MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Asymptotic Theory of Diffuse High Beta Magnetohydrostatic Equilibria in Three Dimensions

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IPP 6/166 January 1978

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Abstract

To demonstrate the possibility of constructing asymptotic diffuse high beta magnetohydrostatic equilibria as solutions of a boundary value problem in arbitrary toroidal domains, an asymptotic equilibrium theory is developed for large aspect ratio and small deviations from axial symmetry with many toroidal periods (old Scyllac scaling). Given two arbitrary profiles (e.g. the pressure ratio β and the rotation number μ as functions of the volume), and given (within the scaling) an arbitrary boundary at which the magnetic field is required to be tangential, there is a formal power series solution of the magnetohydrostatic equations. To leading order, this solution is obtained from a coupled set of quasi-linear elliptic equations in two dimensions. Iteration schemes are described for solving this set numerically. The details depend crucially on how β and μ are scaled. The lowest order pressure surfaces depend on the corrugation of the boundary for high β, but not for low β (in the latter case, the lowest order problem reduces to the well-known equilibrium equation in two dimensions). For high β and large μ the pressure is always constant at the boundary to leading order, while for high β and finite μ the lowest order pressure surfaces in general intersect the boundary, thus causing a high current boundary layer unless μ vanishes in all orders. As a consequence, the corrugation of the wall must be judiciously chosen in high beta stellarators. The present investigation opens up a new theoretical approach to a variety of high beta magnetic confinement problems in three dimensions, such as plasma heating, stability, adiabatic compression, and diffusion.

I. Introduction

Despite their extreme importance to the problem of controlled thermonuclear fusion, the magnetohydrostatic equations

$$\underset{\sim}{B} \times \operatorname{curl} \underset{\sim}{B} + \nabla p = 0, \quad \operatorname{div} \underset{\sim}{B} = 0 \tag{1}$$

are not fully explored mathematically. In particular, it is not known whether they have nontrivial solutions (i.e. solutions with non-constant plasma pressure p) satisfying suitable conditions at the boundary of an arbitrary toroidal domain. The difficulty is due to the presence of real characteristics (viz. the field lines of the magnetic field $\underset{\sim}{B}$ counted twice) in addition to imaginary ones, and it has led to the conjecture[1] that no well-posed problem can be formulated unless attention is restricted to some special symmetry such as axial symmetry or reflection symmetry to a plane.

With axial symmetry, the real characteristics can be integrated out to leave two arbitrary profiles in the remaining elliptic problem. With reflection symmetry the magnetic field lines are automatically closed, and an iteration which alternates between an elliptic problem with one arbitrary profile (a second profile is already assigned by the symmetry) and a quasi-hyperbolic problem along the closed real characteristics converges[2]. Extrapolating from these solvable cases one would anticipate that a general well-posed magnetohydrostatic boundary value problem (if there is any at all) must at least include two prescribable profiles, and that the correct boundary condition is $\underset{\sim}{n} \cdot \underset{\sim}{B} = 0$ rather than p = const (this is a genuine distinction if all field lines are closed).

In view of the practical interest in plasma equilibria without symmetries (e.g. stellarators, effect of discrete coils upon axially symmetric tokamaks, bumpy torus), and lacking a mathematical theory of these, it is natural to ignore the difficulty by trying to compute approximate equilibria regardless of the existence of exact ones. One can either try to solve some discrete version of the full three-dimensional problem numerically[3, 4, 5], or one can simplify the problem by looking for formal asymptotic solutions in the vicinity of a solvable case. Even though these two approaches are complementary, it is hoped that there is sufficient overlap for testing purposes.

The question of convergence is usually not considered in either approach. In the numerical approach this would require an examination of the limit of infinite mesh size, while only finite meshes can be used in practice. In the asymptotic approach this would require an examination of arbitrarily high orders of the expansion, while only the leading order is considered in practice. However, the asymptotic approach has the advantage of yielding, regardless of convergence, results of practical use: If the expansion parameter is small, a truncated asymptotic solution series satisfies the magnetohydrostatic equations approximately. Therefore, the time derivatives in the full magnetohydrodynamic equations are small, and the motion away from an asymptotic equilibrium is slow.

Due to the presence of several parameters, there is a variety of different scalings leading to qualitatively different asymptotic problems. One can distinguish between geometrical parameters (i.e. parameters characterizing the boundary) and profile parameters (i.e. parameters characterizing the prescribable functions). The most significant profile parameters are the characteristic values of the pressure ratio β (defined as $\beta = (p - p_{min})/(p - p_{min} + \frac{1}{2} B^2)$) and of the rotation number μ (defined as the ratio of the numbers of poloidal and toroidal traversals of a field line as one follows it forever). Given a definite scaling of the geometrical parameters, there is still a variety of different asymptotic problems depending on how β and μ are scaled. The following argument indicates that high β and small μ leads to difficulties which otherwise are not present.

According to the equilibrium equations (1), the limit $\mu \to 0$ (closed field lines) is singular unless $\beta = 0$ (constant pressure) because the boundary condition $\underset{\sim}{n} \cdot \underset{\sim}{B} = 0$, in general, implies that p is constant at the boundary, while for $\mu = 0$ it does not unless $\beta = 0$. In axial asymmetry, for instance, p is a function of the distance from the symmetry axis if $\mu = 0$, and thus cannot be constant at any closed surface unless $\beta = 0$. As a consequence, there is a high current boundary layer if μ is small, unless β is small, too. Within an expansion, a boundary layer appears whenever μ is scaled so that it does not enter the leading order but appears in higher orders, and the magnitude of μ which is required to avoid this difficulty increases with β .

