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Field Line Integration for MHD Equilibria Near a Closed Magnetic Field Line

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Abstract

Magnetic field lines near a given closed field line which is taken as a magnetic axis are calculated analytically. Existence properties are obtained for equilibria whose rotational transform on a magnetic axis is an integer multiple of $\frac{1}{2}$. The results are applied to a Tokamak with a perturbation of the main field, the l = 2 stellarator, the stability properties of equilibria with vanishing rotational transform, and the scaling of the equilibrium ß-value in configurations without net longitudinal current and with small rotational transform.

1. Introduction

We consider a closed field line occurring in a toroidal magneto-hydrostatic equilibrium which is a magnetic axis of the configuration, i.e. which is surrounded by a nested set of toroidal magnetic surfaces. This situation may not only occur at the centre of a toroidal equilibrium but also through islation [1] near a rational surface of an equilibrium. In the neighbourhood of such a field line the field line structure may be calculated analytically with the help of the expansion of an equilibrium around a magnetic axis [2,3]. If the rotational transform is rational on the magnetic axis, $\iota = m/n$, there exist integral side conditions on the equilibrium quantities. These conditions are obtained from the evaluation of the integral

$$q = \oint \frac{dl}{B}$$

(l is the arc length along a field line, B is the strength of the field; the integral extends over a closed field line) in the neighbourhood of the magnetic axis by means of the field line equation. Up to the order of the expansion around the magnetic axis which is considered here, the shear of the equilibrium does not enter and two types of integral side conditions on the equilibrium quantities are found. The first type, which is obtained if the rotational transform on axis is integer, follows from the condition that q be stationary on the magnetic axis. The occurrence of these conditions has been discussed before [2,4]. The second type is obtained if the rotational transform on axis is an integer multiple of $\frac{1}{2}$.

These conditions are obtained from the requirement that q be constant on magnetic surfaces to O(V), where V is the volume inside the magnetic surfaces surrounding the magnetic axis.

2. Field line equation and formal expression for q

In order to obtain our results, we have to make full use of the equilibrium calculation of a three-dimensional toroidal MHD configuration near a magnetic axis, as was described in, for example, [3]. Since this formalism requires a great deal of notation, we give details in the Appendix and present here only as much as is necessary to describe the results.

Introducing the coordinate system ρ, ϕ, l linked to the magnetic axis with curvature κ and torsion τ

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 - 2\tau\rho^2 d\phi dl + \left[(1-\kappa\rho\cos\phi)^2 + \tau^2\rho^2\right] dl^2$$

$$\sqrt{g} = \rho(1-\kappa\rho\cos\phi),$$

and describing the contravariant component B^1 of $\vec{B}$ by

$$B^1 = c_o(l) + c_1(\phi,l)\,\rho + c_2(\phi,l)\,\rho^2 + O(\rho^3) ,$$

the volume V inside a magnetic surface by

$$V = V_2(\phi,l)\rho^2 + V_3(\phi,l)\rho^3 + O(\rho^4),$$

and a field line labelled with the constant ψ by

$$\phi = \phi(V,\psi,l) = \phi_o(\psi,l) + \phi_1(\psi,l)\, V^{\frac{1}{2}} + O(V),$$

we obtain the following expansion for q:

$$q = \int \frac{d\vec{l} \cdot \vec{B}}{B^2} = \int \frac{dl}{B^l} = q_o + q_1\, V^{\frac{1}{2}} + q_2\, V + O(V^{\frac{3}{2}}) \quad ,$$

where

$$q_o = \int dl/c_o$$

$$q_1 = -\, 2 \int c_o^{-1}\, \overset{o}{V}_2^{\,-\frac{1}{2}}\, \kappa \cos\phi_o\, dl \tag{1}$$

$$q_2 = \int \frac{dl}{c_o} \Big[\frac{2}{\overset{o}{V}_2^{\,\frac{1}{2}}}\, \kappa\, \phi_1 \sin\phi_o + \kappa \cos\phi_o \Big(\frac{\overset{o}{V}_3}{\overset{o}{V}_2^{\,2}} + \frac{\overset{o}{V}_{2,\phi}\, \phi_1}{\overset{o}{V}_2^{\,\frac{3}{2}}} \Big)$$

$$+ \frac{1}{\overset{o}{V}_2} \Big(4\kappa^2 \cos^2\phi_o - \frac{\overset{o}{c}_2}{c_o}\Big) \Big] \; , \tag{2}$$

and the superscript ^{o} indicates that the function of ϕ considered has to be taken at ϕ_o, e.g. $\overset{o}{c}_2 = c_2\,(\phi_o, l)$. To lowest order the field line is given by the well-known expression [2]

$$\phi_o = -\, \alpha + \arctan \big[e \tan \Psi \big] \; . \tag{3}$$

Here, e is the half-axis ratio of the elliptical (in second order in the distance from the magnetic axis) plasma cross-section, α (for $e > 1$) is the angle between the binormal of the magnetic axis and the major half-axis, and

