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Stability of a Dissipative Gravitating Two-fluid Plasma at Rest

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Abstract

An "energy principle" for stability of two-fluid at rest in a gravitational field and a vacuum magnetic field is given. Viscosity, gyroviscosity, resistivity, compressibility, Hall term and electron inertia are taken into account.

I Introduction

The stability of fluids at rest in a gravitational field and a vacuum magnetic field was revisited by Barston [1] (1970). He assumed the simplest form of Ohm's law

$$\underline{E} + \underline{v} \times \underline{B} = \eta \underline{j}$$

and the viscosity in the form

$$\Delta(\nu \underline{v}) - \underline{v} \Delta \nu - (\nabla \times \underline{v}) \times \nabla \nu$$

ignoring among other effects gyroviscosity, Hall term and compressibility. This was an application of stability theorems found by Barston [2] (1969) which allow one to deal with stability equations of the type

$$N \ddot{\underline{Y}} + M \dot{\underline{Y}} + Q \underline{Y} = 0$$

where N, M and Q are symmetric operators and N and M are positive. A recent extension by Tasso [3] (1976) of these theorems allows one to handle equation of the type

$$N \ddot{\underline{Y}} + (M + P) \dot{\underline{Y}} + Q \underline{Y} = 0$$

where N, M, and Q have the same properties as before but P can now be any antisymmetric operator. The latter type of equation is of course, the one most encountered in physical problems [3,4]. What is important here is that, if M is positive, the operator Q governs stability [3] independently of the existence of P. This technique has already been succesfully applied to fusion plasmas (tokamaks) to study their stability with respect to resistive perturbations [5].

Here we demonstrate how powerful this technique is in finding an "energy principle", i.e. a necessary and sufficient condition for stability of a two-fluid gravitating plasma at rest with respect to all perturbations described within the two-fluid theory. This means that viscosity, gyroviscosity, resistivity, compressibility, Hall term, and electron inertia are taken into account, which represents a significant extension of reference [1].

II Equilibrium

The equations governing the equilibrium are

$$\nabla p_{o_i} = n_o \left(m_i \nabla \phi_o - q_i \underline{E_o} \right) \; ; \; \underline{V_{oi}} = 0 \tag{1}$$

$$\nabla \times \underline{B_o} = \nabla . \underline{B_o} = 0 \tag{2}$$

$$\nabla \times \underline{E_o} = \nabla . \underline{E_o} = 0 \tag{3}$$

where $\underline{B_o}$ and $\underline{E_o}$ are the magnetic and electric fields, ϕ_o denotes the gravitational potential, p_{o_i} is the pressure q_i is the electrical charge, $\underline{V_{oi}}$ and m_i are the velocity and mass of the particle, and n_o is the particle density. The index i denotes the electronic or the ionic fluid .

Taking the curl of eq. (1) together with eq. (3), we obtain

$$\nabla \ln n_o \times \nabla p_{o_i} = 0 \tag{4}$$

III Perturbations and Stability Equations

The perturbations around the static equilibrium are denoted by small letters or indexed by 1, and the displacement vectors are called

The perturbed equations of motion lead to [6]

$$n_o m_i \ddot{\underline{\xi}}_i = -\nabla p_i + n_o q_i (\underline{E}_1 + \dot{\underline{\xi}}_i \times \underline{B}_0) - \nabla\cdot\underline{\underline{\Pi}}_i - n_i m_i \nabla\phi_c - n_o m_i \nabla\phi_1 + n_i q_i \underline{E}_0 + \underline{\Gamma}_i \tag{5}$$

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_o \underline{\xi}_i) = 0 \tag{6}$$

$$p_i = -\gamma P_{o_i} \nabla\cdot\underline{\xi}_i - \underline{\xi}_i \cdot \nabla P_{o_i} \tag{7}$$

where γ is the rate of specific heats.

