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# Feedback Stabilization of Axisymmetric MHD Instabilities in Tokamaks

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Abstract

A stability principle is derived and applied to simple tokamak configurations in order to study the feedback stabilization of unstable vertically elongated tokamak plasmas. For practical applicability it is assumed that the fast instabilities are slowed down by passive conductors so that only slow motions have to be considered. Numerical results are presented for a surface current model of plasma with one conjugate pair of axisymmetric feedback loops. Stabilization is possible, except in a limited region of loop positions. The optimum loop position in the region with possible stabilization is determined.

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## Introduction

Theory predicts that tokamaks without material limiters and with vertically elongated cross-sections such as is desirable for higher ß-values exhibit dangerous axisymmetric instabilites [1 - 21] . Experiments performed on tokamak-type [22 - 25, 27] and belt-pinch-type plasmas [26, 28, 29] have confirmed this prediction.

Theoretically, these instabilites may be stabilized by a superconducting wall surrounding the plasma close enough [5, 7, 11, 21]. Again, experimental results - obtained with conductors of finite conductivity - are at least in qualitative agreement [24, 28-30] , provided the time is short enough to consider the conductors as perfectly conducting. In large fusion devices, however, plasma lifetimes would have to be much larger and, as to the superconducting walls, it is not obvious that they could be placed as close as necessary.

With conductors of finite conductivity full stabilization of unstable modes is impossible [31]. However, a reduction of growth rates may take place [12, 21]. Thus, in large fusion devices a practicable way to suppress the instabilities considered is to reduce the growth rates by means of passive conductors and to control the remaining unstable motions by feedback.

To some extent feedback control of unstable vertical motions has already been tested in several experimental devices. An increase in the lifetime of plasmas with vertically elongated cross-sections was observed [22, 25].

In this paper we investigate feedback stabilization of axisymmetric modes, assuming that it is possible beforehand to reduce the growth rates, this being necessitated by the retarded response and the inertia of practical feedback systems. Feedback currents are assumed to flow in axisymmetric loops which, as an idealization, we consider as "infinitely" thin wires (see Fig. 1).

On these assumptions we encounter the following questions:

Can stability be achieved at all, or are previously stable modes driven unstable under the effect of the feedback currents?

If stabilization is possible, how many loops are required; how sensitive is the stabilization to the position of loops; how strong are the currents required, etc.?

In Section 1 a stability principle is derived. In Section 2 it is evaluated for axisymmetric devices (subsection a), the case of only one pair of feedback loops being specially considered in subsection b. Section 3 contains an analytic preparation of the numerical problem. Numerical results are presented in Section 4.

## 1. General stability principle

We start with the linearized MHD equations for the plasma displacement $\underline{\xi}$ ($\underline{R}$, t), which in standard notation [32] are

$$\rho \ddot{\underline{\xi}} = F(\underline{\xi}), \tag{1.1}$$

$$F(\underline{\xi}) = \nabla(\underline{\xi}\cdot\nabla p + \gamma p \, \mathrm{div}\, \underline{\xi}) + [\mathrm{curl}\, \underline{Q} \times \underline{B}] + [\underline{j} \times \underline{Q}], \tag{1.2}$$

where

$$\underline{Q} = \mathrm{curl}\, [\underline{\xi} \times \underline{B}]. \tag{1.3}$$

Multiplication of eq. (1.1) by $\dot{\underline{\xi}}$ and integration over the plasma volume yields after partial integrations and use of the proper boundary conditions

$$\frac{d}{dt}(K + \delta W_{pl} + \delta W_s + \delta W_{ext}) + \int_{ext} d\tau \, \delta\underline{j}\cdot\delta\underline{E} = 0, \tag{1.4}$$

where

$$K = \frac{1}{2}\int_{pl} d\tau \, \rho \, \dot{\underline{\xi}}^2, \tag{1.5}$$

$$\delta W_{pl} = \frac{1}{2}\int_{pl} d\tau \left\{\underline{Q}^2 + \delta\underline{j}\cdot[\underline{\xi}\times\underline{Q}] + (\underline{\xi}\cdot\nabla p)\,\mathrm{div}\,\underline{\xi} + \gamma p(\mathrm{div}\,\underline{\xi})^2\right\}, \tag{1.6}$$

$$\delta W_s = \frac{1}{2}\int_S dS \, \xi_n^2 \, \underline{n}\cdot\langle p + \frac{1}{2}\underline{B}^2\rangle, \tag{1.7}$$

$$\delta W_{ext} = \frac{1}{2}\int_{ext} d\tau \, (\delta\underline{B}_v)^2, \tag{1.8}$$

and where the domains of integration are the plasma volume (pl), the plasma surface (S) and the region outside the plasma (ext). We have $\xi_n = \underline{n} \cdot \underline{\xi}$ , where the normal vector $\underline{n}$ on S is directed out of the plasma. $\delta \underline{B}_v$ is the perturbational magnetic field outside the plasma.

With

$$\delta \underline{B}_v = \mathrm{curl}\ \underline{A} ; \qquad \delta \underline{E} = - \dot{\underline{A}} \tag{1.9}$$

and the boundary condition

$$\xi_n \underline{B}_v = - [\underline{n} \times \underline{A}] \quad \text{on } S, \tag{1.10}$$

where $\underline{B}_v$ is the equilibrium vacuum field, the expression (1.8) for $\delta W_{ext}$ may be integrated by parts:

$$2\,\delta W_{ext} = \int_S dS\, \xi_n \underline{B}_v \cdot \delta \underline{B}_v + \int_{ext} d\tau\ \underline{A} \cdot \delta \underline{j} . \tag{1.11}$$

Substitution in eq. (1.4) yields

$$\frac{d}{dt}\left( K + \delta W_{pl} + \delta W_S + \frac{1}{2} \int_S dS\, \xi_n \underline{B}_v \cdot \delta \underline{B}_v \right) = \frac{1}{2} \int_c d\tau \left( \delta \underline{j} \cdot \dot{\underline{A}} - \delta \dot{\underline{j}} \cdot \underline{A} \right)$$

$$= \frac{1}{2} \int_p d\tau \left( \underline{A} \cdot \Delta \dot{\underline{A}} - \dot{\underline{A}} \cdot \Delta \underline{A} \right) + \frac{1}{2} \sum_\nu \left( J_\nu \dot{\Phi}_\nu - \dot{J}_\nu \Phi_\nu \right) , \tag{1.12}$$

where the subscript c means integration over all external conductors, p means integration over passive conductors such as field shaping coils, copper shells, etc., and

$\delta \underline{j}$ = curl curl $\underline{A}$, div $\underline{A} = 0$ has been used. The sum on the right-hand side is the contribution of the ("infinitely" thin) feedback loops, where

$$\Phi_\nu = \int_\nu d\underline{s} \cdot \underline{A} = \int_\nu d\underline{F} \cdot \delta \underline{B}_v \qquad (1.13)$$

is the change of flux through the $\nu$-th loop induced by $\delta \underline{B}_v$, and $\mathfrak{I}_\nu$ is the accompanying change of current.

The stabilizing effect of the passive conductors on the fast plasma motions is due to induced mirror currents. We do not discuss the details of this process but assume it to be effective and to reduce the speed of motion below the skin speed. For slow motions on a time scale below the skin time, the passive conductors become less effective since with the dominance of damping the induction of mirror currents is reduced. Alltogether it is for plasma motions on this slow time scale that we investigate active feedback stabilization.

The fluxes $\Phi_\mu$ and currents $\mathfrak{I}_\nu$ are connected by

$$\Phi_\mu = \sum_\nu L_{\mu\nu} \mathfrak{I}_\nu + \hat{\Phi}_\mu , \qquad (1.14)$$

where $L_{\mu\nu}$ are the inductances of the feedback coils, and $\hat{\Phi}_\mu$ are the fluxes due to the plasma motion. With eq. (1.14) the second term on the right-hand side of eq. (1.12) becomes

$$\frac{1}{2}\sum_{\nu}\left( \mathcal{J}_{\nu}\dot{\Phi}_{\nu} - \dot{\mathcal{J}}_{\nu}\Phi_{\nu} \right) = \frac{1}{2}\sum_{\nu}\left( \mathcal{J}_{\nu}\dot{\hat{\Phi}}_{\nu} - \dot{\mathcal{J}}_{\nu}\hat{\Phi}_{\nu} \right). \tag{1.15}$$

The currents $\mathcal{J}_{\nu}$ may, in principle, be adjusted by the feedback system as arbitrary functions of the $\hat{\Phi}_{\mu}$ . Consistently with linear stability theory, however, we shall assume the linear relations

$$\mathcal{J}_{\nu}(t) = \sum_{\mu} c_{\mu\nu}\hat{\Phi}_{\mu}(t) \; ; \qquad c_{\mu\nu} = c_{\nu\mu} \; , \tag{1.16}$$

which imply that in practice the currents $\mathcal{J}_{\nu}$ follow the plasma motion without appreciable phase shift, on the slow time scale considered. The degree of freedom necessary for a feedback system is represented by the constants $c_{\mu\nu}$ which so far may be chosen arbitrarily. From the stability principle to be derived below we shall obtain in a later section inequalities for the currents $\mathcal{J}_{\nu}$ which together with the relations (1.17) yield restrictions for the $c_{\mu\nu}$.

With our assumptions on the feedback mechanism and the slowing down of plasma motions below the skin speed we can approximately assume a quasistatic change of the magnetic field, i.e.

$$\underline{A}(\underline{R},t) = \underline{A}(\underline{R})f(t) \; , \tag{1.17}$$

and thus, according to eqs. (1.15) - (1.17) the r.h.s. of eq. (1.12) vanishes [1] :

$$\frac{d}{dt}\left( K + \delta W_{pl} + \delta W_S + \frac{1}{2}\int_S dS\, \xi_n\, \underline{B}_v \cdot \delta \underline{B}_v \right) = 0. \quad (1.18)$$

As in the usual stability theory without feedback, this yields the necessary and sufficient stability principle

$$\delta W_{pl} + \delta W_S + \frac{1}{2}\int_S dS\, \xi_n\, \underline{B}_v \cdot \delta \underline{B}_v \geq 0. \quad (1.19)$$

$\delta \underline{B}_v$ = curl $\underline{A}$ has the induced current density $\delta \underline{j}_P$ in the passive conductors and the applied current density $\sum_\nu \delta \underline{j}_\nu$ in the feedback loops as sources and must satisfy the boundary condition (1.10). To get rid of the inconvenient dependence on $\delta \underline{j}_P$ , we shall now derive a much simpler sufficient stability principle.