The low β asymptotic equilibrium problem has been treated in the past with great generality[6-9], but the substantially more difficult high β problem has been attacked only with more or less special assumptions such as a sharp boundary[10-14], closed magnetic field lines[15], or concentrically circular unperturbed pressure surfaces[16]. It is the purpose of the present paper to develop a more complete theory of asymptotic high β equilibria with continuous profiles. Thus, we consider (within the old Scyllac scaling) an arbitrary perturbation of an arbitrary axially symmetric domain, and look for formal solutions of the boundary value problem with two arbitrarily prescribed profiles.

As in previous investigations, the problem may be viewed as a perturbation about a unidirectional magnetic field. However, our investigation differs from previous ones in that we consider the perturbed boundary value problem rather than try to perturb a given equilibrium. This is a genuine distinction because the unperturbed boundary value problem is degenerate: There are many solutions in zeroth order, and which of these must be chosen depends on the perturbation of the boundary. Technically, the zeroth order solution is determined by a solvability condition appearing in second order. Consequently, the perturbed problem is two-dimensional even if the unperturbed problem has one-dimensional solutions, assuming a circular unperturbed boundary does not simplify the problem, and assuming concentrically circular unperturbed cross-sections even cripples it because this singles out a cumbersome subclass of data for which a perturbed solution exists. Correspondingly, it is natural to consider arbitrary axially symmetric unperturbed boundaries rather than restrict attention to circular ones.

Even though our investigation indicates that the expansion exists to all orders, we carry it out only as far as is necessary to determine the equilibrium to leading order. Thus, our chief result is a set of nonlinear equations in two dimensions which allows calculation of the lowest order equilibrium from two profiles, from the axially symmetric unperturbed boundary, and from its three-dimensional corrugation. We find significant distinctions depending on how β and μ are scaled. At low β the equilibrium becomes independent of the corrugation to leading order, while at high β the effect of the corrugation competes with that of the profiles (arbitrarily small distortions of the boundary cause finite shifts of the plasma). For large μ and high β the unperturbed boundary turns out to be a pressure surface to leading order for any choice of the corrugation, while for finite μ (which is equivalent to a vanishing lowest order net toroidal current within each pressure surface) and high β this is so only for special corrugations. The practical significance of these facts is that appropriately chosen small distortions of an axially symmetric boundary can be used instead of a net toroidal current in order to confine the plasma. The possibility of high beta stellarators is based on this fact.

2. Surface quantities

The content of this section although not needed for carrying out our asymptotic expansion, is essential for a discussion of the results. The surfaces of constant pressure play an important role in the theory of magnetohydrostatic equilibria. It is often assumed that these surfaces have a simple topology (i.e., that they form one nested set of toroids), and the concept of surface quantities was originally introduced[17] for this case. However, since any reasonably general formulation of the equilibrium problem must allow for complex topologies [18], a somewhat more general formulation of various definitions and relations is necessary. Therefore, we make no assumption about the global structure of the pressure surfaces. In order to include closed-line equilibria, we do not even assume that the boundary is a pressure surface. However, we do assume that the domain is a toroid, and that the magnetic field is tangential at the boundary.

A quantity F satisfying the relation $\nabla p \times \nabla F = 0$ is called a surface quantity. If, in addition, F is a single-valued function of p, we call it a global surface quantity. In a simple topology every surface quantity is global, while in a complex topology a surface quantity can have different values on different parts of a disconnected surface, and therefore be a multi-valued function of p . For our purposes it suffices to consider global surface quantities. It is often convenient to express these as functions of the volume v .

To be specific, we define the volume of the surface $p = p_0$ as the volume of that part of the domain in which $p \geq p_0$. Thus, the volume is a decreasing function of p ; it is zero for $p > p_{max}$, and it equals the volume of the whole domain for $p \leq p_{min}$. As a function in space, the volume is given in the obvious way, i.e. $v(\underset{\sim}{x}) = v(p(\underset{\sim}{x}))$. We now define a surface average (on the surface $v = v_0$) by

$$\langle \cdots \rangle = \iint\limits_{v=v_0} \frac{d^2S}{|\nabla v|} \cdots \quad , \qquad (2)$$

where the integral is over the entire surface regardless of whether it is connected; equivalently,

$$\langle \cdots \rangle = \frac{d}{dv_0} \iiint\limits_{v \leq v_0} d^3\tau \cdots \quad . \qquad (3)$$

Clearly, the surface average of any quantity is a global surface quantity, and any global surface quantity F is characterized by $F = \langle F \rangle$.

We shall occasionally need the average of the Laplacian of a global surface quantity. An elementary calculation involving Gauss's theorem yields

$$\langle \Delta F \rangle = \frac{d}{dv}\left(K \frac{dF}{dv}\right), \tag{4}$$

where

$$K(v) = \langle |\nabla v|^2 \rangle . \tag{5}$$

Clearly, $\langle \Delta F \rangle$ vanishes if F is a constant. The converse is also true unless F has singularities or unless we deal with the extremely special case that both the maximum and the minimum of the pressure are attained at surfaces rather than at field lines. To show this, we note that $\langle \Delta F \rangle = 0$ implies $dF/dv = C/K$ with some constant C. The assertion then follows from the fact that K is zero at the maximum (minimum) of p unless this is attained at a whole surface.