In a gauge $\phi_1 = 0$ we have

$$\underline{E}_1 = -\dot{\underline{A}}_1 \quad ; \quad \underline{B}_1 = \nabla\times\underline{A}_1 \tag{8}$$

$$\nabla\times\nabla\times\underline{A}_1 = \sum_{i=1,2} \mu_o n_o q_i \dot{\underline{\xi}}_i - \frac{1}{c^2}\ddot{\underline{A}}_1 \tag{9}$$

The definition of $\underline{\underline{\Pi}}$ [7] is

$$\begin{aligned}
-\Pi_{xx} &= \alpha(\Gamma_{xx} + \Gamma_{yy}) + \beta\Gamma_{xy} + \frac{\beta^2}{4\alpha}(\Gamma_{xx} - \Gamma_{yy}), \\
-\Pi_{yy} &= \alpha(\Gamma_{xx} + \Gamma_{yy}) - \beta\Gamma_{xy} - \frac{\beta^2}{4\alpha}(\Gamma_{xx} - \Gamma_{yy}), \\
-\Pi_{zz} &= 2\alpha\Gamma_{zz}, \\
-\Pi_{xy} &= -\Pi_{yx} = \frac{\beta}{2}(\Gamma_{yy} - \Gamma_{xx}) + \frac{\beta^2}{2\alpha}\Gamma_{xy}, \\
-\Pi_{xz} &= -\Pi_{zx} = 2\beta\Gamma_{yz} + 2\Gamma_{xz}\frac{\beta^2}{\alpha}, \\
-\Pi_{yz} &= -\Pi_{zy} = \frac{2\beta^2}{\alpha}\Gamma_{yz} - 2\beta\Gamma_{xz}, \\
\Gamma_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial v_m}{\partial x_m}\delta_{ij}, \\
\beta &= \frac{P}{\omega_c} \; ; \; \underline{v} \equiv \frac{\partial\underline{\xi}}{\partial t} \; ; \; \alpha = \frac{2}{3}\beta\omega_c\tau
\end{aligned} \tag{10}$$

where x, y, z are a local system of Cartesian coordinates, z being along the magnetic field, ω_c is the particle cyclotron frequency, and τ is the particle - particle collision time. For simplicity, we have omitted the index i .

The quantity $\underline{P}_{ij}$ is the total momentum transferred to the particles of fluid i per unit volume and time by collisions with the particles of fluid j :

$$\underline{P}_{ij} = \delta (\dot{\underline{\xi}}_j - \dot{\underline{\xi}}_i) \tag{11}$$

From equations (5) and (9) we obtain the following system of equations in matrix operational form

$$N\ddot{Y} + (M+P)\dot{Y} + (\Theta_1 + \Theta_2) Y = 0 \tag{12}$$

$$Y \equiv \begin{pmatrix} \underline{\xi}_1 \\ \underline{\xi}_2 \\ \underline{A}_1 \end{pmatrix} \quad ; N \equiv \begin{pmatrix} n_o m_1 & 0 & 0 \\ 0 & n_o m_2 & 0 \\ 0 & 0 & \varepsilon_o \end{pmatrix}$$

$$M \equiv \begin{pmatrix} S_1 + \delta & -\delta & 0 \\ -\delta & S_2 + \delta & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad P \equiv \begin{pmatrix} As_1 + n_o q_1 \underline{B}_o \times & 0 & n_o q_1 \\ 0 & As_2 + n_o q_2 \underline{B}_o \times & n_o q_2 \\ -n_o q_1 & -n_o q_2 & 0 \end{pmatrix}$$

$$(S_i + As_i) \underline{\xi}_i \equiv \nabla . \underline{\underline{\Pi}}_i \tag{13}$$

where $S_i\,\underline{\dot{\xi}}_i\,(A_{S_i}\,\underline{\dot{\xi}}_i)$ corresponds to the symmetric (antisymmetric) part of $\nabla\cdot\underline{\underline{\Pi}}_i$.

$$\Theta_1 \equiv \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & 1/\mu_0\,\nabla\times\nabla\times \end{pmatrix} \;;\; \Theta_2 \equiv \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_i\,\underline{\xi}_i \equiv \nabla P_i + n_i m_i \nabla\phi_0 - n_i q_i \underline{E_0} \tag{14}$$

$$T_{ij}\,\underline{\xi}_j \equiv -4\pi G\, n_0 m_i m_j \nabla \int dV' K(\underline{r},\underline{r}')(\nabla' n_0 + n_0 \nabla')\cdot\underline{\xi}_j \tag{15}$$

where $K(\underline{r},\underline{r}')$ is the Green's function.

The operator Θ_2 appears because we are considering the perturbation on the self-gravitational field (see Appendix)

IV Boundary Conditions and Symmetry Properties

We assume either that the perturbations decay at infinity for an infinite plasma or that they have to vanish on a surface surrounding the plasma. We now prove that the matrix operators N, M, Θ_1, Θ_2 are symmetric, N and M are positive, and P is antisymmetric.