$$\Psi = \psi + K_o(1) .$$

The function K_o is given by

$$K_o' = \frac{1}{e + \frac{1}{e}} \left(\frac{j}{c_o} + 2\tau + 2\alpha'\right) ,$$

where the prime indicates the derivative with respect to l and j is the current density on the magnetic axis. K_o is related to the rotational transform on axis ι by

$$2\pi\iota = K_o(L) - K_o(0) - \alpha(L) + \alpha(0) - 2\pi m, \tag{4}$$

where $\alpha(L) - \alpha(0) = n\pi$ and n is the number of half-turns of the elliptical cross-section and m the number of full turns of the normal over the length L of the magnetic axis. Using

$$V_2 = \pi\, c_o q_o \left(e \cos^2 u + \frac{1}{e} \sin^2 u\right) ,$$

where $u = \phi + \alpha$, one obtains from the Fourier decomposition of eq. (1) with respect to ψ the conditions for stationary q:

$$-2(\pi q_o)^{-\frac{1}{2}} \oint \kappa\, c_o^{-\frac{3}{2}} \left(e^{-\frac{1}{2}}\cos\alpha \cos K_o + e^{\frac{1}{2}}\sin\alpha \sin K_o\right) dl = 0 , \tag{5}$$

$$-2(\pi q_o)^{-\frac{1}{2}} \oint \kappa\, c_o^{-\frac{3}{2}} \left(e^{\frac{1}{2}} \sin\alpha \cos K_o - e^{-\frac{1}{2}} \cos\alpha \sin K_o\right) dl = 0, \tag{6}$$

which, in their general form although not correctly written out, were obtained in [4].

In the next order the field line is given by

$$\phi_1 = \left[\cos^2\Psi + e^2\sin^2\Psi\right]^{-1} (\pi c_o q_o e)^{-\frac{1}{2}} \cdot \{- e\kappa(\cos\alpha\sin\Psi - e\sin\alpha\cos\Psi) \tag{7}$$

$$+ \frac{\pi}{2L}\left[(12e\bar{S}_c - 3e^2\delta)\sin\Psi - (12\bar{S}_s - 3e\Delta)\cos\Psi + e^2\delta\sin 3\Psi + e\Delta\cos 3\Psi\right]\},$$

where $\bar{S}_c$ and $\bar{S}_s$ are the shifts with respect to the magnetic axis and δ and Δ the triangularities of the third-order flux surfaces, which are described by V_3:

$$V_3 = V_{31c}\cos u + V_{31s}\sin u + V_{33c}\cos 3u + V_{33s}\sin 3u$$

$$V_{31c} = \pi^2 L^{-1} c_o q_o \left[(3e + \frac{1}{e})\bar{S}_c - \delta\right],$$

$$V_{31s} = \pi^2 L^{-1} c_o q_o \left[(\frac{1}{e} + \frac{3}{e^3})\bar{S}_s - \Delta\right],$$

$$V_{33c} = \pi^2 L^{-1} c_o q_o \left[(e - \frac{1}{e})\bar{S}_c + \delta\right],$$

$$V_{33s} = \pi^2 L^{-1} c_o q_o \left[(\frac{1}{e} - \frac{1}{e^3})\bar{S}_s - \Delta\right].$$

By means of these relations and

$$c_2 = \left[2\tau - \frac{1}{2}K_o'(e + \frac{1}{e}) + \alpha'\right] b_o + 3c_o\kappa^2\cos^2\phi$$

$$- c_o\tau^2 - \frac{1}{4}(b_{o,\phi 1} + c_o'') - \dot{p}V_2/c_o,$$

where

$$b_o = c_o\left[\frac{1}{2}K_o'(e + \frac{1}{e}) - \alpha'\right] + \frac{1}{2}c_o K_o'(e - \frac{1}{e})\cos 2u + \frac{1}{2}c_o\frac{e'}{e}\sin 2u,$$

a tedious calculation starting from eq. (2) yields

$$q_2 = \int dl\ (q_{20} + q_{2c}\cos 2\Psi + q_{2s}\sin 2\Psi)$$

with

$$q_{2c} = \frac{1}{8\pi q_o c_o^4 e^3}\{ 12\kappa^2 c_o^2 e^2 [(e^2+1)\cos^2\alpha - e^2]$$

$$+ (e^2 - 1) [4K_o'^2 c_o^2 e^2 - 4(\tau + \alpha')^2 c_o^2 e^2 - 3c_o'^2 e^2 - e'^2 c_o^2] + 4c_o' e' c_o e(e^2+1)$$

$$+ 16L^{-1}\pi c_o^2 e^2 \kappa\ [(-2\bar{S}_c + e\delta)\cos\alpha + (2\bar{S}_s - e\Delta)\sin\alpha] \}, \qquad (8)$$

$$q_{2s} = \frac{1}{2\pi q_o c_o^3 e^2}\{ 6e^2 c_o \kappa^2 \sin\alpha\cos\alpha - 2\tau c_o e' e - 2\alpha' c_o e e'$$

$$+ K_o' e' c_o (e^2+1) - 2ec_o' K_o' (e^2-1) - 8L^{-1}\pi c_o \kappa\ (\bar{S}_s \cos\alpha + e^2 \bar{S}_c \sin\alpha)\} \qquad (9)$$

Thus

$$\oint dl\ (q_{2c}\cos 2K_o + q_{2s}\sin 2K_o) = 0\ , \qquad (10)$$

$$\oint dl\ (q_{2s}\cos 2K_o - q_{2c}\sin 2K_o) = 0 \qquad (11)$$

are obtained as conditions that q be a function of volume alone to O(V).