In a simple topology, the poloidal and toroidal magnetic fluxes (denoted by χ and ψ) and currents (denoted by I and J) can be calculated according to[7]

$$d\chi/dv = \langle \underset{\sim}{B} \cdot \nabla \Theta_p \rangle, \tag{6}$$

$$d\psi/dv = \langle \underset{\sim}{B} \cdot \nabla \Theta_t \rangle, \tag{7}$$

$$dI/dv = \langle \mathrm{curl}\, \underset{\sim}{B} \cdot \nabla \Theta_p \rangle, \tag{8}$$

$$dJ/dv = \langle \mathrm{curl}\, \underset{\sim}{B} \cdot \nabla \Theta_t \rangle, \tag{9}$$

where Θ_p and Θ_t are angle-like coordinates increasing by one in the poloidal and toroidal directions. respectively. The rotation number is then given by

$$\mu = d\chi / d\psi . \tag{10}$$

In a complex topology, or if the boundary is not a pressure surface, we have to define the coordinates Θ_p and Θ_t in a more general way in order to give a meaning to Eqs. (6-10). One possibility is to introduce first the toroidal coordinate Θ_t such that $\underset{\sim}{B} \cdot \nabla \Theta_t > 0$ throughout (if the field lines do not travel around the torus, we put $\underset{\sim}{B} \cdot \nabla \Theta_t \equiv 0$), and then to construct the poloidal coordinate Θ_p such that $\nabla U \cdot (\nabla \Theta_p \times \nabla \Theta_t) \equiv 1$. Each pressure surface is now automatically mapped onto a unit square in the (Θ_p, Θ_t) plane. We note that there may be discontinuities of ∇U , and hence also of Θ_p , but that this does not invalidate our statements.

Equations (6-9) now have an invariant meaning in arbitrary topologies. At a disconnected surface they yield the sums, over its various parts, of the fluxes and currents in each part. Accordingly, the global surface quantity μ defined by Eq. (10) cannot, in general, be interpreted as a rotation number any more. A rotation number can be assigned only to field lines or, in other words, to one part of a disconnected pressure surface (it is defined relative to a magnetic axis), and hence is not a global surface quantity. Nevertheless, we still refer to μ as the "rotation number".

3. Formulation of the asymptotic boundary value problem

Starting from the usual cylindrical coordinates (r, φ, z), we introduce orthogonal toroidal coordinates (ξ, η, ζ) by

$$\xi = r - R, \quad \eta = z, \quad \zeta = R\varphi, \tag{11}$$

where R is a given length. Thus, ξ and η are Cartesian coordinates in the poloidal planes ζ = const, and ζ measures the length along the toroidal circle with radius R. The metric tensor is given by

$$(ds)^2 = g_{ik}\, di\, dk = (d\xi)^2 + (d\eta)^2 + (1+\xi/R)^2 (d\zeta)^2, \tag{12}$$

where the summation convention is used with all indices running over (ξ, η, ζ).

If the magnetic field is represented as $\underset{\sim}{B} = B^i \underset{\sim}{e}_i$, where $\underset{\sim}{e}_i = \partial \underset{\sim}{x}/di = g_{ik} \nabla k$ is a covariant basis vector, Eqs. (1) become

$$p_{/i} + B^k\left[(g_{kl}B^l)_{/i} - (g_{il}B^l)_{/k}\right] = 0, \quad (\sqrt{g}\, B^i)_{/i} = 0, \tag{13}$$

where $g = \det(g_{ik})$, and a subscript $/i$ denotes the derivative with respect to i. To exploit the Cartesian nature of the poloidal coordinates, we split vectors into their poloidal and toroidal parts, thus writing $\underset{\sim}{B} = \underset{\sim}{B}^{\perp} + B^{\zeta} \underset{\sim}{e}_{\zeta}$, and $\nabla = \nabla^{\perp} + \underset{\sim}{e}^{\zeta}\, \partial/\partial\zeta$.

Our equations then are

$$\nabla^{\perp}(p+\tfrac{1}{2}|\underset{\sim}{B}^{\perp}|^{2})-(\underset{\sim}{B}^{\perp}\cdot\nabla)\underset{\sim}{B}^{\perp} \\ +B^{\zeta}[-\underset{\sim}{B}^{\perp}{}_{/\zeta}+\nabla^{\perp}(1+\xi/R)^{2}B^{\zeta}]=0, \tag{14}$$

$$(p+\tfrac{1}{2}|\underset{\sim}{B}^{\perp}|^{2})_{/\zeta}-\underset{\sim}{B}^{\perp}\cdot\nabla^{\perp}(1+\xi/R)^{2}B^{\zeta}=0, \tag{15}$$

$$\nabla^{\perp}\cdot(1+\xi/R)\underset{\sim}{B}^{\perp}+(1+\xi/R)B^{\zeta}{}_{/\zeta}=0, \tag{16}$$

where $\underset{\sim}{B}^{\perp}{}_{/\zeta}=B^{\xi}{}_{/\xi}\underset{\sim}{e}_{\xi}+B^{\eta}{}_{/\eta}\underset{\sim}{e}_{\eta}$ (i.e. the poloidal basis vectors are treated as constants).

