of Eq. (2) reduces to zero, and the equation reduces to the Hain-Lüst equation[5] for a diffuse linear pinch in the incompressible limit.

For finite resistivity, we shall show in the case $\frac{\partial p_0}{\partial r}$ finite, that we obtain the resistive interchange mode which scales as $\eta_0^{1/3}$ and in the limit, $\frac{\partial p_0}{\partial r} = 0$, we obtain the tearing mode which scales as $\eta_0^{3/5}$. Thus, as either resistivity or ω goes to zero $\frac{\eta_0}{-i\omega}$ and ω go to zero, and the left side of the equation goes to 0 at the origin, $\omega = 0$. Hence, at $\omega = 0$, the left side of the equation is zero and the equation becomes identically equal to the marginal stability equation of Newcomb[6].

Noting that F = 0 at the mode rational surface, we see that the $\frac{\partial p_0}{\partial r}\left(\frac{iB_{r1}}{F}\right)$ term on the left side of Eq. (2) identically cancels the

$\frac{\partial p_0}{\partial r}$ term on the right side only at the mode rational surface. This means that for finite ω, the indicated momentum balance equation, written in terms of B_{r_1} or V_{r_1}, is not singular at the mode rational surface.

The form of Eq. (2) is very similar to that of Coppi et al.[7] (their Eq. (1)), except that in their treatment they made the usual tokamak approximation, $B_{\theta_0} < B_{z_0}$ and correspondingly $kr < m$. In fact it was only after reading their paper, that we appreciated that the more exact form here could be derived. The form here gives the resistive interchange scaling (considered next) and reduces to both the Hain Lüst equation and Newcomb's equation in the appropriate limits, while the approximate form derived by Coppi et al.[7] satisfies none of these requirements.

III. HEURISTIC DERIVATION OF THE RESISTIVE INTERCHANGE INSTABILITY

Heuristic derivation of the interchange mode provides the motivation for neglecting terms in the modified ion momentum balance equation, when deriving the dispersion relation, and provides confirmation that the exact momentum balance equation gives the correct resistive interchange scaling. In this derivation of the classical resistive interchange result, we do not include diamagnetic frequencies or the kinetic theory factors as derived by Hazeltine et al.[2] Thus, we write Ohm's law, using Eq. (17), as

$$-i\omega B_{r1} = \frac{a^2}{\tau_s} B_{r1}'' + i F v_{r1} \tag{3}$$

We define the resistive skin time, τ_s, and the Alfven time, τ_a, as

$$\tau_s = \frac{4\pi a^2}{c^2 \eta_0} \quad , \qquad \tau_a = a\left(\frac{4\pi\rho_0}{B_0^2}\right)^{1/2}.$$

In some small vicinity λ of the mode rational surface, r_s, where we use

$$|r - r_s| < \lambda < a,$$

with a the width of the plasma, the terms in Ohm's law, not involving ω, will be comparable. Thus,

$$B_{r1}'' \simeq -i \frac{\tau_s}{a^2} B_0 k_{\parallel}' \lambda v_{r1} \tag{4}$$

where we use $\quad k_{\parallel}(r) \approx k_{\parallel}'(r_s)(r - r_s) \approx k_{\parallel}' \lambda \,,$

with $\quad k_{\parallel} = \dfrac{m}{r}\dfrac{B_{\theta 0}}{B_0} + \dfrac{n}{R}\dfrac{B_{z0}}{B_0} = \dfrac{F}{B_0}\,.$

Rewritting the momentum balance equation, Eq. (2), considering only terms that are important near the mode rational surface, we have

$$-i\omega\rho_0 V_{r_1}'' - \frac{2B_{\theta_0}k\frac{\partial p_0}{\partial r}\left(\frac{V_{r_1}}{-i\omega} + \frac{iB_{r_1}}{k_{\parallel}B_0}\right)}{r^2\,\dfrac{(mB_{z0} + krB_{\theta_0})}{(m^2+k^2r^2)}} \cong ik_{\parallel}B_0B_{r_1}'' - \frac{iB_{r_1}2k^2}{rk_{\parallel}B_0}\frac{\partial p_0}{\partial r} \qquad (5)$$

The left side will be important only in the resistive layer, so that we can assume that the terms on the right side will be comparable at λ.

