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Energy Principle for Dissipative Two-fluid Plasmas

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Abstract

An energy principle for the dissipative two-fluid theory in Lagrangian form is given. It represents a necessary and sufficient condition for stability allowing the use of test functions. It is exact for two-dimensional disturbances but is still correct in terms of perturbation theory for long wavelengths along the magnetic field. This may well find application in tokamak plasmas. A discussion of the general case and its relation to the stability of flows in hydrodynamics is given. This energy principle may be applied for estimating the magnitude of residual tearing modes in tokamaks.

1. Introduction

An energy principle for plasmas with resistivity viscosity and gyroviscosity has already been found by the author [1] within the framework of one-fluid resistive viscous MHD. Hall term and compressibility had been ignored. But at the same time it was recognized that the derivation of a necessary and sufficient condition for stability is possible without looking for eigenmodes for any stability equation of the form [1]:

$$N\ddot{\underline{\xi}} + (P+M)\dot{\underline{\xi}} + Q\underline{\xi} = 0 \tag{1}$$

where N and M are symmetric and positive definite, Q is symmetric, and P is antisymmetric. Then as shown in [1], the necessary and sufficient condition for stability is

$$(\xi, Q\xi) > 0 \tag{2}$$

The problem to solve, for a given physical system, is to find the proper representation $\underline{\xi}$ for which the linearized equation of motion can be put into form (1). This was possible for the one-fluid resistive viscous case [1, 2] and an important question is whether the method can be used for more sophisticated physical situations. In fact, every conservative system can be put in nearly the same form as (1) (the only difference being that $M \equiv 0$) by using a Lagrangian representation. This has been done, for example, for the Vlassov equation by Low [3], for the two-fluid theory by Vuillemin [4], and for the stability of stationary equilibria by Frieman and Rotenberg [5].

Unfortunately, to prove [1] the "necessary" part of condition (2), one needs a positive definite operator M which would correspond to some dissipation. Of course, most of the real systems are dissipative and an interesting question arises: what happens to a conservative system in Lagrangian representation if one introduces dissipative terms? The question is rather easy to answer for fluids at rest, as demonstrated in [6] for dissipative gravitating plasmas, where it is proved that the Lagrangian representation (in this case equivalent to the Eulerian) leads, in fact, to an equation of the form (1) with $\|M\| > 0$.

In a laboratory plasma gravitation is, of course, negligible and an electric current is needed for the confinement. In a two-fluid model, this implies that one of the fluids (more likely the electron fluid) cannot be at rest. In such a case, as we shall see, one obtains an equation of the form:

$$N\ddot{\underline{\xi}} + (P + M)\dot{\underline{\xi}} + (Q + Q_D)\underline{\xi} = 0 \qquad (3)$$

Introduction of dissipation in a Lagrangian system leads to a symmetric positive definite operator M, but also to an additional operator Q_D which in general is not symmetric.

An equation of the type of (3) will be derived here within the two-fluid theory, and cases for which Q_D vanishes or can be neglected will be discussed.

II. Two-fluid theory in Lagrangian representation

The two-fluid theory can be represented by the following equations: [7], [8]

$$n_i m_i \frac{d\underline{v}_i}{dt} = n_i q_i (\underline{E} + \underline{v}_i \times \underline{B}) - \nabla p_i - \nabla \cdot \underline{\underline{\Pi}}_i - \eta q_i^2/c \, n_i (n_i \underline{v}_i - n_e \underline{v}_e) \tag{4}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \underline{v}_i = 0 \tag{5}$$

$$\frac{d}{dt}\left(p_i n_i^{-\gamma}\right) = 0 \tag{6}$$

The index i denotes one of the fluids, ions or electrons, n is the density, m is the mass, $\underline{v}$ is the velocity field, q is the charge, $\underline{E}$ and $\underline{B}$ are the electric and magnetic fields, p is the scalar part of the pressure, $\underline{\underline{\Pi}}$ is the viscous and gyroviscous tensor [8], and γ is the quotient of specific heats.

To eqs. (4), (5) and (6) one must add the Maxwell equation for the electromagnetic field:

$$\varepsilon_0 \nabla \cdot \underline{E} = \sum_i n_i q_i \, , \tag{7}$$

$$\nabla \times \underline{B} = \mu_0 \sum n_i q_i \underline{v}_i + \frac{1}{c^2} \frac{\partial}{\partial t} \underline{E} \tag{8}$$

$$\nabla \times \underline{E} = - \dot{\underline{B}} \tag{9}$$

To study linear stability, one linearizes eqs. (4) to (8) and considers perturbations δn_i, $\delta \underline{v}_i$, δp_i, $\delta \underline{E}$ and $\delta \underline{B}$. These perturbations can be written in terms of Lagrangian displacements. Following Vuillemin [4] this can be done by substituting in equations (4) to (8):