We wish to impose the boundary condition $\underset{\sim}{n}\cdot\underset{\sim}{B}=0$ at the boundary of a domain which we define as follows: Let D be a simply connected domain in the (ξ,η) plane which includes the origin. The domain $(\xi,\eta)\in D$, $0\le\zeta\le 2\pi R$ is then an axially symmetric toroid. We perturb this toroid by applying a corrugation, i.e. by shifting its boundary in the normal direction by a distance d which depends on the position at the unperturbed boundary. We assume that this corrugation is periodic in ζ with period L (the ratio $2\pi R/L$, of course, must be a positive integer).

Our domain is characterized by four lengths: the major radius R , a characteristic radius a of the poloidal domain D , the period L , and a characteristic magnitude of the corrugation d . The associated three non-dimensional parameters are scaled by introducing a small parameter ε , and assuming that

$$L/a = O(1), \; d/L = O(\varepsilon), \; R/a = O(1/\varepsilon^2). \tag{17}$$

The first of these assumptions means that the aspect ratio of one period is finite, and it allows us to look for solutions with equal scale lengths in all three spatial directions by assuming that all three components of the operator $L\nabla$ are $O(1)$. The second assumption implies that the angle between the normals on the unperturbed and the perturbed boundary is small, and it allows the magnetic field to deviate very little from a purely toroidal one, $|\underset{\sim}{B}^{\perp}|/B^{\zeta} = O(\varepsilon)$. The third assumption means that the total aspect ratio is large, and it is made to allow the effect of toroidal curvature to be compensated by that of the corrugation. The scaling (17) has been termed "old Scyllac scaling".

The parameter ε enters the problem through the factor $\xi/R = O(\varepsilon^2)$ and through the boundary condition. We look for formal solutions which are power series in ε , thus writing, for instance, $p = \sum \varepsilon^n p_{(n)}(\xi, \eta, \zeta)$. The boundary condition must be expanded, too, because the corrugation, according to our scaling, can be written as $d = \varepsilon d_{(1)}(\xi, \eta, \zeta)$ with $d_{(1)}/L = O(1)$. As a result we find that the normal component of $\underset{\sim}{B}$ at the unperturbed boundary is given order by order in terms of $d_{(1)}$ and its derivatives. In zeroth order this normal component vanishes. In higher orders it becomes increasingly cumbersome, but is consistent with $\operatorname{div} \underset{\sim}{B} = 0$, i.e. it satisfies the compatibility condition $\iint d^2S\, \underset{\sim}{n} \cdot \underset{\sim}{B} = 0$ to all orders.

As already mentioned, the problem is intrinsically underdetermined from the boundary, and two profiles must be prescribed in each order to specify a unique formal solution. We shall introduce these profiles when appropriate.

4. Zeroth order

In zeroth order our equations are solved by

$$\underset{\sim}{B}_{(0)} = b\, \underset{\sim}{e}_\zeta, \quad p_{(0)} = \text{const} - \frac{1}{2} b^2 \tag{18}$$

with an arbitrary function $b(\xi, \eta)$. This is the most general solution with $\underset{\sim}{B}_{(0)}^{\perp} = 0$. Solutions with $\underset{\sim}{B}_{(0)}^{\perp} \neq 0$ would imply $\mu = O(1/\varepsilon^2)$, and discarding them corresponds to specifying a trivial μ profile in zeroth order.

To avoid complications, we now assume that the function b is positive, thus excluding field reversal. As a consequence, b is a global surface quantity to lowest order. The zeroth order volume can be calculated from the function b unless this is a constant. Since the surfaces are axially symmetric in this order, this volume is proportional to the area in a poloidal cut, i.e. $\upsilon(\xi,\eta)=2\pi R\sigma(\xi,\eta)$, where $\sigma(\xi,\eta)=\sigma(b(\xi,\eta))$, and $\sigma(b_0)$ is the area of that part of the domain in which $b \leq b_0$. This implies that $db/d\upsilon \geq 0$, which is consistent with $dp/d\upsilon \leq 0$, $b>0$, and $dp/d\upsilon + b\, db/d\upsilon = 0$. To compute fluxes using Eqs. (6-9), we construct the required coordinates Θ_p and Θ_t according to $\Theta_t = \zeta/2\pi R$ and $\partial(\sigma,\Theta_p)/\partial(\xi,\eta)=1$. A simple calculation then yields

$$d\psi/d\upsilon = b/2\pi R\,(1+O(\varepsilon)), \tag{19}$$

$$dI/d\upsilon = -2\pi R\, db/d\upsilon\,(1+O(\varepsilon)). \tag{20}$$

The fluxes χ and J vanish in this order.

The function b will be constrained by a solvability condition appearing in second order, and this will leave just enough freedom to prescribe the profile $b(\upsilon)$ or $b(\sigma)$. Thus, our main objective will be to determine the geometry of the lowest order pressure surfaces, or in other words, the function $\upsilon(\xi,\eta)$ or $\sigma(\xi,\eta)$.