Letting
$$D_s \equiv \frac{-2k^2}{(k_{\parallel}'B_0)^2 r}\frac{\partial p_0}{\partial r},$$

gives
$$B_{r_1}'' = -\frac{D_s}{\lambda^2}B_{r_1}. \qquad (6)$$

Here, D_s is the parameter relating to interchange stability of a diffuse pinch. Analyzing Eq. (2) in the ideal MHD limit, it can be shown[8] that the necessary condition for ideal stability is $D_s < \frac{1}{4}$.

To estimate the boundary layer width we allow

$$V_{r_1}'' = -\frac{V_{r_1}}{\lambda^2}. \qquad (7)$$

The physical significance of the negative signs in Eqs. (6) and (7), and the switched signs in the tearing mode case, needs to be explained. In the tearing mode case we have

$$B_{r_1}'' = \frac{\Delta'}{\lambda}B_{r_1} \qquad \text{and} \qquad V_{r_1}'' = \frac{V_{r_1}}{\lambda^2},$$

where Δ' is defined by Eq. (11). When an instability occurs with D_s positive, the driving source of energy is localized in the singular layer. However, when an instability occurs with D_s negative or zero, the driving source of energy is not localized, but outside the singular layer.

Rewritting Eq. (5) at F = 0, we obtain

$$(-i\omega)^2 \rho V_{r1}'' \approx \frac{2k^2}{r} \frac{\partial p_0}{\partial r} V_{r1} . \tag{8}$$

By appropriately combining Eqs. (4), (5), (6), (7), (8) we obtain the resistive interchange growth rate, stability criterion, and dissipative boundary layer width given below. The corresponding values given for the tearing mode can be obtained by setting the pressure gradient terms equal to zero in Eq. (2) and using a similar heuristic analysis. Since this is straightforward, noting the comment following Eq. (7), and is done in detail in Hazeltine et al.[2], where their notation is similar to that here, only the results are repeated:

<u>Interchange Mode</u> (Resistive)

$$-i\omega \approx \frac{\sqrt{D_s}\, F' \lambda}{\sqrt{\rho}} , \tag{9a}$$

stability if $\quad D_s < 0 ,$ (9b)

$$\frac{\lambda^3}{a^3} = \frac{\tau_a}{\tau_s}\frac{\sqrt{D_s}}{(a^2 k_{\parallel}')} \approx \frac{\tau_a}{\tau_s} < 1 , \tag{9c}$$

$$(-i\omega)^3 = \frac{D_s^2}{\tau_s \tau_a^2}\,(a^2 k_{\parallel}')^2 , \tag{9d}$$

Tearing Mode

$$-i\omega \approx \frac{a^2}{\tau_s}\frac{\Delta'}{\lambda} , \tag{10a}$$

$$\text{stability if} \quad \Delta' < 0 , \tag{10b}$$

$$\frac{\lambda^5}{a^5} = \frac{\tau_a^2}{\tau_s^2}\frac{\Delta' a}{(a^2 k_{\parallel}')^2} \approx \frac{\tau_a^2}{\tau_s^2} < 1 , \tag{10c}$$

$$(-i\omega)^5 = \frac{(\Delta' a)^4}{\tau_s^3 \tau_a^2}\,(a^2 k_{\parallel}')^2 , \tag{10d}$$

with
$$\Delta' \equiv \frac{B_{r_1}'(r_s+\epsilon) - B_{r_1}'(r_s-\epsilon)}{B_{r_1}(r_s)} , \tag{11}$$

where $\lambda < \epsilon < a .$

Before proceeding it is useful to list some numerical values. It should perhaps be noted that the growth rates and dissipative boundary layer widths given do not depend on the plasma thickness. For the following set of parameters:

$$B \sim 40\,\mathrm{kG}, \qquad n = 10^{14}\,\mathrm{cm}^{-3}, \qquad Z_{eff} = 1,$$

$$T_i \sim T_e \sim 10^7\,{}^\circ K \sim 1\,\mathrm{Kev}, \quad a = 40\,\mathrm{cm} \quad \frac{R}{a} = 4,$$