$$\delta n_i = - \nabla \cdot n_i \underline{\xi}_i \tag{10}$$

$$\delta \underline{v}_i = \dot{\underline{\xi}}_i + \underline{V}_i \cdot \nabla \underline{\xi}_i - \underline{\xi}_i \cdot \nabla \underline{v}_i \tag{11}$$

$$\delta p_i = - \gamma p_i \nabla \cdot \underline{\xi}_i - \underline{\xi}_i \cdot \nabla p_i \tag{12}$$

where $\underline{\xi}_i$ is the Lagrangian displacement expressed in Euler coordinates of the equilibrium, and the new n_i , $\underline{V}_i$, $\mathbf{P}_i$ are those given by the following equilibrium equations:

$$\nabla \cdot n_i \underline{V}_i = 0 \tag{13}$$

$$n_i m_i \underline{V}_i \cdot \nabla \underline{V}_i = n_i q_i (\underline{E}_o + \underline{V}_i \times \underline{B}_o) - \nabla P_i - \nabla \cdot \underline{\underline{\Pi}}_i - \eta \frac{q_i^2}{c} n_i (n_i \underline{v}_i - n_e \underline{v}_e) \tag{14}$$

$$\underline{V}_i \cdot \nabla (P_i n_i^{-\gamma}) = 0 \tag{15}$$

$$\varepsilon_o \nabla \cdot \underline{E}_o = \sum_i n_i q_i \tag{16}$$

$$\nabla \times \underline{B}_o = \mu_o \sum_i n_i q_i \underline{V}_i \tag{17}$$

$$\nabla \times \underline{E}_o = 0 \tag{18}$$

$\underline{E}_o$ $\underline{B}_o$ being the electric and magnetic fields in stationary equilibrium. It is worthwhile noting that there are not always solutions to eqs. (13) to (18) if one does not add sources to eq. (13), for example. We assume here that we have solutions to these equations, with sources if need be.

III. Perturbations and stability

The perturbed equations in the conservative case have been derived in [4]. Here we just add to the equations of [4] the dissipative contributions in terms of the Lagrangian displacements $\underline{\xi}_i$ by using relations (10) to (12). The perturbed $\underline{E}$ and $\underline{B}$ fields can be written in terms of the perturbed potential vector, which means that the gauge is the one for which the electrostatic potential vanishes. To write the equations in a compact form, let us introduce a matrix vector

$$\underline{\zeta} = \begin{pmatrix} \underline{\xi}_i \\ \underline{\xi}_e \\ \underline{A} \end{pmatrix} \tag{19}$$

In terms of ζ the perturbed equations obtained from (4) to (9) can be written in the following form:

$$\begin{pmatrix} n_i m_i & 0 & 0 \\ 0 & n_e m_e & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix} \ddot{\underline{\zeta}} + \begin{pmatrix} 2 n_i m_i \underline{V}_i \cdot \nabla + e \underline{B}_0 \times + A_{S_i}, & 0 & n_i e \\ 0 & 2 n_e m_e \underline{V}_e \cdot \nabla - e \underline{B}_0 \times + A_{S_e}, & -n_e e \\ -n_i e & n_e e & 0 \end{pmatrix} \dot{\underline{\zeta}} +$$

$$\begin{pmatrix} S_i + \eta \frac{e^2}{c} n_i^2 & -\eta \frac{e^2}{c} n_i n_e & 0 \\ -\eta \frac{e^2}{c} n_i n_e & S_e + \eta \frac{e^2}{c} n_e^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\underline{\zeta}} \quad +$$

$$\begin{pmatrix} Q_i & 0 & -n_i e\, \underline{v}_i \times \nabla \times \\ 0 & Q_e & n_e e\, \underline{v}_e \times \nabla \times \\ -n_i e(\underline{v}_i \cdot \nabla \cdot - \ldots \nabla \underline{v}_i) & n_e e(\underline{v}_e \cdot \nabla \ldots - \ldots \nabla \underline{v}_e) & \nabla \times \nabla \times \\ -e\, \underline{v}_i\, \nabla \cdot n_i \ldots & +e\, \underline{v}_e \cdot \nabla \cdot n_e \ldots & \end{pmatrix} \underline{\xi} +$$

$$Q_D\, \underline{\xi} \quad = 0 \tag{20}$$

where

$$Q_i \underline{\xi}_i = n_i m_i \underline{v}_i \cdot \nabla \underline{v}_i \cdot \nabla \underline{\xi}_i - n_i q_i\, \underline{\xi}_i \cdot \nabla \left[\underline{E}_o + \underline{v}_i \times \underline{B}_o - \frac{1}{n_i q_i} \nabla p_i \right]$$
$$- n_i q_i \left[\underline{v}_i \cdot \nabla \underline{\xi}_i - \underline{\xi}_i \cdot \nabla \underline{v}_i \right] \times \underline{B}_o - \nabla (\underline{\xi}_i \cdot \nabla p_i) + \nabla p_i\, \nabla \cdot \underline{\xi}_i \tag{21}$$
$$- \nabla (\gamma p_i\, \nabla \cdot \underline{\xi}_i)$$

S_i and A_{si} are defined like in [6] .