For this purpose we split

$$\delta \underline{B}_v = \delta \underline{B}_0 + \delta \underline{B}_1 + \delta \underline{B}_2 , \quad (1.20)$$

where $\delta \underline{B}_i$ = curl $\underline{A}_i$ are solutions of the boundary value problems

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1) With the ansatz (1.17) all $\mathfrak{J}_\nu$ would vanish for a motion with $\hat{\Phi}_\mu = 0$, $\mu = 1, 2, \ldots$. However, this case allows arbitrary currents $\mathfrak{J}_\nu$ with still vanishing r.h.s. of eq. (1.12) and can thus be included in the following considerations.

$$\text{curl curl } \underline{A}_o = 0 , \qquad [\underline{n} \times \underline{A}_o] = - \xi_n \underline{B}_v , \tag{1.21a}$$

$$\text{curl curl } \underline{A}_1 = \sum_\nu \delta \underline{j}_\nu , \qquad [\underline{n} \times \underline{A}_1] = 0 , \tag{1.21b}$$

$$\text{curl curl } \underline{A}_2 = \delta \underline{j}_P , \qquad [\underline{n} \times \underline{A}_2] = 0 , \tag{1.21c}$$

the differential equations holding in the region outside the plasma and the boundary conditions holding on the plasma boundary S. Obviously, the stability principle (1.19) separates according to

$$\delta W_0 + \delta W_J + \delta W_P \geqslant 0 , \tag{1.22}$$

where

$$\delta W_0 = \delta W_{pl} + \delta W_S + \frac{1}{2} \int_S dS \, \xi_n \underline{B}_v \cdot \delta \underline{B}_0 , \tag{1.23}$$

$$\delta W_J = \frac{1}{2} \int_S dS \, \xi_n \underline{B}_v \cdot \delta \underline{B}_1 , \tag{1.24}$$

$$\delta W_P = \frac{1}{2} \int_S dS \, \xi_n \underline{B}_v \cdot \delta \underline{B}_2 . \tag{1.25}$$

$\delta W_0$ is the energy variation in the absence of any external conductors as employed in Refs. [10, 13, 17], $\delta W_J$ is a contribution due to the feedback currents and $\delta W_P$ is due to the mirror currents in the passive conductors.

It can now be shown that $\delta W_P$ is non-negative. With

$$\underline{A}^* = \underline{A}_0 + \underline{A}_2 \tag{1.26}$$

one obtains by partial integrations, using eqs. (1.21a, c),

$$\frac{1}{2} \int_{ext} d\tau \, \text{curl}^2 \underline{A}^* - \frac{1}{2} \int_{ext} d\tau \, \text{curl}^2 \underline{A}_0 = \delta W_P + \frac{1}{2} \int_P d\tau \, \underline{A}^* \cdot \delta \underline{j}_P . \tag{1.27}$$

According to a well-known theorem [32], with given fixed boundary values of $[\underline{n} \times \underline{A}]$ the magnetic energy $1/2 \int d\tau \; \text{curl}^2 \underline{A}$

attains its minimum for a vacuum field. Thus, since $\underline{A}_o$ and $\underline{A}^*$ satisfy the same boundary condition on S, the left-hand side of eq. (1.27) cannot be negative and

$$\delta W_P \geqslant - \frac{1}{2} \int_P d\tau \, \underline{A}^* \cdot \delta \underline{j}_P \, . \tag{1.28}$$

The current density $\delta \underline{j}_P = - \sigma (\dot{\underline{A}}_o + \dot{\underline{A}}_1 + \dot{\underline{A}}_2) = - \sigma (\dot{\underline{A}}_1 + \dot{\underline{A}}^*)$ has two origins: induction from feedback currents, $\dot{\underline{A}}_1$, and induction from the plasma motion, $\dot{\underline{A}}^*$. According to our assumption that the typical times of unstable plasma motions are slowed down below the skin time of the passive conductors, these will be practically absent for the slow field variations due to feedback currents, and we may neglect the $\underline{A}_1$ contribution against the $\underline{A}^*$ contribution, which is responsible for the slow down, thus obtaining

$$\delta W_P \geqslant \frac{1}{2} \int_P d\tau \, \sigma \, \underline{A}^* \cdot \dot{\underline{A}}^* \, . \tag{1.29}$$

Together with our assumption (1.14) of quasistatic field change it follows that $\delta W_P \geqslant 0$ for unstable motions. Omission of $\delta W_P$ from our criterion (1.19) finally yields the sufficient stability principle

$$\delta W_o + \delta W_J \geqslant 0 \, . \tag{1.30}$$

For very slow instabilities with $\delta W_P \longrightarrow 0$ it also becomes necessary.

In the stability criterion (1.30) the evaluation of $\delta W_0$ , eq. (1.23), which by definition is the same as without feedback, proceeds as in previous work on axisymmetric stability [13, 17] and is not repeated here. The evaluation of the feedback contribution $\delta W_J$ , however, is carried out in the next section.

## 2. Application to axisymmetric devices

a) Many feedback loops

In axisymmetric devices for axisymmetric perturbations the vacuum field $\delta \underline{B}$ is purely poloidal and we consider only feedback loops which leave $\delta \underline{B}$ poloidal, i.e. axisymmetric toroidal loops. Later on we shall use the fact that the arrangement of feedback loops may always be considered to be symmetric with respect to the equatorial plane. If in reality it were not, loops in conjugate positions with zero current could be added formally. We shall refer to the conjugate upper and lower positions by using upper and lower bars; see, for example, eq. (2.13).

Using cylindrical coordinates R, $\vartheta$ , z and a unit vector $\underline{e}_\vartheta$ in the $\vartheta$ -direction, with $dl^2 = dR^2 + R^2 \, d\vartheta^2$ and

$$y = [\underline{n} \times \underline{e}_\vartheta] \cdot \delta \underline{B}_1 \,, \tag{2.1}$$

we get from eq. (1.23)

$$\delta W_J = \pi \oint dl \, R B \xi_n y \,, \tag{2.2}$$

where the integration extends over a poloidal cut of S, and B stands for $[n \times e_\vartheta] \cdot B_v$.

y(l) is determined from an integral equation on S [13, 33]

$$y(l) = \frac{1}{2\pi}\oint dl' K(l,l')\, y(l') + \frac{1}{2\pi}\sum_{\nu}\left([\underline{n}\times\underline{e}_{\vartheta}]\cdot \mathrm{curl}\int d\tau' \frac{1}{r}\delta \underline{j}_{\nu}(\underline{R}')\right), \quad (2.3)$$

where the boundary condition in (1.19b) has been used. K(l,l') is defined as in Ref. [13]. For "infinitely" thin feedback loops one obtains from eq. (2.3)

$$y(l) = \frac{1}{2\pi}\oint dl' K(l,l')\, y(l') - \frac{1}{2\pi}\sum_{\nu} J_{\nu}\, K(R(l), z(l); R_{\nu}, z_{\nu}), \quad (2.4)$$

where

$$\begin{aligned} K(R,z;R',z') &= -R'\int_0^{2\pi} d\vartheta' \left([\underline{n}\times\underline{e}_{\vartheta}]\cdot\left[\nabla\frac{1}{r}\times\underline{e}_{\vartheta'}\right]\right) \\ &= \frac{4R'}{r_0^3(1-k^2)}\left\{-n_R R' E + \frac{1}{k^2}\left[n_R R + n_z(z-z')\right]\right. \quad (2.5) \\ &\qquad \left.\cdot\left[(2-k^2)E - 2(1-k^2)K\right]\right\}. \end{aligned}$$

E(k) and K(k) are elliptic integrals of the first and second kinds, and

$$r^2 = (R-R')^2 + (z-z')^2; \quad r_0^2 = (R+R')^2 + (z-z')^2; \quad k^2 = \frac{4RR'}{r_0^2}. \quad (2.6)$$

An ambiguity in the solution of eq. (2.4) (see Refs. [13] and [33, 34]) is removed by adding a side condition (corresponding to flux conservation) which is a straight forward generalization of the side condition derived in [13] to the case $j \neq 0$ outside the plasma:

$$\oint dl\, V(R(l),z(l))\, y(l) \quad + \oint dl\, K(R(l),z(l);\, R_a, 0) RB\xi_n$$

$$= \sum_\nu J_\nu V(R_\nu, z_\nu)\,, \tag{2.7}$$

where

$$V(R,z) = -R\int_0^{2\pi} d\vartheta' \frac{1}{|\underline{R}-\underline{R}'_a|}(\underline{e}_\vartheta \cdot \underline{e}_{\vartheta'})$$

$$= \frac{4R}{r_a k_a^2}\left[2E(k_a) - (2-k_a^2)K(k_a)\right]. \tag{2.8}$$

$\underline{R}'_a$ is the position vector of a circular loop located at arbitrary $R' = R_a$, $z' = 0$ inside the plasma,

$$r_a^2 = (R + R_a)^2 + z^2\,; \qquad k_a^2 = \frac{4RR_a}{r_a^2}\,. \tag{2.9}$$

For plasma cross-sections which are symmetric with respect to the z = 0 plane it is useful to split $\xi_n$ and $y$ into their symmetric and antisymmetric components, respectively,

$$\xi_n = \xi_n^s + \xi_n^a\,; \qquad y = y^s + y^a\,. \tag{2.10}$$

The symmetry properties of K and V are

$$K(R,z;\, R_\nu, z_\nu) = K(R,-z;\, R_\nu, -z_\nu)$$

$$V(R_\nu, z_\nu) = V(R_\nu, -z_\nu) \tag{2.11}$$

and with the above-mentioned symmetry of the feedback loops eq. (2.4) then splits into a pair of equations for $y^s$ and $y^a$, respectively. In condensed notation we have

$$y^{s,a} = \frac{1}{2\pi} \oint dl' K(l,l') \, y^{s,a}(l') - \frac{1}{2\pi} \sum_{\bar{\nu}} J_\nu^{s,a} K_\nu^{s,a} , \qquad (2.12)$$

where either the first or the second superscript is valid, and where

$$J_\nu^s = \frac{1}{2}( J_{\bar{\nu}} + J_{\underline{\nu}} ) ; \quad K_\nu^s = K(l; R_\nu, z_\nu) + K(l; R_\nu, -z_\nu) ,$$

$$J_\nu^a = \frac{1}{2}( J_{\bar{\nu}} - J_{\underline{\nu}} ) ; \quad K_\nu^a = K(l; R_\nu, z_\nu) - K(l; R_\nu, -z_\nu) . \qquad (2.13)$$

Equation (2.7) becomes a side condition for $y^s$ only.