Since $\beta = (b_{max}^2 - b^2)/b_{max}^2 + O(\varepsilon)$, $\beta = O(\varepsilon)$ or $\beta = O(1)$, depending on whether $b(\upsilon)$ is a constant or not. We are primarily interested in high β equilibria. But, since we also want to examine the transition to low β, we make no assumption about the profile $b(\upsilon)$ at this stage.

5. First order

The first order of our equations is

$$\nabla^{\perp} q_{(1)} - b\,\underset{\sim}{B}^{\perp}_{(1)/\zeta} = 0,$$

$$q_{(1)/\zeta} - b\,B^{\zeta}_{(1)/\zeta} - \underset{\sim}{B}^{\perp}_{(1)} \cdot \nabla b = 0, \qquad (21)$$

$$\nabla \cdot \underset{\sim}{B}^{\perp}_{(1)} + B^{\zeta}_{(1)/\zeta} = 0,$$

where $p_{(1)}$ has been eliminated in favor of the quantity

$$q_{(1)} = p_{(1)} + b\,B^{\zeta}_{(1)}. \qquad (22)$$

The boundary condition is

$$\underset{\sim}{n} \cdot \underset{\sim}{B}^{\perp}_{(1)} = b\,d_{(1)/\zeta} \qquad (23)$$

in this order (we recall that d is the corrugation of the boundary).

To reduce the system (21) to one single equation, we now split each quantity into its ζ -average (denoted by an overbar) and its varying part (denoted by a tilde). Thus, $u = \bar{u} + \tilde{u}$, and $\bar{\tilde{u}} = 0$. We also introduce an integration operator, applicable only to quantities with vanishing average, such that the average of the integral vanishes, and denote this by a hat. Thus, for $\bar{u} = 0$, $\hat{u}_{/\zeta} = u$, and $\bar{\hat{u}} = 0$.

The average of the system (21) is

$$
\begin{aligned}
&\nabla^{\perp}\bar{q}_{(1)} = 0,\\
&\bar{\underset{\sim}{B}}^{\perp}_{(1)}\cdot\nabla b = 0,\\
&\nabla^{\perp}\cdot\bar{\underset{\sim}{B}}^{\perp}_{(1)} = 0.
\end{aligned}
\tag{24}
$$

Hence $\bar{q}_{(1)}$ is an arbitrary constant, and

$$\bar{\underset{\sim}{B}}^{\perp}_{(1)} = \underset{\sim}{e} \times \nabla\Phi, \tag{25}$$

where $\Phi(\xi,\eta)$ is an arbitrary function satisfying

$$\nabla b \times \nabla\Phi = 0, \tag{26}$$

and $\underset{\sim}{e}$ is the unit vector in the toroidal direction. There is no restriction on Φ if b is a constant; if b is not a constant, Φ is a surface quantity to lowest order. The averaged boundary condition,

$\underset{\sim}{n} \cdot \overline{\underset{\sim}{B}}{}^{\perp}_{(1)} = 0$, requires that ϕ be constant at the boundary. If ϕ is not a constant, this implies that b is constant at the boundary, too, so that the boundary is a pressure surface. If ϕ is a constant, no such conclusion can be drawn. The averaged part of the toroidal magnetic field remains undetermined in this order.

Applying the integration operator to the varying part of the first two equations of the system (21) and introducing the abbreviation $Q = \hat{\hat{q}}_{(1)}$ allows us to express the varying part of the first order magnetic field in terms of Q :

$$\underset{\sim}{\tilde{B}}{}^{\perp}_{(1)} = \frac{1}{b} \nabla^{\perp} Q_{/\zeta} , \quad \tilde{B}^{\zeta}_{(1)} = \frac{1}{b}\left(Q_{/\zeta\zeta} - \frac{1}{b} \nabla b \cdot \nabla Q \right). \tag{27}$$

The varying part of the third equation is then equivalent to

$$b(\Delta^{\perp} Q + Q_{/\zeta\zeta}) - 2 \nabla b \cdot \nabla Q = 0 , \tag{28}$$

and the boundary condition (23) is equivalent to

$$n \cdot \nabla^{\perp} Q = b^2 \tilde{\alpha}_{(1)} . \tag{29}$$

The problem (28-29), being a Neumann problem, has a solution which is unique within an added constant. This constant can be determined by applying the constraint $\overline{Q} = 0$ (which is compatible with the boundary condition), but this is unnecessary because the constant does not contribute to the magnetic field or the pressure. It should be stressed that the Neumann problem, having coefficients which do not depend on ζ, separates when Fourier-decomposed. Therefore, it is in fact a problem in two dimensions.

Collecting results, we now state that, given the surface quantities b and Φ, the first order fields and pressure can be calculated from the solution Q of the Neumann problem according to

$$\underset{\sim}{B}^{\perp}_{(1)} = \underset{\sim}{e} \times \nabla\Phi + \frac{1}{b}\nabla^{\perp} Q_{/\zeta},$$

$$B^{\zeta}_{(1)} = T + \frac{1}{b}\left(Q_{/\zeta\zeta} - \frac{1}{b}\nabla b\cdot\nabla Q\right), \tag{30}$$

$$p_{(1)} = -bT + \frac{1}{b}\nabla b\cdot\nabla Q + \mathrm{const},$$

where T is an arbitrary function of ξ and η.