To obtain this result, we consider scalar products (Y_2, AY_1), where A is one of the matrix operators defined before. The parentheses mean the integral over the volume of the product:

$$Y_2\, A\, Y_1 \equiv (\underline{t}_1 \;\; \underline{t}_2 \;\; \underline{t}_3)\, A \begin{pmatrix} \underline{\xi}_1 \\ \underline{\xi}_2 \\ \underline{A}_1 \end{pmatrix} \tag{16}$$

The operator N, of course, is symmetric and positive definite. The operator $\nabla\cdot\underline{\underline{\Pi}}_i$ has symmetric and positive definite terms in α and β^2/α and antisymmetric terms in β [8]. The operator $m_o q_i \underline{B}_o \times$ is antisymmetric. The operator $\begin{pmatrix} \delta & -\delta & 0 \\ -\delta & \delta & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is symmetric and positive.

The operator $\begin{pmatrix} 0 & 0 & m_o q_1 \\ 0 & 0 & m_o q_2 \\ -m_o q_1 & -m_o q_2 & 0 \end{pmatrix}$ is, of course, antisymmetric.

The operator $\nabla\times\nabla$ is symmetric. We prove in the Appendix that the operators T_{ij} are also symmetric.

From eqs. (1), (7), (14) and (16) we can write

$$(\underline{\phi}_i , R_i \underline{\xi}_i) = -\int dV \left(P_i \nabla + \frac{m_i \nabla P_o}{m_o}\right) \cdot \underline{\phi}_i =$$

$$= \int dV \Big[\gamma P_{o_i} \nabla\cdot\underline{\xi}_i \nabla\cdot\underline{\phi}_i + (\underline{\xi}_i \cdot \nabla P_{o_i}) \nabla\cdot\underline{\phi}_i + (\underline{\phi}_i \cdot \nabla P_{o_i}) \nabla\cdot\underline{\xi}_i$$

$$+ (\underline{\phi}_i \cdot \nabla P_{o_i})(\nabla \ln m_o \cdot \underline{\xi}_i)$$

In the last equation the first term and the sum of the second and third terms are symmetric. The last term is also symmetric because from eq. (4) we can write

$$(\underline{\phi}_i \cdot \nabla P_{o_i})(\nabla \ln m_o \cdot \underline{\xi}_i) = (\nabla P_{o_i} \cdot \nabla \ln m_o)\, \xi_{N_i} \phi_{N_i}$$

where

$$F_{N_i} \equiv \frac{\underline{F} \cdot \nabla P_{o_i}}{|\nabla P_{o_i}|} \qquad (17)$$

According to [3] these properties allow us to state that the necessary and sufficient condition for exponential stability is

$$\left(Y, (\Omega_1+\Omega_2) Y\right) > 0 \tag{18}$$

V Explicit Criterion

$$\left(Y, (\Omega_1+\Omega_2)Y\right) = \int dV \sum_{i=1,2} \left[\gamma P_{o_i} |\nabla\cdot\underline{\xi}_i|^2 + 2(\underline{\xi}_i\cdot\nabla P_{o_i})\nabla\cdot\underline{\xi}_i + \nabla P_{o_i}\cdot\nabla\ln n_o |\xi_{N_i}|^2 \right.$$
$$+ \int dV\,(\nabla\times\underline{A}_1)^2 + 4\pi G \sum_{\substack{i=1,2\\ j=1,2}} m_i m_j \int dV\left(\underline{\xi}_i\cdot\nabla n_o + n_o\nabla\cdot\underline{\xi}_i\right)\int dV'$$
$$k(\underline{r},\underline{r}')\left(\underline{\xi}_j\cdot\nabla' n_o + n_o\nabla'\cdot\underline{\xi}_j\right) \tag{19}$$

where the last term appears because we are considering the perturbation on the self-gravitational field (see Appendix)

a) If we neglect the self-gravitational perturbation and consider an incompressible fluid, we find essentially that

$$\nabla P_{o_i}\cdot\nabla\ln n_o > 0$$

is necessary and sufficient for stability as in Ref. [1]. The Hall effect does not affect the stability condition. So we think that the stabilization and destabilization due to the Hall effect reported in the literature [9] appear because infinitely conducting media have been considered.