we obtain

$$\tau_a = .05\,\mu sec, \qquad \tau_s = 8\,sec, \qquad S = 1.6 \times 10^8, \tag{12a}$$

$$(-i\omega)\ \text{tearing} \sim .2\,\mathrm{kHz}, \tag{12b}$$

$$(-i\omega)\ \text{interchange} \sim 3\,\mathrm{kHz}, \tag{12c}$$

$$\lambda_T \equiv \lambda\ \text{tearing (Eq 9c)} \sim .05\,\mathrm{cm}, \tag{12d}$$

$$\lambda_I \equiv \lambda\ \text{interchange (Eq 10c)} \sim .06\,\mathrm{cm}, \tag{12e}$$

$$r_i \equiv \text{ion gyroradius} \sim .08\,\mathrm{cm}, \tag{12f}$$

$$\omega_{*i} \equiv \text{ion diamagnetic freq.} = \frac{c}{eB_0 n_i}\frac{m}{r}\nabla p_i \sim -2\,\mathrm{kHz}, \tag{12g}$$

$$\omega_{*e} \equiv \text{elect. diamagnetic freq.} = \frac{-c}{eB_0 n_e}\frac{m}{r}\nabla p_e \sim 2\,\mathrm{kHz}, \tag{12h}$$

$$\omega_{*T} \equiv -\frac{c}{eB_0}\frac{m}{r}\nabla T_e, \tag{12i}$$

$$\nu_c \equiv \text{coulomb collision frequency} \sim 200\ \mathrm{kHz}, \tag{12j}$$

$$\Omega_e \tau_e \simeq 5 \times 10^6 > 1, \quad \Omega_i \tau_i \simeq 10^5 > 1, \tag{12k}$$

$\Omega_{e,i} \equiv$ electron / ion gyrofrequency, $\tau_{e,i} \equiv$ electron / ion collision time.

It should be noted that the resistive interchange growth rate is of the order of the diamagnetic frequency and that the tearing mode growth rate is about $.1\,\omega_*$. The conventional definitions[2] of the various diamagnetic frequencies are used here.

Drake and Lee divided the resistive drift instabilities into three regimes according to the collisionality of the plasma, as follows:

$$
\begin{array}{lll}
\text{collisional} & r_i < \lambda & \omega_* < \nu_c \, , \\
\text{semi-collisional} & r_i > \lambda & \omega_* < \nu_c \, , \\
\text{collisionless} & r_i > \lambda & \omega_* > \nu_c \, .
\end{array}
\qquad (13)
$$

Drift effects increase the dissipative layer widths; thus, using Eq. (22), or more specifically Eqs. (31) and (34), with the numbers here, the λ's calculated are larger than the ion gyroradius. Therefore, the collisional regime is appropriate here. However, future experimental devices will operate in the semicollisional regime as indicated by the analysis of Drake and Lee[3]. Nevertheless, their statement, that the dynamics of the layer are relatively insensitive to the geometry under consideration, so that slab results can be extended to a cylindrical geometry, is shown to be invalid in the collisional regime treatment here. The inclusion of pressure gradient should also change the dynamics of the semi-collisional regime, particularly, since the interchange mode scales the same as the semi-collisional growth rate calculated by Drake and Lee.

In deriving the drift ion momentum balance equation, we use Braginskii's two-fluid equations[9]. These equations apply where $\nu_c > \omega$ and when the characteristic gradient lengths are larger than the mean free paths, that is, in the collisional regime. The inequalities in Eq. (12k) indicate that it is quite appropriate to use Braginskii's strong magnetic field case.

IV. FINITE GYRORADIUS EFFECTS INCLUDED

As device plasma temperatures increase, classical resistivity becomes smaller, so that the resistive growth rate becomes comparable with or less than the electron diamagnetic frequency. In fact for present plasmas, $\gamma_T << \omega_{e*}$ and $\gamma_I \sim \omega_{e*}$, so that diamagnetic drift effects become important.