If we introduce the quantities $Y^{s,a}$ as solutions of the equations

$$\mathcal{Y}_\nu^{s,a} = \frac{1}{2\pi} \oint dl' K(l,l') \, \mathcal{Y}_\nu^{s,a}(l') - \frac{1}{2\pi} K_\nu^{s,a} , \qquad (2.14)$$

then the solutions of eq. (2.12) are given by

$$y^{s,a}(l) = \sum_{\bar{\nu}} J_\nu^{s,a} \, \mathcal{Y}_\nu^{s,a}(l) . \qquad (2.15)$$

Using the symmetry properties of $y^{s,a}$ and $\xi_n^{s,a}$ and inserting eq. (2.15) in eq. (2.2), it is readily seen that $\delta W_J$ splits according to

$$\delta W_J = \delta W_J^s + \delta W_J^a , \qquad (2.16)$$

where

$$\delta W_J^{s,a} = \sum_{\bar{\nu}} J_\nu^{s,a} \oint dl\, \pi R B y_\nu^{s,a} \xi_n^{s,a} . \qquad (2.17)$$

According to Refs. [13, 17] $\delta W_0$ splits similarly into contributions for symmetric and antisymmetric perturbations, $\delta W_0 = \delta W_0^s + \delta W_0^a$ . Using this and inserting eq.(2.16) into eq. (1.30) finally yields the stability criteria

$$\delta W^{s,a} \equiv \delta W_0^{s,a} + \delta W_J^{s,a} \geq 0 . \qquad (2.18)$$

Since the quantities $J_\nu$ , eq. (2.13), can always be controlled so that $\delta W_J^{s,a} \geq 0$ , it is obvious that the feedback contribution has a stabilizing effect.

b) One pair of feedback loops

We consider some further consequences for the simplest case with one pair of loops only, positioned symmetrically with respect to the equatorial plane. Furthermore, we consider only the stabilization of antisymmetric instabilities since for elongated cross-sections these are the more important ones; see Ref. [17]. A treatment of symmetric instabilities would be largely analogous. Care has to be taken, however, to ensure that stable symmetric modes are not unintentionally driven to instability by the feedback current. This is avoided by applying antisymmetric currents

$$J_{\bar{1}} = - J_{\underline{1}} \equiv J \qquad (2.19)$$

so that $J_1^a = J$ and $J_1^s = 0$ and hence $\delta W_J^s = 0$. Otherwise $\xi_n^s$ could always adjust itself so that $\delta W_J^s < 0$, unless $J_{\bar{1}}$ and $J_{\underline{1}}$ are controlled separately.

With eq. (2.19) $\delta W_J^a$ is obtained as

$$\delta W_J^a = J \oint dl\, \pi R B Y_1^a \xi_n^a . \qquad (2.20)$$

If for any $\xi_n^a$ which is unstable in the problem without feedback we have

$$\oint dl\, R B Y_1^a \xi_n^a = 0 , \qquad (2.21)$$

then obviously no feedback stabilization is possible, whatever the current J may be. On the other hand, if the expression is non-zero for all possible $\xi_n^a$, stabilization is possible. Which of the two cases is present depends through $Y_1^a$ on the position of the feedback loops. This suggests the possible existence of ineffective loop positions into which the loops should not be placed because they would be inoperative there.

The problem now is to determine for a given plasma configuration and position of the feedback loops the minimum $|J|$, if it exists, for which the stability criterion is satisfied for all antisymmetric $\xi_n$. This problem can only be solved numerically. For this purpose, as in Refs. [13, 17], $\xi_n^a$, $y^a$, etc. are discretized, i.e. $\xi_n^a(\ell) \longrightarrow \xi_{n_i}^a$, i = 1, ..N, etc. Using the numerical results for the problem without feedback, it is possible to obtain further reaching analytical simplification for the problem with feedback.

## 3. Analytic preparation of the numerical problem

Once an explicit numerical representation of the quadratic form $\delta W_0^a(\xi_n^a, \xi_n^a)$,

$$\delta W_0^a = \sum_{\mu,\nu=1}^{N} W_{\mu\nu} \xi_{n\mu}^a \xi_{n\nu}^a , \tag{3.1}$$

for the energy variation without feedback is given $\delta W^a$ may be expressed in a particularly simple form. If we denote the eigenvalues of the matrix $W_{\mu\nu}$ by $W_i$ , i = 1, .. N (where N is the order of $W_{\mu\nu}$ ), and the corresponding eigenvectors by $\xi_\nu^{(i)}$ , we may expand an arbitrary $\xi_{n_\nu}^a$ with suppression of $\nu$ as follows:

$$\xi_n^a = \sum_{i=1}^{N} x_i \xi^{(i)} . \tag{3.2}$$

This reduces $\delta W^a$ to

$$\delta W^a = Z + \mathfrak{I} B , \tag{3.3}$$

where

$$Z = \sum_{i=1}^{N} W_i x_i^2 ; \qquad B = \sum_{i=1}^{N} b_i x_i ,$$

$$b_i = \oint dl \pi R B y_1^a \xi^{(i)} . \tag{3.4}$$

If $B = 0$ for any set $\{ x_i, i = 1, ..N \}$ with $Z < 0$, corresponding to eq. (2.21), no stabilization is possible.

Conversely, if for all sets $\{ x_i, i = 1, ..N \}$ with $Z < 0$ we have $B \neq 0$, stabilization is possible and the conditions for stability are

$$\mathfrak{J}\, B > 0 \tag{3.5}$$

and

$$|\mathfrak{J}| \geq \mathfrak{J}_o = \sup_{\sum_{i=1}^{N} x_i^2 = 1;\; Z < 0} \left( \frac{-Z}{B} \right) . \tag{3.6}$$

The side condition $Z < 0$ means that we are interested only in plasmas which are unstable without feedback. In linear stability theory a common factor of the $x_i$, $i = 1, ..N$, is undetermined in principle. This "amplitude" factor is arbitrarily fixed by the side condition $\sum x_i^2 = 1$ or equivalently $2\pi \langle (\xi_n^a)^2 \rangle = \int_0^{2\pi} du (\xi_m^a)^2 = 1$ . Note that Z/B is linear in this amplitude factor, and that $J_o$ is thus related to a unit perturbation. If smaller or larger amplitudes are considered $J_o$ must be multiplied by the corresponding amplitude factor.

The problem of finding the "ineffective"loop positions with $B = 0$ for $Z < 0$, on the one hand, and of determining the minimum stabilizing current $J_o$ for the effective positions, on the other, can still be simplified under the special conditions which are present for the surface current model at moderate plasma elongations (below ≈ 4-5). Under these conditions for all values of the aspect ratio the existence of only one negative eigenvalue $W_1$ and additionally the validity of the relation

$$W_1 + W_2 \geq 0 \tag{3.7}$$

were found. +)

It is shown in Appendix A that under these circumstances a loop position is effective if, and only if, the expression $-Z/B$ in eq. (3.6) has an extremum for $Z < 0$, $\sum x_i^2 = 1$ , or, equivalently, if, and only if, the equation

$$\sum_{i=1}^{N} b_i^2 \frac{Z - W_i}{(Z - 2W_i)^2} = 0 \tag{A.4}$$

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+) The existence of just one negative eigenvalue for moderate elongations is not a peculiarity of the surface current model but is found for diffuse current models as well (see Refs. [4 , 35]). Condition (3.7) which may not hold for diffuse profiles is not necessary but only convenient for the subsequent simplification, and much weaker but more complicated conditions on the $\{b_i, W_i, i = 1, \ldots N\}$ could do the same.

has at least one negative solution $Z = \bar{Z}$. If there exist solution(s) $\bar{Z}_\nu$, then $J_o$ is given by

$$J_o = \sup_{\bar{Z}_\nu} \left[ \sum_{i=1}^{N} \left( \frac{b_i}{\bar{Z}_\nu - 2W_i} \right)^2 \right]^{-1/2} . \qquad (A.6)$$

Note that in the case $\bar{Z}_\nu \rightarrow 0$ according to eq. (A.6) $J_o$ stays finite, which implies that $B \rightarrow 0$ as well. Hence, in the limit of marginally stabilizable cases the feedback current $J_o$ stays finite.

The calculation of the feedback current $J_o$ is now widely reduced to the problem without feedback whose solution yields the $W_i$ and $\xi^{(i)}$. Once the integral equation (2.14) is solved with the methods of Ref. [13] the $b_i$ are obtained from eq. (3.4) and thus all quantities needed for the calculation of $J_o$ are determined.

We conclude this section with a few remarks about the situation arising when more than one eigenvalue is negative. Consider a case where two (or more) eigenvalues, say $W_1$ and $W_2$, are negative, and a test mode $\{x_i\}$ with the properties $b_1x_1 + b_2x_2 = 0$, $x_1$ and $x_2 \neq 0$, $x_{i \geq 3} = 0$. For this mode, with $\delta W^a = W_1x_1^2 + W_2x_2^2 < 0$ for all J, the plasma is always unstable. It may be shown that for each additional negative eigenvalue at least one additional pair of feedback loops is required for stabilization.