The quantity q_{2o} is given by

$$q_{2o} = \frac{1}{8\pi q_o c_o^4 e^3} \{ 4\kappa^2 c_o^{\,2} e^2 \left[(e^2 - 1)\cos^2\alpha - e^2 \right]$$

$$+ (e^2+1) \left[4K_o'^2 c_o^{\,2} e^2 + 4(\tau+\alpha')^2 c_o^{\,2} e^2 + 3c_o'^2 e^2 + e'^2 c_o^{\,2} \right]$$

$$- 4c_o' e' c_o e(e^2-1) - 16 c_o^{\,2} e^3 K_o'(\tau+\alpha') \qquad (12)$$

$$+ 16 L^{-1} \pi c_o^{\,2} e^2 \kappa \left[4(\bar{S}_c \cos\alpha + \bar{S}_s \sin\alpha) - e\,(\Delta \sin\alpha + \delta\cos\alpha) \right] \} + \frac{\dot{p}}{c_o^3} ,$$

so that one obtains on the magnetic axis

$$\frac{\dot{q}}{q} = \frac{\int q_{2o} dl}{q_o} ,$$

which is identical with the expression for $-\ddot{\Phi}/\dot{\Phi}$ given in [3]. Considering eqs. (5,6) and eqs. (10,11) together with eq. (4), we see that eqs. (5,6) and eqs. (10,11) yield nontrivial conditions only for integer values of ι and values of ι which are integer multiples of $\frac{1}{2}$, respectively.

3. Applications

a) Tokamak with toroidal field divertor

As an illustration we consider one of the simplest situations in which eqs. (5,6) show that an equilibrium does not exist. An axially symmetric equilibrium is characterized by $\tau = \alpha' = e' = \kappa' = 0$. Let us now assume $\iota = 1$ (so that $K_o = 1/R$, where R is the major radius) and a perturbation of the main magnetic field so that $c_o' \not\equiv 0$ (e.g. by a toroidal field divertor). An equilibrium of this type does not exist, as is seen from eqs. (5,6) if c_o contains the perturbation with the longest periodicity length compatible with the periodicity condition ($\sim \cos l/R$). The equilibrium adjusts itself in such a way that at least one of the quantities κ and e varies.

b) The l = 2 stellarator

The simplest case of an l = 2 stellarator is described by $\tau = e' = \kappa' = c_o' = 0$, and $\alpha' = n/(2\ R)$, where n is the number of field periods and R the major radius. One then has

$$K_o = \alpha + \iota \frac{1}{R} = \alpha \ (1 + \frac{2}{n}\iota) \quad ,$$

so that eqs. (5,6) give nontrivial conditions only for $\iota = 0$, $\iota = -n$ and eqs. (10,11) only for $\iota = 0$, $\iota = -n$, $\iota = -\frac{n}{2}$ because $\bar{S}_c$, $\delta \sim \cos\alpha$, $\bar{S}_s$, $\Delta \sim \sin\alpha$ (see, for example, [5]). Since

$$\iota = -\frac{n}{2} \frac{(e-1)^2}{(e^2+1)}$$

for j = 0, there are no conditions in the case of vanishing longitudinal current.

c) Stability of $\iota = 0$ equilibria near the θ - pinch

The stability of equilibria with vanishing rotational transform, a plane magnetic axis ($\tau \equiv 0$), **reflexional symmetry**, and $c_o' = \alpha' = j = 0$ was investigated in [6]. There, it was found that the side condition obtained from eq. (10)

$$\oint q_{2c} dl = 0$$

imposed a severe restriction on stable equilibria of this type. Here, we consider general equilibria with vanishing rotational transform which are near an equilibrium with unidirectional field and show that they are unstable for $\dot{p} < 0$. We describe the neighbourhood of an (elliptical) θ-pinch by

$$
\begin{aligned}
e &= e_o + \tilde{e} , \\
c_o &= 1 + \tilde{c} , \\
\tau &= \tau_o + \tilde{\tau} , \\
\alpha &= \alpha_o + \tilde{\alpha} , \qquad \tau_o + \alpha_o' = 0 , \\
j &= \tilde{j} , \qquad K_o' = \tilde{K}' , \\
\kappa &= \tilde{\kappa} , \\
\bar{S}_c &= \tilde{S}_c , \quad \bar{S}_s = \tilde{S}_s , \quad \delta = \tilde{\delta} , \quad \Delta = \tilde{\Delta} ,
\end{aligned}
$$