In the following we assume that ${}_{m}\Phi$ is a global surface quantity, thus ignoring possible ramifications in complex topologies. The poloidal magnetic flux and the toroidal current are then given in terms of the profile $\Phi(v)$:

$$d\chi/dv = \varepsilon\, 2\pi R\, d\Phi/dv\,(1 + O(\varepsilon)), \tag{31}$$

$$dJ/dv = \frac{\varepsilon}{2\pi R}\frac{d}{dv}\left(K\frac{d\Phi}{dv}\right)(1 + O(\varepsilon)). \tag{32}$$

These relations show that $\mu = O(1/\varepsilon)$ if Φ is not a constant, but it is otherwise smaller, and that $J = 0$ implies constant Φ. Therefore, we shall refer to equilibria with $\mu = O(1)$ (or smaller) as "stellarators". This is consistent with the usual definition of a stellarator as an equilibrium with $J = 0$ (the rotational transform is provided by external currents rather than by the plasma current).

It will turn out that the constraint appearing in second order allows us to prescribe not only the profile $b(v)$, but also the profile $\Phi(v)$. Therefore, four different scalings of profiles are possible within the present theory (viz. $\beta = O(\varepsilon)$ or $\beta = O(1)$, and $\mu = O(1)$ or $\mu = O(1/\varepsilon)$, depending on whether b is a constant or not, and on whether Φ is a constant or not.

6. Second order

The second order of our equations is

$$\begin{aligned}
&\nabla^{\perp} q_{(2)} - b\, \underset{\sim}{B}^{\perp}_{(2)/\zeta} = \underset{\sim}{A}^{\perp}, \\
&q_{(2)/\zeta} - b\, B^{\zeta}_{(2)/\zeta} - \underset{\sim}{B}^{\perp}_{(2)} \cdot \nabla b = A^{\zeta}, \qquad (33) \\
&\nabla \cdot \underset{\sim}{B}^{\perp}_{(2)} + B^{\zeta}_{(2)/\zeta} = 0,
\end{aligned}$$

where $q_{(2)} = p_{(2)} + b B^{\zeta}_{(2)}$, and the inhomogeneous terms are given by

$$\underset{\sim}{A}^{\perp} = -\frac{1}{2}\nabla^{\perp}|\underset{\sim}{B}^{\perp}_{(1)}|^{2} + (\underset{\sim}{B}^{\perp}_{(1)}\cdot\nabla)\underset{\sim}{B}^{\perp}_{(1)}$$

$$-2\alpha b\nabla^{\perp}b\xi/a + B^{\zeta}_{(1)}(\underset{\sim}{B}^{\perp}_{(1)/\zeta} - \nabla^{\perp}B^{\zeta}_{(1)}), \tag{34}$$

$$A^{\zeta} = -\frac{1}{2}|\underset{\sim}{B}^{\perp}_{(1)}|^{2}_{/\zeta} + \underset{\sim}{B}^{\perp}_{(1)}\cdot\nabla^{\perp}B^{\zeta}_{(1)},$$

where $\alpha = a/\varepsilon^{2}R$ is the leading order of the inverse aspect ratio. Since we merely wish to determine under what conditions the system (33) has solutions rather than actually solving it, the second order boundary condition is not explicitly needed. The homogeneous system corresponding to our system (33) is identical with the system (21) which we have already discussed. Thus, its varying part has only the trivial solution, and it suffices to consider the averaged part of the inhomogeneous system:

$$\nabla^{\perp}\overline{q}_{(2)} = \overline{\underset{\sim}{A}}^{\perp}, \tag{35}$$

$$\overline{\underset{\sim}{B}}^{\perp}_{(2)}\cdot\nabla b = -\overline{A}^{\zeta}, \tag{36}$$

$$\nabla\cdot\overline{\underset{\sim}{B}}^{\perp}_{(2)} = 0. \tag{37}$$

The unknowns $\overline{q}_{(2)}$ and $\overline{\underset{\sim}{B}}^{\perp}_{(2)}$ are decoupled in these equations. Equation (35) has a solution $\overline{q}_{(2)}$ only if

$$\overline{A}^{\xi}_{/\eta} = \overline{A}^{\eta}_{\xi}. \tag{38}$$

This condition is also sufficient because the domain D is simply connected; otherwise there would be additional periodicity conditions. When expressed in terms of the quantities b , Φ , and Q , the condition (38) takes the form

$$\nabla b \times \nabla F = 0,$$

$$F = \frac{d\Phi}{db}\Delta\Phi + \frac{2\alpha b\xi}{a} - \frac{1}{b^3}\overline{|\nabla^\perp Q_{,\zeta}|^2 + Q_{,\zeta\zeta}^2} . \tag{39}$$

The first, second, and third terms in Eq. (39) are due to the rotational transform, toroidal curvature, and corrugation, respectively. If the corrugation vanishes, the last term is zero, and we are left with a large aspect ratio version of the equilibrium equation in axial symmetry. In the limit of infinite aspect ratio ($\alpha \to 0$) the second term is zero, too, and Eq.(39) reduces to the equilibrium condition in plane symmetry.