## 4. Numerical results

The methods just described were applied to the feedback stabilization of the surface current equilibria described in Ref. [13] with elliptically, triangularly and rectangularly shaped cross-sections represented by (see Fig. 1)

$$e^2(R-1)^2 + (1+\tau_3^2)z^2 - 2A\tau_3(R-1)z^2 - A^2\tau_4(R-1)^2z^2$$

$$= \frac{e^2}{A^2} \,. \qquad (4.1)$$

The results are shown in Figs. 2-8. The safety factor q does not enter the discussion since the stability with respect to antisymmetric axisymmetric perturbations is independent of q.

In Figs. 2 and 3 the dotted area is the plasma, the axis of symmetry being to the left. The position of the pair of feedback loops - the dots in Fig. 2a - was varied in small steps on surfaces concentric with the plasma surface. The quantity d = (plasma aspect ratio)/(aspect ratio of concentric surface) is used as a measure of the loop distance from the plasma. In the hatched areas no stabilization is possible ("ineffective" regions), while everywhere in the blank area a stabilizing current can be found. The position where the minimum feedback current is lowest (optimum position) is marked by arrows.

The left-right asymmetries due to toroidal effects, Figs. 2a - 2c, are reduced at higher aspect ratio, A = 10, Figs. 3a - 3c.

The ratio of J and $J_{opt}$ (= current at the optimum position) versus positional angle u (see Fig. 2a) is shown in Fig. 4. Since the curves exhibit flat minima, there exist relatively broad regions of approximately equal effectiveness of the loops.

In Figs. 5a, 5b, $J_{opt}$ is plotted versus d, plasma cross-sections corresponding to Figs. 2a, 2b, 2c being indicated by symbols. Dimensionless units have been used: $J_{opt}$ is the ratio of the feedback current and toroidal plasma current, and the average displacement on the plasma surface has been arbitrarily taken to be one-tenth of the minor radius a. Defining $\zeta$ by $\zeta = \sqrt{\langle \xi_n^2 \rangle} / 0.1\, a$, the currents required have to be multiplied by $\zeta$ if $\zeta \neq 1$. The currents increase with aspect ratio and are of the order of one per cent.

The parameter $\beta_p$ (poloidal beta) has almost no effect on the optimum loop position and on the currents required, as is shown in Fig. 6.

Figure 7 shows that $J_{opt}$ increases with increasing plasma elongation e.

A good illustration of the feedback process is obtained by examining the minimizing perturbations $\xi_n^a$ . In Figs. 8a and 8b arrows represent the plasma displacement on the boundary in a case with feedback currents switched on. The broken line indicates where the arrows would end with the feedback currents switched off. Figure 8a shows a stabilized case, the feedback loops being up and down. One can see that as a result of the feedback currents the displacement in the vicinity of the loops is reduced. In Fig. 8b the loops are in a position where no stabilization is possible. The displacement in the vicinity of the loops is again reduced. This reduction, however, is unable to compensate the virtually free or even enhanced unstable motion of the bulk of the plasma.

For practical reasons the currents J applied for feedback stabilization would be larger than the minimum currents $J_o$ calculated above. In Appendix B it is explained that if $J/J_o$ is increased above a certain threshold the minimizing perturbation "locks in" and becomes independent of J (apart from the total amplitude).

## 5. Conclusions

On the assumption that large MHD growth rates may be reduced by passive conductors feedback stabilization of the remaining slow plasma motions was investigated. Detailed numerical studies were performed for the surface current plasma model. It was shown that for elliptical, triangularly and rectangularly shaped cross-sections feedback stabilization of the axisymmetric modes is possible with a single pair of feedback loops positioned symmetrically with respect to the midplane, provided the loops are not positioned in certain well-defined and not very spacious "ineffective" regions.

The feedback current required for stabilization is not a very sensitive function of plasma elongation and of distance from the plasma surface. This supports our previous argument that for distributed current profiles the results will be qualitatively similar for moderate elongations (below 4-5).

If, however, the elongation is very large (belt pinch type), feedback stabilization becomes more difficult: With the appearance of additional negative eigenvalues corresponding to unstable symmetric and antisymmetric motions independent control of the loop currents and additional feedback loops become necessary.

The theory of feedback stabilization as presented in this paper can, in principle, be applied to diffuse current models as well since the feedback currents cause only an additional term in the vacuum contribution to $\delta W$, while $\delta W_{pl}$ and $\delta W_S$ remain unchanged.

The possibility of stabilization with localized currents, the existence of optimum loop positions and the existence of ineffective regions appears to have consequences also for passive stabilization with extended conductors surrounding the plasma. Apparently, the currents induced in these are more or less effective or even completely ineffective, depending on their position, and wall sections placed at the most effective positions would attain about the same objective and provide great technical advantages over closed walls.

## Appendix A

We repeat that a loop position is effective, i.e. stabilization is possible, if, and only if, $B \neq 0$ for $Z < 0$.

Now, if $B \neq 0$ for all $\{x_i, i = 1, ..N\}$ with $Z < 0$, then, according to the definitions (3.4), $- Z/B$ is an analytic function of the $x_i$ in the domain of the supersphere $\sum x_i^2 = 1$ marked off by the condition $Z < 0$. Since one has $Z/B = 0$ on the boundary $Z = 0$ of this domain and since with $W_1 < 0$ one certainly has $\sup_{\sum x_i^2 = 1,\, Z < 0} - Z/B > 0$,

and in view of the analyticity it follows that $- Z/B$ assumes a largest value at $Z < 0$, and that this is an extremum.

Next we prove the reverse conclusion that $B \neq 0$ for $Z < 0$ if $- Z/B$ has an extremum for $Z < 0$. Defining $Z^* = \inf_{\sum x_i^2 = 1,\, B = 0} Z$ and $\bar{Z}_o$ to be the smallest Z-value

for which $- Z/B$ assumes an extremum on $\sum x_i^2 = 1$, it is equivalent to prove that we cannot have both $Z^* < 0$ and $\bar{Z}_o < 0$. Note that obviously $Z^*$ must also be a stationary value.

Taking care of the side conditions $\sum x_i^2 = 1$ and $B = 0$ by Lagrangian parameters $\lambda$ and $\mu$, respectively, we find that $Z$ is made stationary by

$$2(W_i + \lambda) x_i + \mu b_i = 0. \qquad \text{(A.1)}$$

Multiplication of eq. (A.1) by $x_i$ and summation over all i yields $\lambda = -Z$. Solving eq. (A.1) for $x_i$ and using $\sum_{i=1}^{N} x_i^2 = 1$, $Z^*$ is obtained as the smallest solution of

$$\sum_{i=1}^{N} \frac{b_i^2}{Z - W_i} = 0. \qquad \text{(A.2)}$$

Considering now $-Z/B$ and again taking care of the side condition $\sum x_i^2 = 1$ by a Lagrangian parameter $\lambda$, we find stationary values for

$$2B(W_i + \lambda B) x_i - Z b_i = 0. \qquad \text{(A.3)}$$

Similarly to above, we obtain $\lambda B = -Z/2$, and, solving eq. (A.3) for $x_i$ and using $\sum_i (W_i - Z) x_i^2 = 0$, we find the Z values which correspond to stationary values of $-Z/B$ as solutions of the equation

$$\sum_{i=1}^{N} b_i^2 \frac{Z - W_i}{(Z - 2W_i)^2} = 0. \qquad \text{(A.4)}$$

Since $Z^*$ and $\bar{Z}_o$ are solutions of eqs. (A.2) and (A.4) respectively, a condition for the $b_i^2$, $i \geqslant 2$ is obtained if $Z^*$ and $\bar{Z}_o$ are plugged into and $b_1^2$ is eliminated from these equations, reading

$$\sum_{i=2}^{N} \alpha_i (1-\beta_i) b_i^2 = 0 , \qquad \text{(A.5)}$$

where

$$\alpha_i = \frac{1}{Z^*-W_i} \; ; \qquad \beta_i = \frac{Z^*-W_i}{Z^*-W_1} \cdot \frac{\bar{Z}_0-W_i}{\bar{Z}_0-W_1} \cdot \left(\frac{\bar{Z}_0-2W_1}{\bar{Z}_0-2W_i}\right)^2 .$$

Let us consider the consequences of the assumption that both $Z^* < 0$ and $\bar{Z}_0 < 0$. For $i \geqslant 2$ according to inequality (3.7) we have $W_i \geqslant - W_1 > 0$ and hence $\alpha_i < 0$. Considering the derivatives of $\beta_i$ with respect to $Z^*$ and $\bar{Z}_0$ it is easily seen that $\beta_i$ becomes minimal for $Z^* \rightarrow \bar{Z}_0 \rightarrow 0$ with $\beta_i \rightarrow 1$ so that $\alpha_i(1-\beta_i) > 0$ for $i \geqslant 2$. Thus, eq. (A.5) cannot be satisfied, i.e. the assumption cannot be valid.

Finally, a convenient expression for the stationary values of $- Z/B$ is obtained by plugging the $x_i$ from eq. (A.3) into the relation $\sum x_i^2 = 1$, thus yielding for the extremum (extrema)

$$\overline{\left(\frac{-Z}{B}\right)} = \left[ \sum_{i=1}^{N} \left( \frac{b_i}{\bar{Z} - 2W_i} \right)^2 \right]^{-1/2} . \qquad \text{(A.6)}$$

## Appendix B

We consider a current J of given amplitude $|J|$ with sign adjusted to favour stability, i.e.

$$\delta W^a = Z + |J| \cdot |B| \tag{B.1}$$

and determine the set $\{x_i, i = 1, ..N\}$ which minimizes $\delta W^a$. For simplicity, consider an idealized case of two modes only. Eliminating $x_2$ with $x_1^2 + x_2^2 = 1$ from eq. (B.1) we get two branches, the more unstable of which is (assuming $b_1, b_2 > 0$ without loss of generality)

$$\delta W^a = W_2 - (W_2 - W_1)x_1^2 + |J| \cdot |b_1 x_1 - b_2 \sqrt{1 - x_1^2}|. \tag{B.2}$$

In Fig. 9 $\delta W^a(x_1)$ is plotted for $J_3 > J_2 > J_1 > 0$ in a case with $\delta W^a(\hat{x}_1) > 0$, where $B(\hat{x}_1) = 0$. For small J $\delta W^a$ assumes a negative minimum. The current $J_o$ for which this minimum becomes zero defines the minimum feedback current required. If the current is further increased, there appears a threshold current above which $\delta W^a$ has no minimum but an infimum at $x_1 = \hat{x}_1$, the infimizing perturbation $x_1 = \hat{x}_1$, $x_2 = \sqrt{1 - \hat{x}_1^2}$ thus becoming independent of J.