where the quantities with tilde are of first order. By means of the equilibrium formulae given in the Appendix the following identity may be proved:

$$
\int q_{2o}(e_o^2 + 1)\, dl = \int \left[q_{2o}(e_o^2 + 1) + q_{2c}(e_o^2 - 1) \right] dl
$$

$$
= \frac{1}{8\pi q_o e_o^3} \int \Big\{ 4 e_o^4 \tilde{\kappa}^2 + 2 e_o^2 \Big[(e_o^4 + 1)\Big(\tilde{K}' - \frac{e_o^2 + 1}{e_o^4 + 1} (\tilde{\tau} + \tilde{\alpha}') e_o\Big)^2
$$

$$
+ (\tilde{\tau} + \tilde{\alpha}')^2 e_o^2 \frac{(e_o^2 - 1)^2}{e_o^4 + 1} \Big] + 3 \tilde{c}'^2 e_o^4 + \tilde{e}'^2 e_o^2 \Big\} dl
$$

$$
+ 4\dot{p} \int (\tilde{k}_1^2 + e_o^2 \tilde{k}_2^2)\, dl + \dot{p}(e_o^2 + 1) \int \frac{dl}{c_o^3} ,
$$

where $\tilde{k}_1$ and $\tilde{k}_2$ are given by

$$\tilde{\kappa} \cos\alpha_o = \tilde{k}_1' \ ,$$

$$\tilde{\kappa} \sin\alpha_o = \tilde{k}_2' \ ; \qquad \int \tilde{k}_\nu dl = 0, \quad \nu = 1,2.$$

Combination of this equation with the necessary stability criterion (see, for example, [2,6]) yields the result (for details, see Appendix)

$$- \dot{p}(e_o^2+1) \left\{ - \frac{\int q_{2o} dl}{q_o} + \dot{p}\, q_o^2 \left[\int \frac{g_{\theta\theta}}{|\nabla V|^2} d\zeta - \frac{\left(\int \frac{g_{\theta\zeta}}{|\nabla V|^2} d\zeta \right)^2}{\int \frac{g_{\zeta\zeta}}{|\nabla V|^2} d\zeta} \right] \right\} \tag{13}$$

$$= \frac{\dot{p}}{8\pi q_o e_o^3} \int \Big\{ 4 e_o^4 \tilde{\kappa}^2 + 2 e_o^2 \Big[(e_o^4+1) \Big(\tilde{K}' - \frac{e_o^2+1}{e_o^4+1} (\tilde{\tau} + \tilde{\alpha}') e_o \Big)^2$$

$$+ (\tilde{\tau} + \tilde{\alpha}')^2 e_o^2 \frac{(e_o^2-1)^2}{e_o^4+1} \Big] + 3\, \tilde{c}'^2 e_o^4 + \tilde{e}'^2 e_o^2 \Big\} dl$$

$$- \frac{4 \dot{p}^2}{e_o \cos^2 2\pi\theta + \frac{1}{e} \sin^2 2\pi\theta} \Big[e_o \cos 4\pi\theta \Big(\int \tilde{k}_2^2 dl - \int \tilde{k}_1^2 dl \Big)$$

$$- (e_o^2+1) \sin 4\pi\theta \int \tilde{k}_1 \tilde{k}_2 dl \Big] \ ,$$

which, for $\dot{p} < 0$, can be made negative by appropriately choosing θ.

4. Conclusion

In this paper we have considered the analytic field line integration in the neighbourhood of a magnetic axis and the existence properties which we obtained for rational values of the rotational transform on this magnetic axis. Several remarks appear appropriate.

The expansion has only been carried to third order in the distance from the magnetic axis. As a consequence existence conditions occur only for values of the rotational transform which are integer multipler of $\frac{1}{2}$. It is clear that an arbitrary rational value of ι on axis will lead to integral side conditions on equilibrium quantities if the expansion is carried to sufficiently high order. Considering the expansion as asymptotic, we conclude that one has to satisfy existence conditions for small values of m, n, where $\iota = m/n$, if one wants to avoid drastic changes in equilibrium properties if one passes these values of ι.

Another remark concerns the relation between the existence properties found and the current density parallel to the magnetic field. Writing the current density as

$$\vec{j} = (\vec{B} \times \nabla p)/B^2 + h \vec{B} \quad ,$$

one obtains the following relation (see, for example, [7]) between the variation of q on a rational magnetic surface and the increase of h along the lines of force

$$\Delta h = \frac{\dot{p}}{\dot{\Phi}} \frac{\partial q}{\partial \psi}$$

Thus, we may conclude, for example, that a singularity appears in $O(\rho^2)$ in the parallel current density if eqs. (10), (11) are not satisfied.