The ion momentum balance equation is derived from two-fluid theory. The set of Eqs. (1) are now generalized to a set appropriate for ions, however, we need to include the pressure tensor to obtain the finite gyroradius terms which give the ion diamagnetic frequency. In particular, the ion momentum balance equation is

$$\rho_i \frac{d\vec{v}_i}{dt} = \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi} - \nabla p_i - \nabla \cdot \left[\Pi_{io} + \Pi_{FGR}\right] , \tag{14}$$

where

$$\Pi_{\alpha\beta} \cong p\tau \left[-W_{0\alpha\beta} + \left(\frac{W_{3\alpha\beta} + 2W_{4\alpha\beta}}{2x} \right) \right] .$$

Here $W_{\alpha\beta}$ are the Braginskii [9] ion stress tensors, and we use his expansion in the strong field case with $x = \Omega_i \tau_i > 1$, so that terms up to Π_{FGR}, the finite gyroradius term which corresponds to the $1/x$ term, need to be included.

Considering only lowest order term and singular terms, since we are concerned with perturbations localized near the mode rational surface, yields the modified ion momentum balance equation:

$$\frac{\partial}{\partial r}\left[\frac{r\rho_0(-i\omega+i\omega_{*i})}{m^2+k^2r^2}\frac{\partial}{\partial r}(rV_{r1})\right]+\frac{2B_{\theta 0}k\frac{\partial p_0}{\partial r}\left(\frac{V_{r1}}{-i\omega+i\sigma\omega_{*i}}+\frac{iB_{r1}}{F}\right)}{(mB_{z0}-krB_{\theta 0})}$$

$$+\text{ higher order terms} \qquad (15)$$

$$=irF\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r}{m^2+k^2r^2}\frac{\partial}{\partial r}(rB_{r1})\right]\right.$$

$$\left.-\frac{rB_{r1}}{Fr^2}\left[\frac{G}{m^2+k^2r^2}+\frac{\partial}{\partial r}\left[\frac{r}{m^2+k^2r^2}\frac{\partial}{\partial r}(rF)\right]\right]\right\},$$

where, again

$$G=F(m^2+k^2r^2-1)+\frac{2k^2r^2}{m^2+k^2r^2}\left(kB_{z0}-\frac{mB_{\theta 0}}{r}\right)+\frac{2k^2r}{F}\frac{\partial p_0}{\partial r}$$

and σ is a factor less than unity [10]. When $\omega_{*i}=0$ and with the inclusion of lower order terms, Eq. (15) reduces to Eq. (2). However, Eq. (15) is not singular when both ω and $F\to 0$, remembering that the B_{r1}/F terms identically cancel at the mode rational surface. Thus, the marginal stability equation of Newcomb [6] no longer applies. In general, the inclusion of diamagnetic drift effects alters the growth rates and stability criteria of both ideal and resistive MHD, and results in new drift mode stability criteria and growth rates. Although, the incompressible case is still being considered, we now expect over-stable or complex growth rates, since there are new terms analogous to zero-order velocities. The existence of overstable modes with non-zero equilibrium velocities was recognized by Furth et al.[11] Recently

Dobrott et al.[12] have pointed out that the conventional neglect of radial diffusion velocity V_{r0} in resistive calculations is incorrect and find that the stability criterion for tearing modes is no longer $\Delta' < 0$, and that overstable tearing modes have to be considered.

The set of equations represented by Eqs. (1) are of course not complete even when generalized to two-fluid theory. We, however, do not use two-fluid theory in obtaining the electron momentum balance equation, but take that result directly from the work of Hazeltine et al.[2] They used the guiding-center kinetic equation with Fokker-Planck collision term to obtain

$$\eta_{\parallel 0} J_{\parallel} = \alpha_1 E_{\parallel} + \alpha_1 \left(\frac{B_{r1}}{B_0} + \frac{k_{\parallel} V_{rE}}{\omega} \right) \left(\frac{P'_{0e}}{e\, n_{0e}} + \frac{\alpha_2 T'_{0e}}{e} \right) , \tag{16}$$

where
$$\alpha_1 = 0.98 \frac{(1 - .54 i \omega/\nu_c)}{1 - 2.97 i \frac{\omega}{\nu_c} - 1.04 \frac{\omega^2}{\nu_c^2}} ,$$