For more than two $x_i$ the situation is analogous.

## Acknowledgement

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## Figure captions

Fig. 1: Coordinate system.

Fig. 2: Effective (blank) and ineffective (hatched) loop positions for A = 3.

Fig. 3: Effective and ineffective loop positions for A = 10.

Fig. 4: Dependence of feedback current on poloidal angle u.

Fig. 5: Optimum feedback current versus "distance" d for various shapes ($e = 2$; $\tau_3 = -0.5$, $\tau_4 = 0.7$, $\tau_3 = \tau_4 = 0$).

Fig. 6: Optimum loop positions and feedback currents for several $\beta_p$ .

Fig. 7: Dependence of $J_{opt}$ on elongation e for various shapes ($\tau_3 = -0.5$, $\tau_4 = 0.7$, $\tau_3 = \tau_4 = 0$).

Fig. 8: Minimizing plasma displacement for a stabilized case (a) and a case which cannot be stabilized (b). Feedback loops are at position of dots.

Fig. 9: Energy $\delta W^a$ versus mode amplitude $x_1$ for several values of feedback current J.

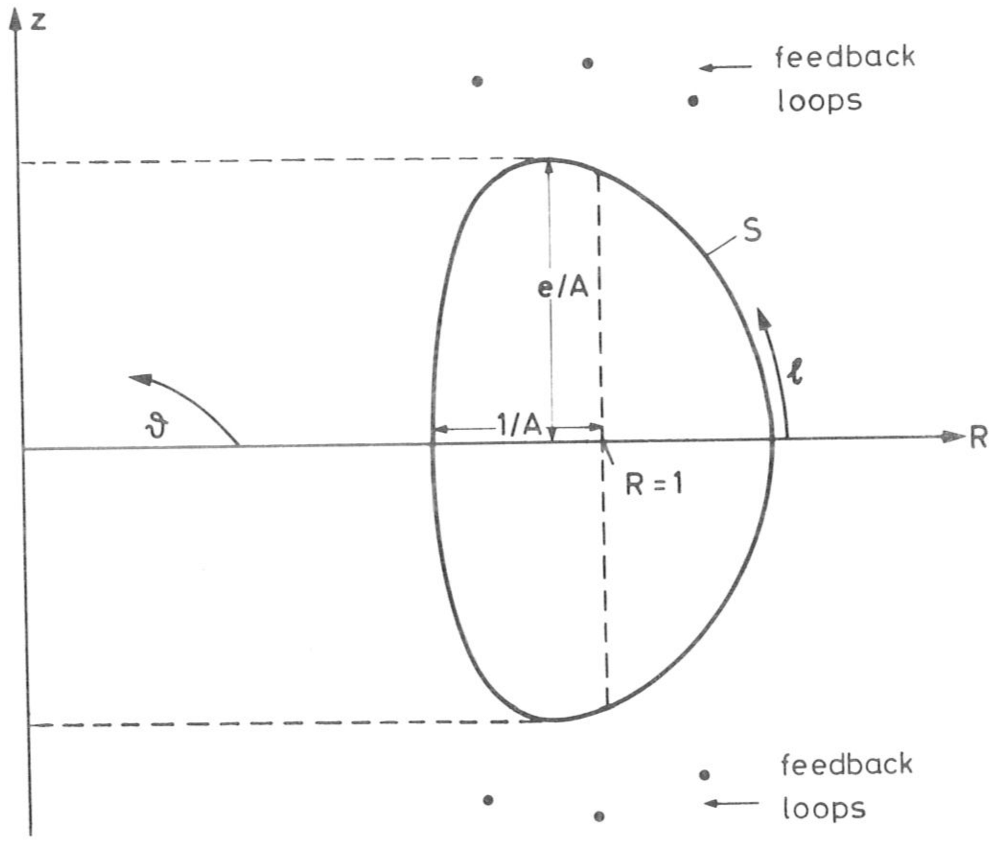



Fig. 1: Coordinate system.

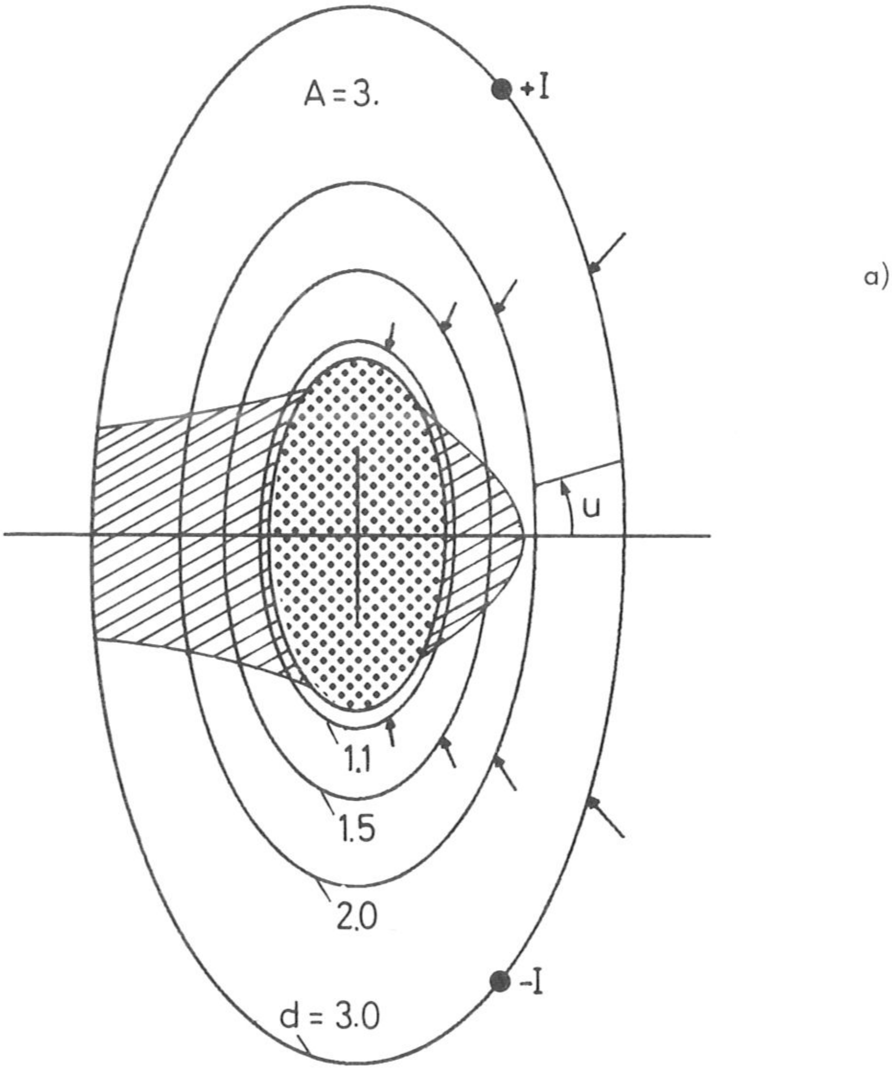
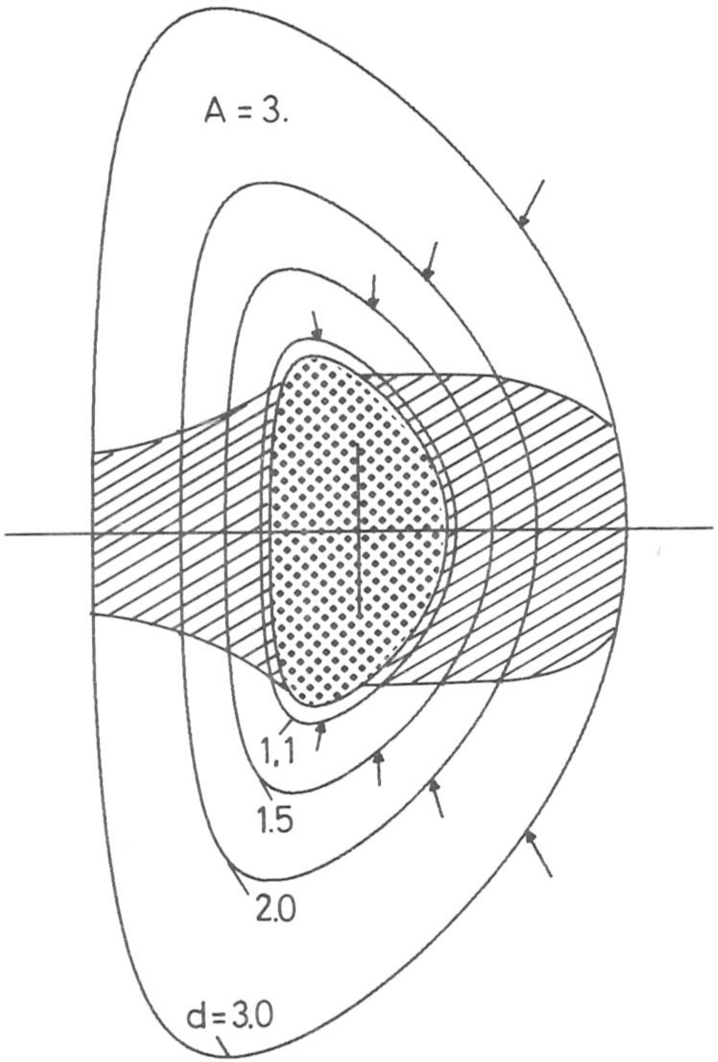
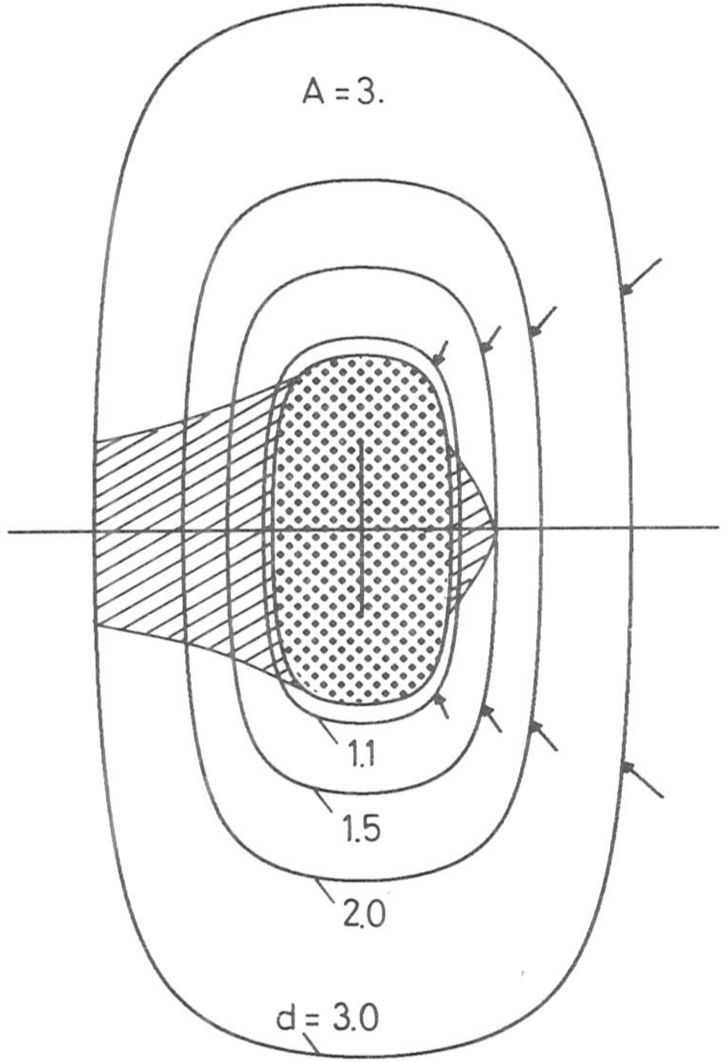