To discuss the remaining Eqs. (36-37), we integrate Eq. (37) to obtain

$$\overline{B}^\perp_{(2)} = \underset{\sim}{e} \times \nabla G \tag{40}$$

with an arbitrary function $G(\xi, \eta)$. Equation (36) is now an equation for G . If we use Eqs. (30) and (34) to express the inhomogeneity $\overline{A}^\zeta$ in terms of first order quantities, this equation takes the form

$$\nabla b \times \nabla G + \nabla\Phi \times \nabla T = 0. \tag{41}$$

If b is not a constant, there are solutions G for arbitrary functions T. However, it can be shown that the averaged part of the appropriate second order boundary condition then requires that T be constant at the boundary unless ϕ is a constant (this is analogous to the first order result that b is constant at the boundary). Nevertheless, as far as the lowest order pressure surfaces are concerned, Eq. (39) is the only constraint if $\beta = O(1)$. The problem then consists in determining these surfaces so that the solution Q of the Neumann problem (28-29) satisfies the constraint (39). As already mentioned, we shall show that this is possible with any two arbitrarily prescribed profiles $b(\upsilon)$ and $\phi(\upsilon)$. If b is a constant ($\beta = O(\varepsilon)$), the constraint (39) reduces to

$$\nabla\phi \times \nabla\Delta\phi = 0, \tag{42}$$

leaving only one profile to be prescribed (e.g. $\Delta\phi$ as a function of ϕ). However, Eq. (41) now is

$$\nabla\phi \times \nabla T = 0, \tag{43}$$

thus being a genuine constraint upon the function T unless ϕ is a constant. In the latter case $(\beta = O(\varepsilon), \mu = O(1))$ one has to proceed to higher orders in order to determine pressure surfaces.

We have no doubt that the expansion in powers of ε can be carried to arbitrary orders although this has not yet been demonstrated. We conjecture that, given two profiles in each order, the n-th order of our equations determines up to the (n-1)-th order all quantities except for the averaged toroidal field. The latter plays an extraordinary role in that it is also determined up to the (n-1)-th order if $\beta = O(\varepsilon)$, but only up to the (n-2)-th order if $\beta = O(1)$; in both cases this is the (n-2)-th nontrivial order.

7. Low beta

We have seen that the lowest order theory depends on whether $\beta = O(1)$ or $\beta = O(\varepsilon)$.
In particular, the lowest order pressure surfaces are the level surfaces of b if $\beta = O(1)$, but those of T if $\beta = O(\varepsilon)$. At first glance, these two sets of surfaces seem unrelated because there is no constraint on T if $\beta = O(1)$, while b is constant if $\beta = O(\varepsilon)$. However, the pressure surfaces are also the level surfaces of Φ in both cases, and since the constraint on Φ is universally valid, there is a smooth transition from one case to the other. In other words, first assuming that $\beta = O(1)$, and then expanding the

result in powers of β yields, to leading order, the same pressure surfaces as assuming $\beta = O(\varepsilon)$. Hence it suffices to consider the scaling $\beta = O(\varepsilon)$.

The various lowest order quantities are then constructed as follows: First choose a function $\Phi(\xi, \eta)$ which is constant at the boundary and whose level surfaces coincide with those of $\Delta\Phi$. Again ignoring possible generalizations in complex topologies, we may determine such a function by solving

$$\Delta\Phi = f(\Phi) \tag{44}$$

with a given function f (which, incidentally, equals the toroidal current density). Given Φ , the first order pressure $p_{(1)}$ may be arbitrarily assigned to each surface of constant Φ (if $f = 0$, Φ is constant, and one must proceed to higher orders to determine the pressure surfaces). Finally, the first order magnetic field is given by

$$\begin{aligned} \underset{\sim}{B}^{\perp}_{(1)} &= \underset{\sim}{e} \times \nabla\Phi + \frac{1}{b}\nabla^{\perp} Q_{/\zeta} , \\ B^{\zeta}_{(1)} &= \frac{1}{b}\left(-p_{(1)} + Q_{/\zeta\zeta}\right) + \text{const}, \end{aligned} \tag{45}$$

where Q is found from the Neumann problem (28-29).

It seems noteworthy that the lowest order pressure surfaces are independent of the corrugation in this low β scaling; the latter enters only the first order magnetic field (viz. through the boundary condition on Q). The low β scaling is also distinguished by the fact that the toroidal curvature (viz. the parameter α) does not enter the lowest order calculation at all. This reflects the fact that for $\beta = O(\varepsilon)$ we could have assumed $R/a = O(1/\varepsilon)$ rather than $R/a = O(1/\varepsilon^2)$, while for $\beta = O(1)$ this would have implied that b is a function of ξ only, thus necessitating a high current boundary layer. With the scaling $R/a = O(1/\varepsilon)$ we would have obtained a large aspect version of the well-known equilibrium equation in axial symmetry instead of Eq. (44), which is the corresponding equation in plane symmetry.

8. High beta

We now turn to the general case of nonconstant profiles $b(\upsilon)$ and $\Phi(\upsilon)$, or in other words, $\beta = O(1)$ and $\mu = O(1/\varepsilon)$. Again ignoring possible ramifications, we assume that any surface quantities are global. The constraint (39) is then written as $F = \langle F \rangle$, where the average (as defined by Eq. (2)) is over the surfaces of constant b . Using Eq. (4), we write this constraint as

$$\Delta \Phi = \frac{d}{dv}\left(K \frac{d\Phi}{dv}\right) + \frac{db}{d\Phi}\left(H - \langle H \rangle\right),$$
$$H = \frac{1}{b^3} \overline{|\nabla^{\perp} Q_{,\zeta}|^2 + Q^2_{,\zeta\zeta}} - \frac{2\alpha b \xi}{a} \qquad (46)$$

With the boundary condition Φ = const Eq. (46) constitutes a Dirichlet problem in two dimensions, provided the right-hand side is given. This problem is coupled with the Neumann problem (28-29).