$Q_D \underline{\xi}$ can be defined in terms of $\underline{\xi}_i$ and $\underline{\xi}_e$, of $\underline{v}_i$ and $\underline{v}_e$ and of the transport coefficients. The explicit form of $Q_D \underline{\xi}$ will, of course, depend on the transport coefficients. For simplicity, we are going to assume 2-d equilibria, the ions at rest and the velocity field of the electrons $\underline{v}_e = f(x,y)\, \underline{e}_z$, and we ignore the viscosity and gyroviscosity of the electrons, which is plausible because of their smaller mass. This means that in equilibrium we have among other relations

$$e n_i \underline{E}_o - \nabla P_i + \eta \frac{e^2}{c} n_i n_e \underline{V}_e = 0 \tag{22}$$

$$-e n_e \underline{E}_o - \nabla P_e - \eta \frac{e^2}{c} n_e^2 \underline{V}_e = 0 \tag{23}$$

In the $\underline{e}_z$ direction these equations reduce to

$$E_{z_o} \underline{e}_z = -\eta \frac{e^2}{c} n_e \underline{V}_e \tag{24}$$

which is the simplest form of Ohm's law.

We are now able to calculate Q_D by collecting the appropriate terms from eq. (4):

$$\begin{aligned} Q_D^{d} \cdot \underline{\xi}_e = & -e\,\delta n_e \left(E_{z_o} \underline{e}_z + \eta \frac{e}{c} n_e \underline{V}_e \right) \\ & - \frac{e^2}{c} n_e \underline{V}_e\, \delta(\eta n_e) - \frac{e^2}{c} \eta n_e \left(\underline{V}_e \cdot \nabla \underline{\xi}_e - \underline{\xi}_e \cdot \nabla \underline{V}_e \right) \end{aligned} \tag{25}$$

The first set of parentheses in eq. (25) vanishes because of eq. (24). It remains for us to calculate $\delta(\eta n_e)$. We know [7] that $\eta \simeq \frac{1}{T_e^{3/2}}$. This fact combined with the adiabaticity relation for the electrons (6) now allows us to state that

$$\frac{d}{dt}\left(n_e^{1-\gamma} T_e \right) = 0$$

or also that $\quad \frac{d}{dt}\left(n_e^{1-\gamma} T_e \right)^{-3/2} = 0$

which for $\gamma = \frac{5}{3}$ leads to

$$\frac{d}{dt}\left(\eta n_e \right) = 0 \tag{26}$$

From eq. (26) we can conclude that the Euler perturbation

$$\delta(\eta n_e) = -\underline{\xi}_e \cdot \nabla(\eta n_e) \tag{27}$$

Using eqs. (27) and (18), and adding the ion contribution, we obtain

$$Q_D \underline{\xi} = \begin{pmatrix} 0 & \frac{e^2}{c}\eta n_i \underline{V_e} \cdot \nabla \cdots & 0 \\ 0 & -\frac{e^2}{c}\eta n_e \underline{V_e} \cdot \nabla & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{\xi} \tag{28}$$

This operator is thus, unfortunately, not symmetric, and then eq. (20) is not of the type (1) unless $\underline{V_e} \cdot \nabla \underline{\xi} = 0$, which implies 2-dimensional perturbations. This would be the direct generalization of reference [1] to the two-fluid theory. There is no need to prove in eq. (20) the symmetry of P and Q because of the Lagrangian [4] representation used and the definite positivity and symmetry of M because it is done in, for example, [9] and [6].

IV. Discussion of Q_D and conclusion

In the sense of perturbation theory Q_D could be neglected in the case of long wavelength along $\underline{e}_z$. This case is quite useful because of the direct applications to tokamak plasmas for which the wavelength in the toroidal direction is much larger than in the poloidal one.

It seems plausible that, if the ion fluid is at rest or has a small flow, the residual Q_D will not, in general, be significant. This should be looked at more carefully by making estimates on the basis of convergent perturbation techniques.

The fact that Q_D cannot be ignored in all cases is largely demonstrated in hydrodynamics, where for large viscosity Q_D must have an important stabilizing contribution. But in hydrodynamics instabilities are primarily due to the kinetic energy in the flow. In plasma physics the instabilities are essentially of a potential character, and so the influence of dissipation will act primarily on the operator M but not significantly on Q_D. This, as said before, should be justified with a more profound analysis. In this sense the problem of marginal stability in the dissipative two-fluid theory and for long wavelength and nearly parallel currents could be considered as solved. The necessary and sufficient condition would be $(\xi, Q\xi) > 0$ with the Q found in eq. (20).

The advantages of this stability analysis is that it permits us to understand the different energy reservoirs by just using test functions in $(\xi, Q\xi)$

and to make estimates of the maximum possible energy conversion from one reservoir to another. One possible example could be an estimate of the amplitude of a tearing deformation of the magnetic field when the free energy corresponds to the one of drift waves or of rippling modes.

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