Fig. 2: Effective (blank) and ineffective (hatched) loop positions for A = 3.




2b)

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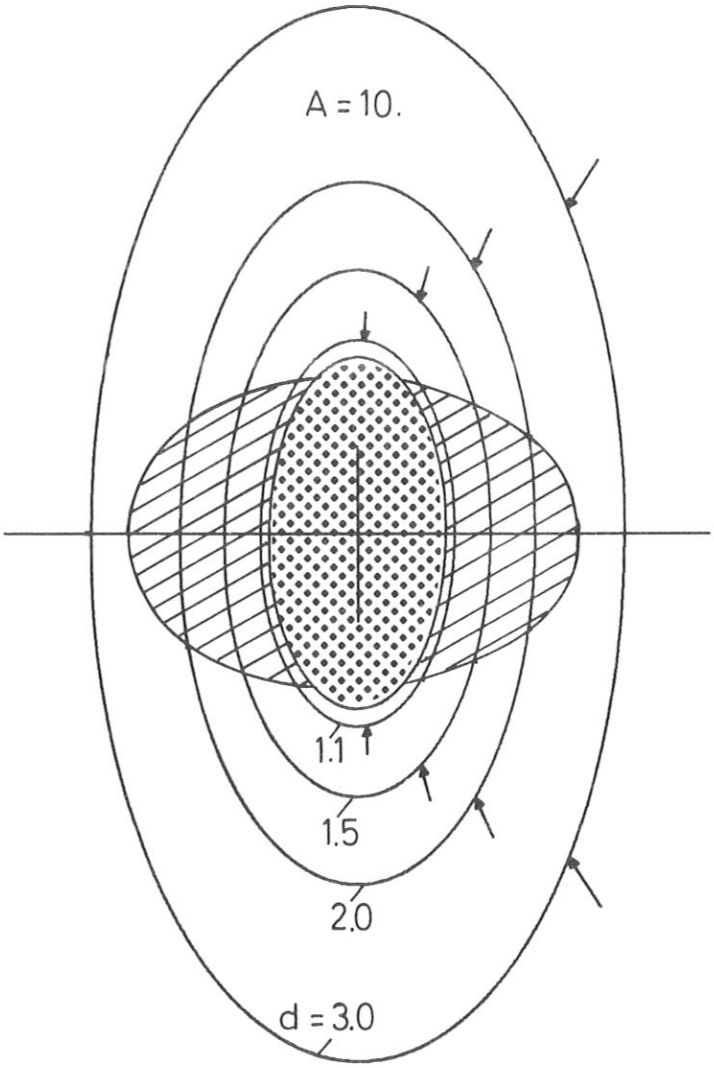
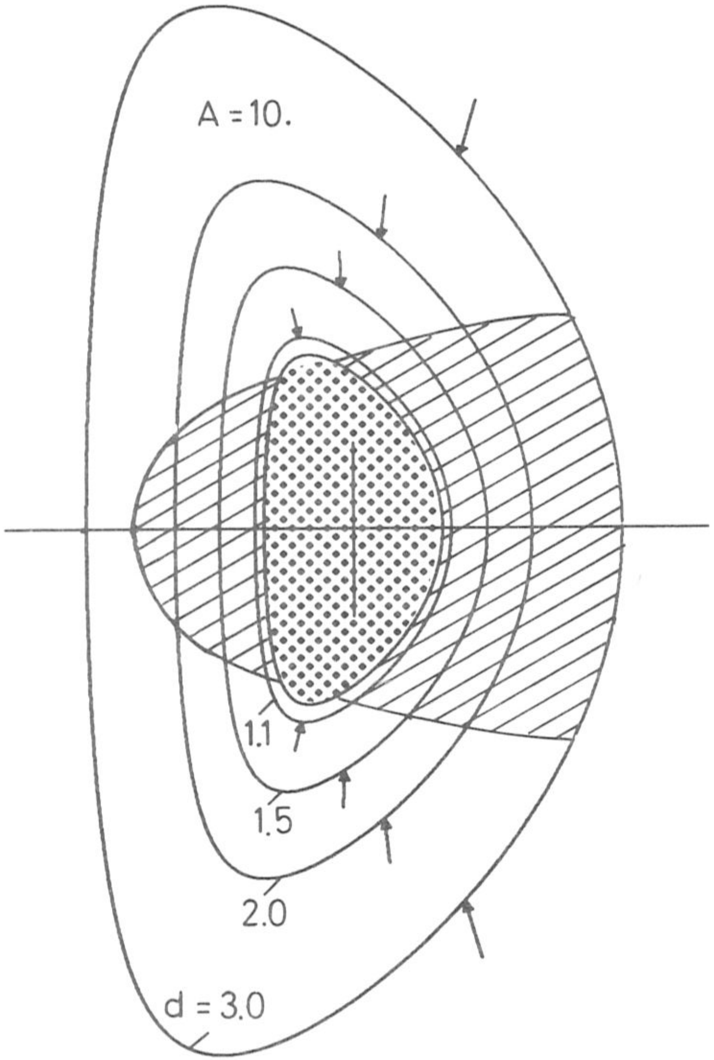



a)

Fig. 3: Effective and ineffective loop positions for A = 10.

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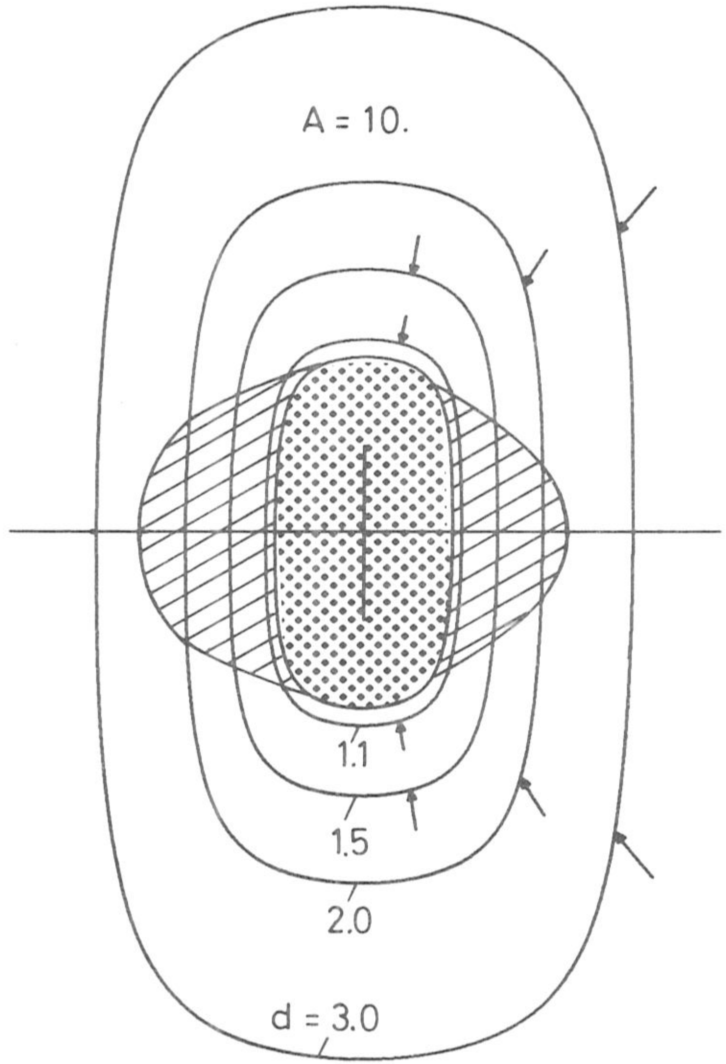