Finally, we may add a piece of speculation concerning the scaling of the equilibrium ß-value in configurations without net longitudinal current obtainable for small values of the rotational transform depending on whether eqs. (5,6) and eqs. (10,11) are satisfied or not. For small values of the rotational transform one has

$$\vec{j}\cdot\vec{B}/B^2 \sim \frac{L}{\iota} \frac{\dot{p}}{c_o} \frac{\partial q}{\partial \psi}$$

Substituting ι/L for the left-hand side (which means that the parallel current density is allowed to have a value equivalent to that which generates the same amount of rotational transform in a configuration with longitudinal current, i.e. a Tokamak - like configuration), we obtain

$$\beta \sim \frac{L\,\iota^2}{c_o A^2 \frac{\partial q}{\partial \psi}} ,$$

where A is the aspect ratio. In the case of the simplest 1 = 2 stellarator (see, for example, [5] one has

$$\frac{\partial q}{\partial \psi} \sim q_1 V^{\frac{1}{2}} \sim \frac{L}{c_o A} \quad ,$$

so that $ß \sim \iota^2/A$,

as is well known [2,8]. If eqs. (5,6) are satisfied, it follows that

$$\frac{\partial q}{\partial \psi} \sim q_2 V \sim \frac{L}{c_o A^2} \quad ,$$

so that

$$ß \sim \iota^2 \ .$$

Finally, if eqs. (10,11) are satisfied, one may expect

$$ß \sim \iota^2 A \ .$$

The above scalings have, however, to be substantiated by configurational studies analogous to that performed for the 1 = 2 stellarator [5].

Appendix

Here, we go as much into the details of the equilibrium calculation (see, for example, [3]) as is necessary to obtain the field line equations (3) and (7) and the stability result in eq. (13). We denote the contravariant components of $\vec{B}$ by

$$B^\rho = a_1\rho + a_2\rho^2 + O(\rho^3) \quad ,$$
$$B^\phi = b_o + b_1\rho + O(\rho^2) \quad ,$$
$$B^l = c_o + c_1\rho + c_2\rho^2 + O(\rho^3),$$

and a field line labelled with the constant ψ by

$$\phi = \phi_o(\psi,l) + \phi_1(\psi,l)\, V^{\frac{1}{2}} + O(V).$$

Employing the differential equation of the field line

$$\frac{d\phi}{dl} = \frac{B^\phi}{B^l} \quad , \tag{A1}$$

we obtain to lowest order

$$\left.\frac{\partial\phi_o}{\partial l}\right|_\psi = \frac{\overset{o}{b}_o}{c_o} \quad , \tag{A2}$$

where the superscript ^{o} indicates that $b_o(\phi, l)$ has to be taken at ϕ_o. Using the equilibrium relation

$$b_o = c_o\left[\frac{1}{2}K_o'(e + \frac{1}{e}) - \alpha'\right] + \frac{1}{2}c_oK_o'(e - \frac{1}{e})\cos 2u + \frac{1}{2}c_o\frac{e'}{e}\sin 2u \;,$$

one can easily verify that eq. (A2) is solved by

$$\phi_o = -\alpha + \arctan\,(e \tan \Psi) \quad , \tag{A3}$$

where

$$\Psi = \psi + K_o \quad . \tag{A4}$$

Equation (A3) entails the following relations:

$$\cos \mathring{u} \left[e \cos^2 \mathring{u} + \frac{1}{e} \sin^2 \mathring{u} \right]^{-\frac{1}{2}} = e^{-\frac{1}{2}} \cos\Psi \quad ,$$

$$\sin \mathring{u} \left[e \cos^2 \mathring{u} + \frac{1}{e} \sin^2 \mathring{u} \right]^{-\frac{1}{2}} = e^{\frac{1}{2}} \sin\Psi \quad ,$$

$$\mathring{V}_2 = \pi c_o\, q_o \left[\frac{1}{e} \cos^2\Psi + e \sin^2\Psi \right]^{-1} \quad , \tag{A5}$$

$$\frac{\mathring{b}_{o,\phi}}{c_o} = \frac{c_o}{\mathring{V}_2} \left. \frac{\partial}{\partial l} \right|_\psi \frac{\mathring{V}_2}{c_o} = - \frac{\mathring{V}_2}{c_o} \left. \frac{\partial}{\partial l} \right|_\psi \frac{c_o}{\mathring{V}_2} \quad .$$

We further note the following relations which are valid for any function $f(\phi_o, l)$:

$$\left. \frac{\partial}{\partial l} \right|_\psi = \left. \frac{\partial}{\partial l} \right|_{\phi_o} + \frac{\mathring{b}_o}{c_o} \left. \frac{\partial}{\partial \phi_o} \right|_l \; , \quad \left. \frac{\partial}{\partial \phi_o} \right|_l = \frac{\pi c_o q_o}{\mathring{V}_2} \left. \frac{\partial}{\partial \psi} \right|_l \quad ,$$

$$\left. \frac{\partial}{\partial \phi_o} \right|_l \left. \frac{\partial}{\partial l} \right|_\psi = \frac{c_o}{\mathring{V}_2} \left. \frac{\partial}{\partial l} \right|_\psi \frac{\mathring{V}_2}{c_o} \left. \frac{\partial}{\partial \phi_o} \right|_l \quad , \tag{A6}$$

where the second identy follows from eq. (A5).