$$\alpha_2 = \frac{0.80}{1 - .54 i \frac{\omega}{\nu_c}} .$$

If we had used two-fluid theory from Braginskii[9], and the adiabatic assumption as a closure condition for the two fluids, we would have obtained the above equation with $\alpha_1 = 1.0$ and $\alpha_2 = 0.71$. Thus, in the extreme collisional limit $\omega \lll \nu_c$, there is a correspondence between kinetic theory and two-fluid theory, since then Hazeltine's et al. α's would reduce to $\alpha_1 = 0.98$ and $\alpha_2 = 0.80$. The advantages in using the kinetic theory model are that it allows for a range of $\frac{\omega}{\nu_c}$ values, which is important since in present tokamaks ω_* is not much smaller than ν_c, and because the use of kinetic theory complex factors result in the prediction of instabilities, such as the temperature electron gradient instability, which cannot be predicted by two-fluid theory.

The electron momentum balance equation, or generalized Ohm's law, follows from Hazeltine et al. as

$$B_{r1} \approx iF\xi_r + \frac{i a^2 B_{r1}''}{\alpha_1 \tau_s (\omega - \omega_{*eT})}, \tag{17}$$

with $\omega_{*eT} \equiv \omega_{*e} + \alpha_2 \omega_{*T}$,

and where only the lowest order terms important near the singular layer are retained.

V. DISPERSION RELATION

Now rewritting the momentum balance equation, Eq. (15) considering only terms that are important near the mode rational surface, we have

$$4\pi\rho_0(-i\omega+i\omega_{*i})V_{r_1}'' + \frac{2B_{\theta_0}k\frac{\partial p_0}{\partial r}(m^2+k^2r^2)}{r^2(mB_{z_0}-krB_{\theta_0})}\left(\frac{V_{r_1}}{-i\omega+i\sigma\omega_{*i}} + \frac{iB_{r_1}}{F}\right)$$

$$\simeq iFB_{r_1}'' - \frac{iB_{r_1}2k^2}{rF}\frac{\partial p_0}{\partial r}. \tag{18}$$

Within the singular layer, using $V_{r_1} = \frac{\partial \xi_r}{\partial t} = -i\omega\xi_r$, and $\psi = -iB_{r_1}$ yields

$$4\pi\rho_0\omega(\omega-\omega_{*i})\xi_r'' - \frac{D_s F'^2\omega\xi_r}{(\omega-\sigma\omega_{*i})} = F\psi''. \tag{19}$$

Correspondingly, Eq. (17) yields

$$\frac{a^2\psi''}{\tau_s} + i\alpha_1(\omega-\omega_{*}e_T)(\psi-F\xi_r) = 0. \tag{20}$$

Since the $\frac{1}{F}\frac{\partial p_0}{\partial r}$ terms cancel out at F = 0, we assume that $\psi = \psi(r_s) = \psi_s$ as constant, within the boundary layer. Combining Eqs. (18) and (19), we obtain

$$\xi_r'' + \left[\frac{1}{\tilde{x}^2} - \frac{(r-r_s)^2}{\tilde{\lambda}^2}\right]\xi_r = -\frac{(r-r_s)\psi_s}{k_{\parallel}' B_0\tilde{\lambda}^4}, \tag{21}$$

where $$\tilde{\lambda}^4 \equiv \frac{\tau_a^2 \omega(\omega - \omega_{*i})}{(k_{\parallel}')^2 \tau_s i\alpha_1(\omega - \omega_{*eT})}, \qquad \tilde{\chi}^2 \equiv -\frac{\tau_a^2(\omega - \omega_{*i})(\omega - \sigma\omega_{*i})}{(k_{\parallel}')^2 D_s a^2}.$$

The definitions of $\tilde{\lambda}$ and $\tilde{\chi}$ are predicated on the fact that when the diamagnetic frequencies go to zero, $\tilde{\lambda}$ reduces to λ_T and $\tilde{\chi}$ reduces to λ_I as in Eqs. (10c) (12d) (9c) and (12e). Although both $\tilde{\lambda}$ and $\tilde{\chi}$ are, in general, complex, the dissipative boundary layer widths, respectively, are

$$\begin{aligned} &\text{tearing mode} \qquad && \lambda_T = |\tilde{\lambda}|, \\ &\text{interchange mode} \qquad && \lambda_I = |\tilde{\chi}|. \end{aligned} \tag{22}$$