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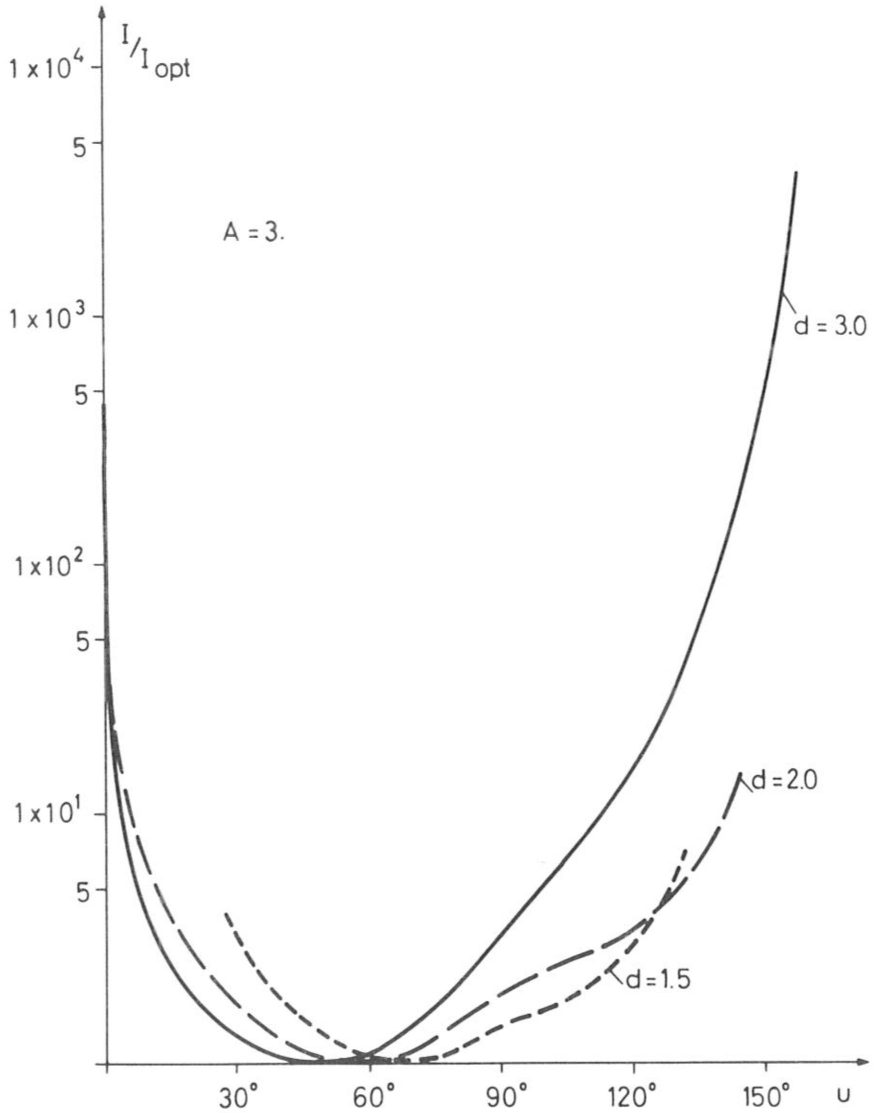



Fig. 4: Dependence of feedback current on poloidal angle u.

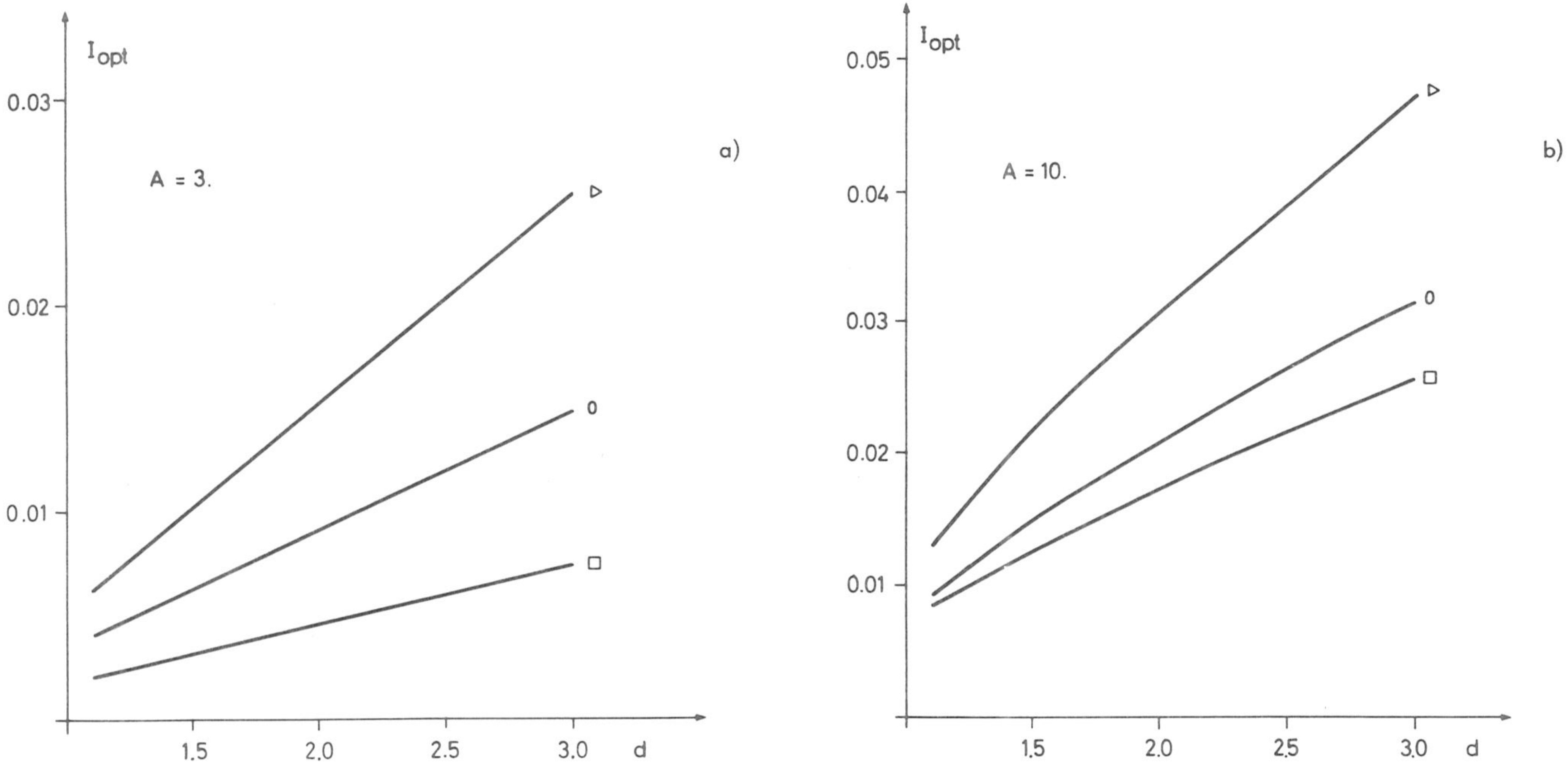



Fig. 5: Optimum feedback current versus "distance" d for various shapes ($e = 2$; $\tau_3 = -0.5$, $\tau_4 = 0.7$, $\tau_3 = \tau_4 = 0$).

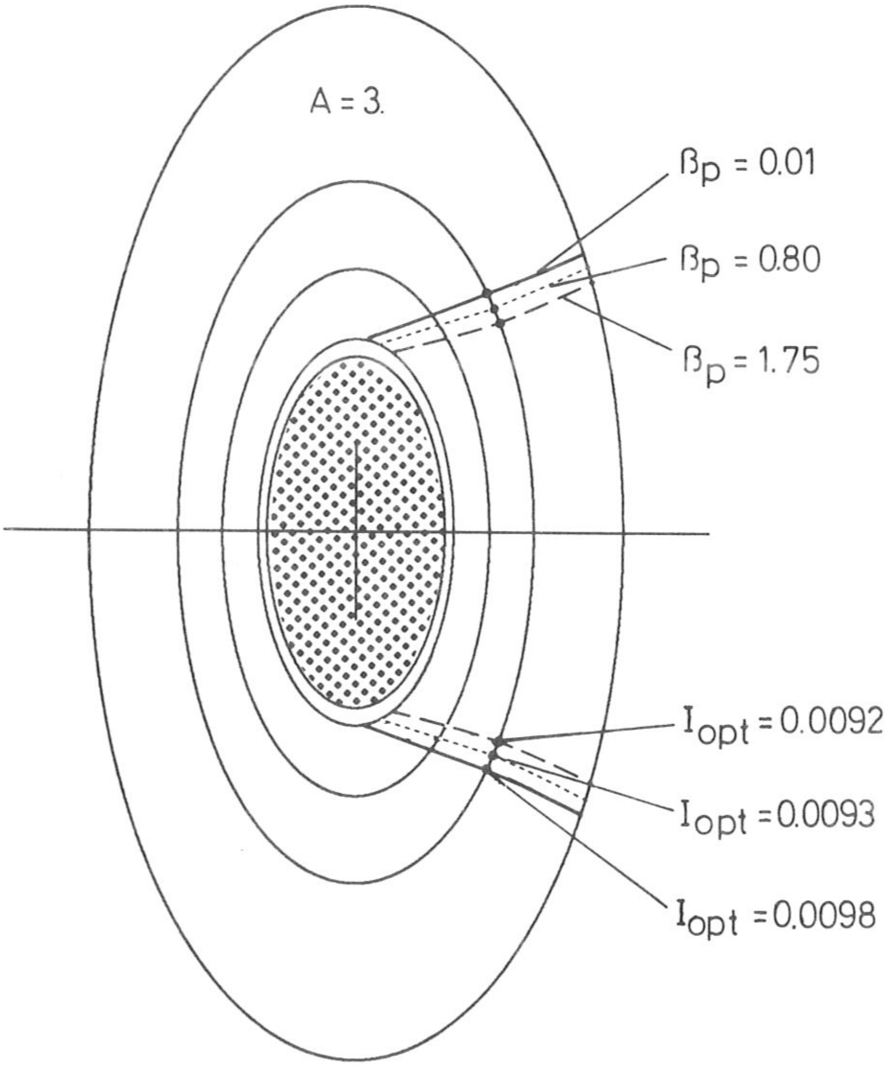



Fig. 6: Optimum loop positions and feedback currents for several $\beta_p$ .