To order $O(V^{\frac{1}{2}})$ eq. (A1) yields

$$\left. \frac{\partial \phi_1}{\partial l} \right|_\psi = \frac{\mathring{b}_{o,\phi}}{c_o}\, \phi_1 + \frac{\mathring{V}_2^{\,-\frac{1}{2}}}{c_o} \left(\mathring{b}_1 - 2\mathring{b}_o\, \kappa \cos\phi_o \right) \quad ,$$

where the equilibrium relation

$$c_1 = 2\kappa c_o \cos\phi$$

has been used. Introducing

$$\phi^* = \frac{c_o}{\overset{o}{V}_2}\,\phi_1 \tag{A7}$$

and employing eq. (A5), or we obtain

$$\left.\frac{\partial\phi^*}{\partial 1}\right|_\psi = \overset{o}{V}_2^{-\frac{3}{2}}\,(\overset{o}{b}_1 - 2\overset{o}{b}_o\,\kappa\,\cos\overset{o}{\phi}_o) \ . \tag{A8}$$

We now use the following equilibrium relations (see, for example, [3]):

$$2a_1 + b_{o,\phi} + c_o' = 0 \ , \tag{A9}$$

$$3a_2 + b_{1,\phi} - 3a_1\kappa \cos\phi + b_o\kappa \sin\phi - \kappa \cos\phi\, b_{o,\phi} + (c_o\kappa)' \cos\phi = 0, \tag{A10}$$

$$2a_1 V_2 + b_o V_{2,\phi} + c_o V_{2,1} = 0 \ , \tag{A11}$$

$$3a_1 V_3 + b_o V_{3,\phi} + c_o V_{3,1} + 2a_2 V_2 + b_1 V_{2,\phi} + c_1 V_{2,1} = 0 \ . \tag{A12}$$

The following relation is obtained from eqs. (A5,6,9) for any function $f(\phi_{o,1})$ and any m:

$$\left.\frac{\partial}{\partial 1}\right|_\psi \overset{o}{V}_2^{-\frac{m}{2}} f = c_o^{-1}\,\overset{o}{V}_2^{-\frac{m}{2}}\,(c_o\, f_{,1} + \overset{o}{b}_o\, f_{,\phi_o} + m a_1 f) \ . \tag{A13}$$

We now use eqs. (A10,12) to eliminate a_2. Using eq. (A13), we may write the resulting equation in the following form:

$$\frac{\partial}{\partial\phi_o}\left[\overset{o}{V}_2^{-\frac{3}{2}}(b_1 - 2b_o\,\kappa\cos\phi_o)\right] = c_o\overset{o}{V}_2^{-1}\frac{\partial}{\partial 1}\bigg|_\psi\left[\overset{o}{V}_2^{-\frac{3}{2}}(\tfrac{3}{2}\overset{o}{V}_3-\overset{o}{V}_2\kappa\cos\phi_o)\right].$$

Considering this equation together with eqs. (A6,8), we obtain

$$\frac{\partial\phi^*}{\partial\phi_o}\bigg|_1 = c_o\,\overset{o}{V}_2^{-\frac{5}{2}}(\tfrac{3}{2}\overset{o}{V}_3 - \overset{o}{V}_2\,\kappa\cos\phi_o)$$

and, together with eq. (A6),

$$\frac{\partial\phi^*}{\partial\psi}\bigg|_1 = (\pi q_o)^{-1}\,\overset{o}{V}_2^{-\frac{3}{2}}(\tfrac{3}{2}\overset{o}{V}_3 - \overset{o}{V}_2\kappa\cos\phi_o).$$

Employing the identities

$$\cos\overset{o}{u}\left[e\cos^2\overset{o}{u} + \frac{1}{e}\sin^2\overset{o}{u}\right]^{-\frac{3}{2}} = \frac{1}{4}e^{-\frac{3}{2}}\left[(3+e^2)\cos\Psi + (1-e^2)\cos3\Psi\right],$$

$$\sin\overset{o}{u}\left[e\cos^2\overset{o}{u} + \frac{1}{e}\sin^2\overset{o}{u}\right]^{-\frac{3}{2}} = \frac{1}{4}e^{-\frac{1}{2}}\left[(3e^2+1)\sin\Psi + (1-e^2)\sin3\Psi\right],$$

$$\cos3\overset{o}{u}\left[e\cos^2\overset{o}{u} + \frac{1}{e}\sin^2\overset{o}{u}\right]^{-\frac{3}{2}} = \frac{1}{4}e^{-\frac{3}{2}}\left[3(1-e^2)\cos\Psi + (1+3e^2)\cos3\Psi\right],$$