We now describe an iteration scheme which indicates that there is a solution $v(\xi,\eta)$ once the profiles $b(v)$ and $\Phi(v)$ are specified. Clearly, both b and Φ must be monotonic in v ; b is increasing by definition, and Φ must be monotonic because we have divided by $d\Phi/db$. The iteration proceeds in this way: Starting with some function $v(\xi,\eta)$ which satisfies $\iint d^2S/|\nabla v| = 1$ at each of its level surfaces and which is constant at the boundary, we construct the function $b(\xi,\eta)$ according to $b(\xi,\eta) = b(v(\xi,\eta))$. With this function we solve the Neumann problem to obtain a function $Q(\xi,\eta,\zeta)$. Knowing the functions v , b , and Q , we can compute the right-hand side of Eq. (46) as a function of ξ and η , thus being able to compute a solution $\Phi(\xi,\eta)$ of the Dirichlet problem. A new function $v(\xi,\eta)$ is then constructed as the volume of the surfaces Φ = const (in the same way as we ~~have~~

constructed the volume of the pressure surfaces in Sec.2). The lowest order problem is solved if this iteration converges. Since elliptic operators have smoothing properties, we are confident that it does converge although we have not proven this.

Clearly, the above iteration is appropriate only if the profiles $b(\upsilon)$ and $\phi(\upsilon)$ are given, and different schemes have to be used if different pieces of information are given. In general, any three surface quantities may be prescribed as functions of each other, and which surface quantities one chooses depends on the application.

9. The high beta stellarator case

If $\mu = O(1)$, ϕ is a constant, and the preceding iteration scheme breaks down. Instead we must couple the Neumann problem (28-29) to the constraint $H = \langle H \rangle$. Given the profile $b(\upsilon)$, the following iteration is now appropriate: Starting from some function $\upsilon(\xi, \eta)$, we again construct $b(\xi, \eta)$ and then solve the Neumann problem to obtain a function Q. From b and Q we can compute $H(\xi,\eta)$ according to Eq. (39). The level surfaces of H determine a new function $\upsilon(\xi,\eta)$.

In contrast to the case $\mu = O(1/\varepsilon)$, there is now no boundary condition which guarantees that the boundary is a pressure surface to lowest order and there is no freedom to impose such a condition. If the corrugation vanishes, for instance, Q is zero, and Eq. (39) implies that the lowest order pressure surfaces are the cylindrical surfaces of constant ξ. However, if μ is not zero in all orders, there will be boundary conditions on the pressure in higher orders, requiring a boundary layer unless the present lowest order pressure surfaces do not intersect the unperturbed boundary. Since a boundary layer is undesirable both theoretically and experimentally, one would like to specify the corrugation so that the boundary is a pressure surface in leading order. This should be possible, but has the disadvantage that the required corrugation depends on the pressure profile, so that an arbitrarily small deviation of this profile from the assumed one again causes a boundary layer.

This difficulty can be overcome by assuming that the high β plasma is surrounded by a low β region. This is consistent with the actual setup in high beta stellarator experiments. Accordingly, we specify a profile $b(\sigma)$ which increases in some interval $0 \leq \sigma \leq \sigma_0$, but remains constant for $\sigma > \sigma_0$. If a corrugation is then specified so that the high β plasma is well separated from the wall in the solution this solution should be sufficiently insensitive to changes of data.

Let us finally comment on the stability of these equilibria. A fully relevant investigation of this problem must include a normal mode analysis. Needless to say, lacking even a general equilibrium theory in the past, such an analysis has not yet been carried out. However, there are two approaches to the stability problem which, although of limited relevance, give some indication of what is to be expected. Firstly, a great deal of information has been obtained[14] within the sharp boundary model, i.e. in the limit of a step function $b(\sigma)$. However, all unstable modes except for the ones with no radial modes are lost in this limit, and it is not clear if a theory of these remaining modes is representative of the general case. Secondly, stability criteria[19-22] have been evaluated[23] at the magnetic axis of general confined equilibria (i.e. at a magnetic field line at which the pressure has a local maximum). Since the stability problem is global, it cannot be fully solved by such a local analysis; in particular, growth rates cannot be obtained or even estimated in this way. Both approaches indicate that the present high beta stellarator equilibria are unstable, but that the present high beta equilibria with large rotation number contain stable ones.

On the other hand, the expansions about a magnetic axis also indicate that stable asymptotic high beta stellarator equilibria can be constructed if one perturbs a helically symmetric domain (rather than an axially symmetric one) by introducing toroidal curvature and appropriate corrugations. The asymptotic determination of such equilibria[24], though more involved than the present calculation, follows the same lines.

Acknowledgments

Helpful discussions with J. Nührenberg and D. Lortz are gratefully acknowledged.

This work was completed under the terms of the agreement on association between Max Planck Institut für Plasmaphysik and EURATOM.

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