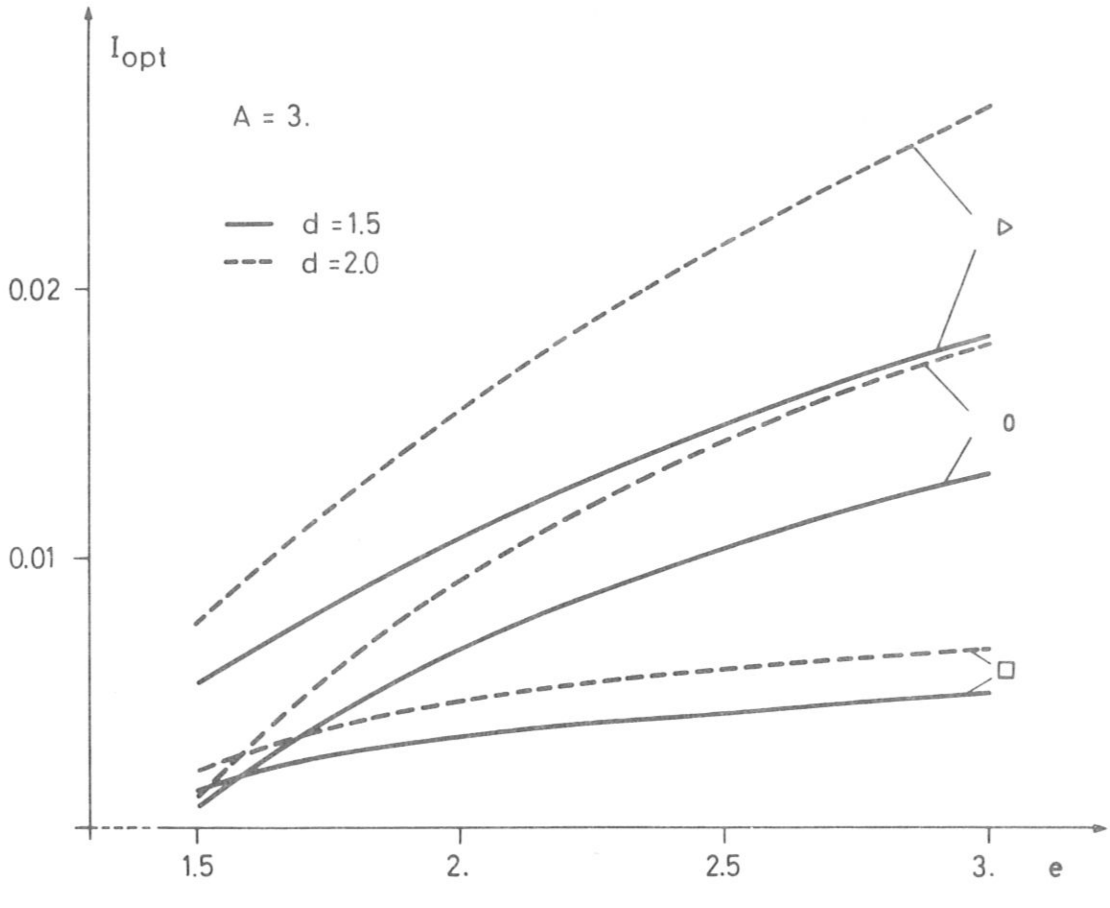
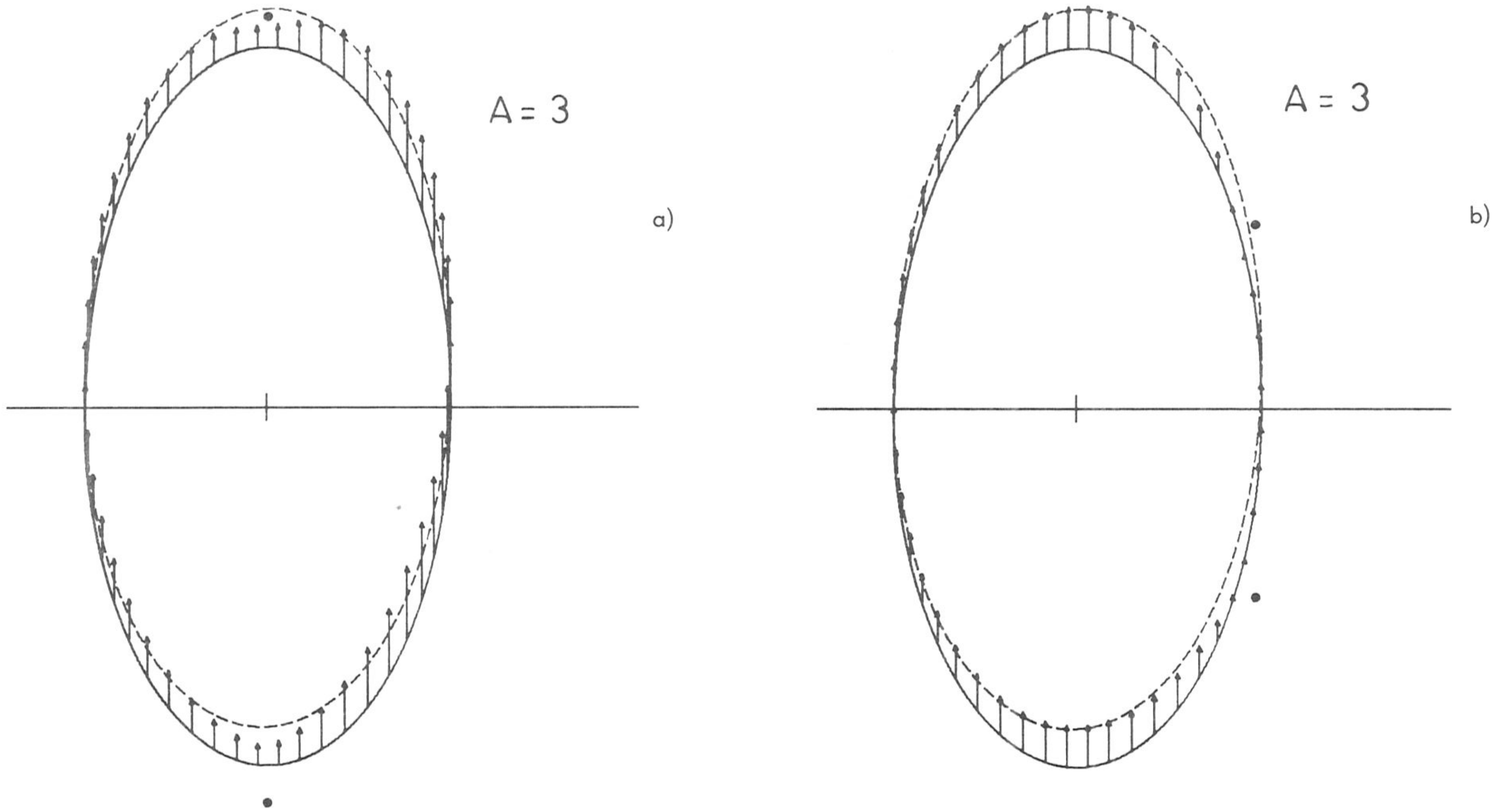



Fig. 7: Dependence of $J_{opt}$ on elongation e for various shapes ($\tau_3 = -0.5$, $\tau_4 = 0.7$, $\tau_3 = \tau_4 = 0$) .

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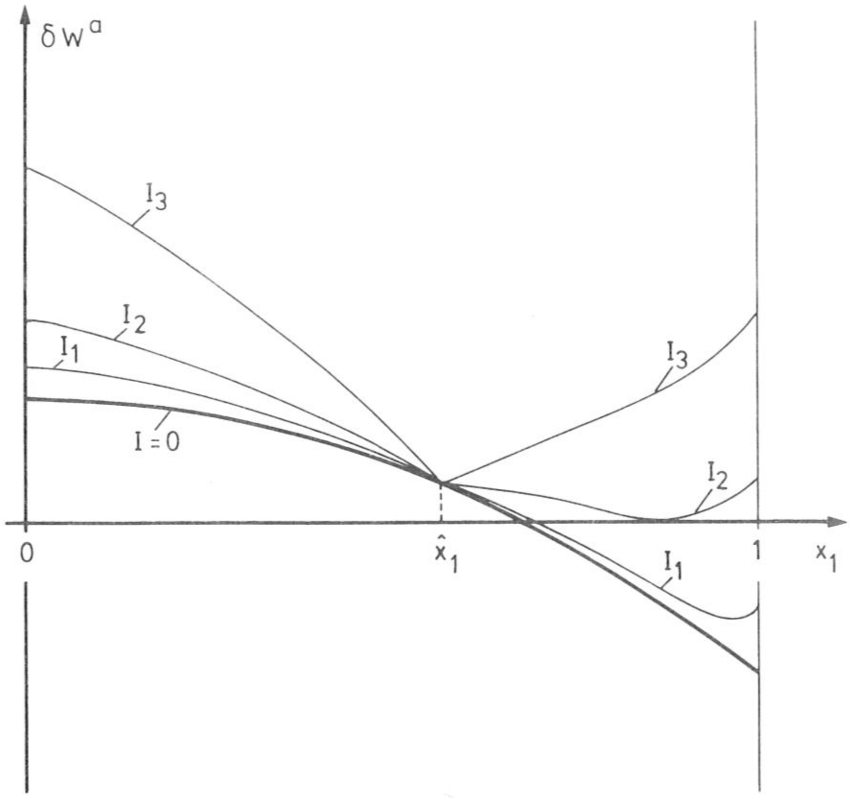


Fig. 9: Energy $\delta W^{a}$ versus mode amplitude $x_1$ for several values of feedback current J.

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