$$\sin3\overset{o}{u}\left[e\cos^2\overset{o}{u} + \frac{1}{e}\sin^2\overset{o}{u}\right]^{-\frac{3}{2}} = \frac{1}{4}e^{-\frac{1}{2}}\left[3(1-e^2)\sin\Psi + (3+e^2)\sin3\Psi\right],$$

which follow from eq. (A5) and the notation used in [3] which describes V_3

$$V_3 = V_{31c}\cos u + V_{31s}\sin u + V_{33c}\cos3u + V_{33s}\sin3u,$$

where

$$V_{31c} = \pi^2L^{-1}c_oq_o\left[(3e + \tfrac{1}{e})\,\bar{S}_c - \delta\right],$$

$$V_{31s} = \pi^2L^{-1}c_oq_o\left[(\tfrac{1}{e} + \tfrac{3}{e^3})\,\bar{S}_s - \Delta\right],$$

$$V_{33c} = \pi^2 L^{-1} c_o q_o \left[(e - \frac{1}{e}) \bar{S}_c + \delta \right] ,$$

$$V_{33s} = \pi^2 L^{-1} c_o q_o \left[(\frac{1}{e} - \frac{1}{e^3}) \bar{S}_s - \Delta \right] ,$$

we obtain

$$\left.\frac{\partial \phi^*}{\partial \psi}\right|_1 = (\pi q_o e)^{-\frac{3}{2}} c_o^{-\frac{1}{2}} \{ - e \kappa (\cos\alpha \cos\Psi + e \sin\alpha \sin\Psi)$$

$$+ \frac{3\pi}{2L} \left[(4e\bar{S}_c - e^2 \delta) \cos\Psi + (4\bar{S}_s - e\Delta) \sin\Psi + e^2 \delta \cos 3\Psi - e\Delta \sin 3\Psi \right] \} .$$

The final result, eq. (7), is then obtained with eq. (A7).

We now turn to the derivation of eq. (13). The equilibrium equations relating $\bar{S}_c$, $\bar{S}_s$, δ, and Δ are

$$(e+\frac{3}{e}) \left[- \frac{1}{2}(\frac{c_o'}{c_o} + \frac{e'}{e}) \bar{S}_c + \bar{S}_c' + \frac{1}{e} K_o' \bar{S}_s \right] + \frac{1}{2} (\frac{c_o'}{c_o} - \frac{e'}{e}) \delta - \delta' - e K_o' \Delta = R_1 ,$$

$$(3e+\frac{1}{e}) \left[- \frac{1}{2}(\frac{c_o'}{c_o} + \frac{3}{2} \frac{e'}{e}) \bar{S}_s + \bar{S}_s' - e K_o' \bar{S}_c \right] - e^2 \left[- \frac{1}{2}(\frac{c_o'}{c_o} + \frac{e'}{e}) \Delta + \Delta' \right] + e K_o' \delta = R_2 ,$$

$$R_1 = \frac{L\kappa}{8\pi} \{ 2 \sin\alpha \left[-K_o' + \alpha' (3e - \frac{2}{e}) + 4\tau e \right]$$

$$+ \cos\alpha \left[\frac{c_o'}{c_o} (e - \frac{2}{e}) + \frac{2\kappa'}{\kappa} (e + \frac{2}{e}) - \frac{e'}{e} (3e + \frac{2}{e}) \right] + \frac{16}{3} \frac{e}{\kappa c_o} \bar{b}_{11s} \},$$

$$R_2 = e^2 \frac{L\kappa}{8\pi} \{ 2 \cos\alpha \left[K_o' + \alpha' (2e - \frac{3}{e}) - 4\tau / e \right]$$

$$+ \sin\alpha \left[\frac{c_o'}{c_o} (\frac{1}{e} - 2e) + \frac{2\kappa'}{\kappa} (2e + \frac{1}{e}) + \frac{e'}{e} (2e + \frac{3}{e}) \right] - \frac{16}{3} \frac{1}{e c_o \kappa} \bar{b}_{11c} \} ,$$

where

$$\bar{b}_{11c} = \frac{3}{2}\dot{p}\pi q_o c_o^{\frac{3}{2}} e^{\frac{1}{2}} (b_i \cos\alpha - b_r \sin\alpha),$$

$$b_{11s} = \frac{3}{2}\dot{p}\pi q_o c_o^{\frac{3}{2}} e^{-\frac{1}{2}} (b_r \cos\alpha + b_i \sin\alpha),$$

and $b = b_r + ib_i$ satisfies

$$b' + i(K_o' - \alpha')b = -\exp(i\alpha)c_o^{-\frac{3}{2}} \kappa \left(e^{-\frac{1}{2}} \cos\alpha - i\, e^{\frac{1}{2}} \sin\alpha\right).$$