Introducing dimensionless variables x and y we may write Eq. (21) as

$$\frac{d^2 y}{d^2 x} + \left[\frac{\tilde{\lambda}^2}{\tilde{\chi}^2} - x^2\right] y + x = 0, \tag{23}$$

where

$$y \equiv \left(\frac{\tilde{\lambda} k_{\parallel}' B_0}{\psi_s}\right) \xi_r, \qquad x \equiv \frac{(r - r_s)}{\tilde{\lambda}}.$$

Equation (23) has the solution

$$y = \frac{x}{2}\int_0^1 dt\,[1 - t^2]^{-1/4}\, e^{-\frac{x^2}{2}t} \left[\frac{1+t}{1-t}\right]^{\frac{\tilde{\lambda}^2}{4\tilde{\chi}^2}}, \tag{24a}$$

or equivalently, letting $t = \cos\theta$,

$$y = \frac{x}{2}\int_0^{\frac{\pi}{2}} d\theta\, \frac{\sin^{1/2}\theta}{\left(\tan\frac{\theta}{2}\right)^{\frac{\tilde{\lambda}^2}{4\tilde{\chi}^2}}}\, e^{-\frac{x^2\cos\theta}{2}}, \tag{24b}$$

as can be verified by substitution and integration by parts. This equation differs from the standard equation developed by Rutherford and Furth [13] only due to the presence of the $\frac{\tilde{\lambda}^2}{4\tilde{\chi}^2}$ term. The dispersion relation is now obtained by requiring the dissipative layer solution $\Delta(\omega)$ and the solution in the external region, evaluated using Eq. (11), to be the same

$$\Delta(\omega) = \Delta' . \tag{25}$$

It should be pointed out that with the inclusion of pressure gradient and diamagnetic frequency, it is no longer appropriate to use the standard tearing mode analysis [14] to calculate Δ'. There is no satisfactory way to calculate Δ' at the present time. However, assuming that the solution for Δ' can be obtained, the boundary result can be found from integration of Eq. (20) as

$$\frac{i\Delta a^2}{(\omega - \omega_{*eT})\tau_s \tilde{\lambda} \alpha_1} = \int_{x_s-\epsilon}^{x_s+\epsilon} [1 - xy]\, dx . \tag{26}$$

Substituting for y using Eq. (24a), and integrating over x first and then t, easily gives the dispersion relation we have been seeking

$$\Delta = -i\frac{\alpha_1(\omega - \omega_{*eT})2\pi\tau_s\tilde{\lambda}}{a^2} \frac{\Gamma\left[\frac{1}{4}\left(3 - \frac{\tilde{\lambda}^2}{\tilde{\chi}^2}\right)\right]}{\Gamma\left[\frac{1}{4}\left(1 - \frac{\tilde{\lambda}^2}{\tilde{\chi}^2}\right)\right]} , \tag{27}$$

where for clarity

$$\tilde{\lambda}^4 \equiv \frac{\tau_a^2\,\omega(\omega-\omega_{*i})}{i\alpha_1 k_{\parallel}'^2 \tau_s(\omega-\omega_{*eT})}, \qquad \tilde{\chi}^2 \equiv -\frac{\tau_a^2(\omega-\omega_{*i})(\omega-\sigma\omega_{*i})}{k_{\parallel}'^2 D_s a^2} .$$

For a given mode (m, n) and equilibrium profile, Δ' can be determined in the external region, and ω solved for using Eq. (27).

VI. SUMMARY AND CONCLUSIONS

We have derived, from the ion momentum balance equation obtained here using two-fluid theory and from the electron momentum balance equation obtained by Hazeltine et al. using guiding-center kinetic theory, the dispersion relation, Eq. (27), which holds uniformly for all values of ω/ν_c in the collisional regime.