Linearizing the above equations according to Sec. 3c) and introducing

$$\tilde{\kappa} \cos\alpha_o = k_1' , \qquad \tilde{\kappa} \sin\alpha_o = k_2' , \qquad k_\nu = \tilde{k}_\nu + \bar{k}_\nu , \quad \int \tilde{k}_\nu dl = 0, \ \nu = 1,2 ,$$

we obtain

$$b = -\exp(i\alpha_o)\left(e_o^{-\frac{1}{2}} k_1 - ie_o^{\frac{1}{2}} k_2\right)$$

$$(e_o^2 + 3)\, \tilde{S}_c' - e_o \tilde{\delta}' = \frac{L}{4\pi}\left[k_1'' (e_o^2 + 2) - 4\dot{p}\pi q_o e_o k_1 \right] ,$$

$$(3e_o^2 + 1)\, \tilde{S}_s' - e_o^3 \tilde{\Delta}' = e_o^2 \frac{L}{4\pi}\left[k_2'' (2e_o^2 + 1) - 4\dot{p}\pi q_o e_o k_2 \right] .$$

The last two equations show that $\bar{k}_1 = \bar{k}_2 = 0$.

In order to evaluate the necessary criterion (see, for example, [2], [6]) on has to evaluate metrical quantities of the Hamada coordinate system whose metrical coefficients are given by the following expressions:

$$g_{VV} = (4\pi q_o c_o V)^{-1} (N_2 + 4c_o{}^3 Z_2{}^2) + O(V^{-\frac{1}{2}}) ,$$

$$g_{\theta\theta} = (q_o c_o)^{-1} 4\pi V (N_1 + 4c_o{}^3 Z_1{}^2) + O(V^{\frac{3}{2}}),$$

$$g_{\zeta\zeta} = q_o{}^2 c_o{}^2 + O(V^{\frac{1}{2}}),$$

$$g_{V\theta} = (q_o c_o)^{-1} (N_3 + 4c_o{}^3 Z_1 Z_2) + O(V^{\frac{1}{2}}),$$

$$g_{V\zeta} = q_o{}^{\frac{1}{2}} (\pi V)^{-\frac{1}{2}} c_o{}^2 Z_2 + O(V^o),$$

$$g_{\theta\zeta} = 4(\pi q_o V)^{\frac{1}{2}} c_o{}^2 Z_1 + O(V),$$

$$g^{VV} = 4\pi q_o c_o V N_1 + O(V^{\frac{3}{2}}) ,$$

$$g^{\theta\theta} = q_o c_o (4\pi V)^{-1} N_2 + O(V^{-\frac{1}{2}}),$$

$$g^{\zeta\zeta} = (q_o c_o)^{-2} \left[1 + 4c_o{}^3 (eB_1{}^2 + \frac{1}{e} B_2{}^2) \right] + O(V^{\frac{1}{2}}) ,$$

$$g^{V\theta} = - q_o c_o N_3 ,$$

$$g^{V\zeta} = 4q_o{}^{-\frac{1}{2}} (\pi V)^{\frac{1}{2}} c_o (eB_1 \cos\Psi - \frac{1}{e} B_2 \sin\Psi) ,$$

$$g^{\theta\zeta} = -(\pi q_o V)^{-\frac{1}{2}} c_o (e B_1 \sin\Psi + \frac{1}{e} B_2 \cos\Psi) ,$$

where

$$Z_1 = B_1 \sin\Psi + B_2 \cos\Psi ,$$

$$Z_2 = B_2 \sin\Psi - B_1 \cos\Psi \ ,$$

$$B_1 = b_r \cos\alpha + b_i \sin\alpha \ ,$$

$$B_2 = b_i \cos\alpha - b_r \sin\alpha \ ,$$

$$N_1 = e \cos^2\Psi + \frac{1}{e} \sin^2\Psi \ ,$$

$$N_2 = e \sin^2\Psi + \frac{1}{e} \cos^2\Psi \ ,$$

$$N_3 = (e - \frac{1}{e}) \sin\Psi \cos\Psi \ ,$$

$$\Psi = K_o + \psi = K_o + 2\pi(\theta - \iota\ \zeta).$$

Here, we need the quantities

$$\frac{g_{\theta\theta}}{|\nabla V|^2} \ , \ \frac{g_{\theta\zeta}}{|\nabla V|^2} \ , \ \frac{g_{\zeta\zeta}}{|\nabla V|^2} \ .$$

The leading order terms of Z_1 and N_1 are

$$\tilde{Z}_1 = e_o^{\frac{1}{2}} \tilde{k}_2 \cos 2\pi\theta - e_o^{-\frac{1}{2}} \tilde{k}_1 \sin 2\pi\theta \ ,$$

$$N_1 = e_o \cos^2 2\pi\theta + \frac{1}{e_o} \sin^2 2\pi\theta \ ,$$

so that

$$\int \frac{g_{\theta\zeta}}{|\nabla V|^2} \, d\zeta = O(V^o)$$

and the final result, eq. (13), is readily obtained.

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