A self consistency requirement is that

$$\mathrm{Re}\,(\tilde{\lambda}^2) > 0. \tag{28}$$

This follows from the observation that the asymptotic solution of the original differential equation, Eq. (23), is $y \sim 1/x$. Examination of Eq. (24a) indicates that, for the integral solution of y to have the correct asymptotic dependence, it is necessary that Eq. (28) be satisfied. We also see that if this condition is not satisfied, the unstable mode is no longer spatially localized. Rutherford and Furth [13] have shown that it is mathematically possible to continue the dispersion relation beyond the branch cut represented by $\mathrm{Re}\,(\tilde{\lambda}^2) = 0$, by inclusion of an electrostatic term. Since electrostatic terms are included in kinetic theory in a natural way, analytical continuation of the dispersion curve beyond the branch curve should not represent a problem.

The dispersion relation can be solved analytically in the two limits $\omega_* \gg \gamma$ and $\omega_* \ll \gamma$, where

$$\omega = \omega_r + i\gamma,$$

and $\gamma > 0$ for an instability. Our previous assumptions also require $\omega_* < \nu_c$, $\gamma < \nu_c$ and $\lambda > r_i$.

A. The case $\omega_* >> \gamma$.

This ordering is closest to present tokamak experiments. The significant root has $\omega \approx \omega_{*eT}$ so that the term $(\omega - \omega_{*eT})$ in Eq. (27) is small. Thus, we write $\omega = \omega_0 + \omega_1$, where ω_0 is defined to be a solution to

$$\omega_0 - \omega_{*e} - \alpha_2\left(\frac{\omega_0}{\nu_c}\right)\omega_{*T} = 0. \tag{29}$$

Here we consider two subcases: D_s negative, and $D_s = 0$.

(i) D_s negative.

For this case, we see that the argument of the gamma function, $z = \frac{-\tilde{\lambda}^2}{4\tilde{x}^2}$, can be large due to the presence of the $(\omega - \omega_{*eT})$ term and we can expand

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + O\left(\frac{1}{z^2}\right)\right],$$

valid for $|z| \to \infty$, provided we stay away from the negative real axis [15].

Thus, we obtain

$$\omega = \omega_0 + \omega_1 = \omega_{*eT} + .43\,\frac{i\,\omega_{*eT}\,\omega_{*T}}{\nu_c} - i\,\frac{\Delta'^2 a^2(\omega_{*eT} - \sigma\omega_{*i})}{D_s\,\pi^2\,\tau_s\,\omega_{*eT}}, \tag{30}$$

and instability of the last term with $D_s < 0$ and $\Delta' > 0$. The two modes here rotate in the same sense as ω_{*eT}.

It is of course incorrect to take the limit $D_s \to 0$, since the derivation is based on $|\omega_1/\omega_0| \ll 1$. However, the equation does suggest the possibility that an evolving quasi-equilibrium, where line tying of regions of good curvature to bad curvature breaks down, could result in an enhancement of growth rate.

It should be pointed out that while D_s is nearly always positive in a straight system, it is multiplied by a factor $1 - q^2$, in the simplest tokamak approximation. Here, $q = \frac{r B_{z0}}{R B_{\theta 0}}$ is the safety factor, so that for $q > 1$, D_s is usually negative in tokamaks. Thus, Eq. (30) is more appropriate for tokamaks than for screw pinches. For this mode

$$\frac{\lambda}{a} \approx \left| \frac{\tau_a^{1/2} \, \omega_{*eT}^{1/2}}{\tau_s^{1/4}} \; \frac{\gamma_I^{3/8}}{\gamma_T^{5/8}} \right| . \tag{31}$$

(ii) $D_s = 0$.

In this case $\frac{\tilde{\lambda}^2}{\tilde{\chi}^2} = 0$, and we obtain the tearing-drift dispersion relation of Hazeltine et al.[2]

$$\omega(\omega - \omega_{*i})(\omega - \omega_{*e} - \alpha_2 \omega_{*T})^3 \alpha_1^3 = i \gamma_T^5, \tag{32}$$

where

$$\gamma_T = \left[\frac{a \Delta' \Gamma(\frac{1}{4})}{2\pi \, \Gamma(\frac{3}{4})} \right]^{4/5} \frac{(a^2 k_{\parallel}')^{2/5}}{\tau_s^{3/5} \tau_a^{2/5}}$$

Now, we, as did Hazeltine et al., obtain

$$\omega = \omega_0 + \omega_1 = \omega_{*eT} + \frac{.43 \, i \, \omega_{*eT} \, \omega_{*T}}{\nu_c} + \left[\frac{i \gamma_T^5}{\omega_{*eT}(\omega_{*eT} - \omega_{*i})} \right]^{1/3} \tag{33}$$

and instability from the last term with $D_s = 0$ and $\Delta' > 0$.

Correspondingly, Drake and Lee[3] give Eq. (33) with .43 replaced by $\frac{315}{32}$, apparently because they did not include electron-electron collisions. In both Eqs. (30) and (33), the electron temperature gradient instability, the ω_{*T} term, occurs. The ω_{*T} term is believed to be important in explaining enhanced particle and energy transport in tokamaks[16]. The difference between Eqs. (30) and (33) exists in the last term, where the interchange term of Eq. (30) is larger than the tearing mode contribution of Eq. (33). In both cases, the energy source lies outside the singular layer.

For the mode here

$$\frac{\lambda}{a} \approx \left| \frac{1}{\tau_s} \frac{\omega_{*eT}^{2/3}}{\gamma_T^{5/3}} \right| . \tag{34}$$

B. The case $\omega_* << \gamma$.

Again, we consider two subcases: D_s positive, and $D_s = 0$.

(i) D_s positive.

In the zero drift limit and with $\alpha_1 = 1$, the dispersion relation, Eq. (27), reduces to

$$\Delta = \frac{(-i\omega)^{5/4}\, 2\pi\, \tau_a^{1/2}\, \tau_s^{3/4}}{a\,(k_{\parallel}' a^2)^{1/2}} \; \frac{\Gamma\left[\frac{3}{4} - \frac{D_s}{(-i\omega)^{3/2}} \frac{(k_{\parallel}' a^2)}{4\tau_a \tau_s^{1/2}}\right]}{\Gamma\left[\frac{1}{4} - \frac{D_s}{(-i\omega)^{3/2}} \frac{(k_{\parallel}' a^2)}{4\tau_a \tau_s^{1/2}}\right]} . \tag{35}$$

This is identical to Eq. (92) of Johnson et al.[4] and is the classical resistive dispersion relation for the incompressible cylindrical pinch case. We note the interchange scaling in the arguments of the gamma functions and the tearing mode scaling outside. The assertion made with respect to Eq. (2), that there are only real resistive growth rates in the incompressible zero drift case has been reaffirmed here.

The gamma function $\Gamma(z)$ is analytic over the finite part of the z plane with the exception of points $z = 0, -1, -2$, etc., where it has simple poles. In the zero drift case, given by Eq. (35), when D_s is positive regardless of the value of Δ', there is always an instability. When Δ' vanishes, the growth rates are given by

$$(-i\omega)^3 = \frac{D_s^2 \, (a^2 k_{\parallel}')^2}{(4n+1)^2 \tau_s \tau_a^2} \, , \text{ where } n = 0, 1, 2, \text{etc.} \tag{36}$$

and which is similar to Eq. (9d).

With diamagnetic frequency, ω becomes complex, and the result depends sensitively on how small ω_{real} is, that is, how close z is to the negative real axis. The $\gamma \gg \omega_*$ limit is of practical interest, because, for $\gamma \sim \omega_{*eT}$, one would normally not expect drift effects to reduce γ below the zero drift limit. In fact, Cordey[17] has recently pointed out that resistive interchange instabilities occur in the Levitron experiment with experimental parameters reasonably close to those given here and with γ_I of the order of ω_{*eT}. However, since D_s is normally negative in tokamaks including the m = 1, n = 1 mode, in which case D_s is small, the limit here is primarily of interest for screw pinch type experiments.

(ii) $D_s = 0$.

For this case, we obtain from Eq. (32), where we do not include kinetic theory corrections,

$$-i\omega = \gamma_T , \tag{37}$$

for $\Delta' > 0$, and the classical tearing mode results.

Finally, we note that the drift dispersion relation derived here can be numerically studied to determine stability regimes and growth rates of the drift-interchange and drift-tearing modes.

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