b) The first integral of eq. (19) can be rewritten in the following way:

$$\int dV \sum_{i=1,2}\left[\nabla P_{o_i}\cdot\nabla\ln n_o\left(\xi_{N_i} + \frac{\nabla\cdot\underline{\xi}_i\,|\nabla P_{o_i}|}{\nabla P_{o_i}\cdot\nabla\ln n_o}\right)^2 - \frac{(\nabla\cdot\underline{\xi}_i)^2\,|\nabla P_{o_i}|^2}{\nabla P_{o_i}\cdot\nabla\ln n_o} + \gamma P_{o_i}(\nabla\cdot\underline{\xi}_i)^2\right]$$

If $\nabla P_{o_i}\cdot\nabla\ln n_o > 0$, we might have instabilities in the compressible case unless

$$\frac{\nabla\ln P_{oi}}{\nabla\ln n_o} < \gamma$$

which is well known from the stability of atmospheres.

c) If the plasma satisfies the conditions

$$m_o = \text{constant}, \quad P_{o_i} = \text{constant} \tag{20}$$

we can write

$$\Delta\Phi_1 = -4\pi G m_o \sum_{j=1,2} m_j \nabla \cdot \underline{\xi}_j \tag{21}$$

Defining

$$\underline{\eta}_j(\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int d\underline{r}\, e^{-i\underline{k}\cdot\underline{r}}\, \underline{\xi}_j(\underline{r})$$

we obtain

$$\nabla\Phi_1 = \frac{-4\pi G m_o}{(2\pi)^{3/2}} \sum_{j=1,2} m_j \int d\underline{k}\, e^{i\underline{k}\cdot\underline{r}} (\underline{\eta}_j \cdot \underline{k}) \frac{\underline{k}}{k^2} \tag{22}$$

$$(Y, \Theta_2 Y) = \int d\underline{k} \sum_{\substack{i=1,2\\ j=1,2}} (\underline{\eta}_i^* \cdot \underline{k})(\underline{\eta}_j \cdot \underline{k}) \left[\frac{-4\pi G m_o^2 m_i m_j}{k^2} + \gamma P_{o_i} \delta_{ij} \right] \tag{23}$$

Considering

$$(\underline{\eta}_i^* \cdot \underline{k})(\underline{\eta}_j \cdot \underline{k}) = (\underline{k} \cdot \underline{D}_i^*)(\underline{k} \cdot D_j)\, \delta(\underline{k} - \underline{k}_o) \tag{24}$$

the condition (18) becomes

$$\sum_{\substack{i=1,2\\ j=1,2}} (\underline{D}_i^* \cdot \underline{k}_o)(\underline{D}_j \cdot \underline{k}_o) \left[\frac{-4\pi G m_o^2 m_i m_j}{k_o^2} + \gamma P_{o_i} \delta_{ij} \right] \tag{25}$$

Considering the perturbation of one fluid only, we obtain the Jean's condition for stability [10]:

$$k_o > n_o m \sqrt{\frac{4\pi G}{\gamma P_o}} \tag{26}$$

Appendix

We consider here the contributions to eq. (12) from the perturbed self-gravitational field. They are

$$n_o m_i \nabla \phi_1$$

where ϕ_1 satisfies the equation

$$\Delta \phi_1 = -4\pi G \sum_{j=1,2} n_j m_j$$

The solution of this equation in terms of the Green's function is

$$\phi_1 = 4\pi G \sum_{j=1,2} \int dV' \, K(\underline{r}, \underline{r}') n_i(\underline{r}') m_i$$

From eq. (6) we can write

$$n_i(\underline{r}') = -(\nabla' n_o + n_o \nabla') \cdot \underline{\xi}_i$$

Therefore, we obtain

$$n_o m_i \nabla \phi_i = 4\pi G n_o m_i \sum_{j=1,2} m_j \nabla \int dV' K(\underline{r}, \underline{r}') (\nabla' n_o + n_o \nabla') \cdot \underline{\xi}_j = \sum_{j=1,2} T_{ij} \underline{\xi}_j$$

We can write for the operator Q_2

$$(Y, Q_2 Y) = \sum_{\substack{i=1,2 \\ j=1,2}} \int dV \, \underline{\psi}_i \cdot T_{ij} \, \xi_j =$$

$$= 4\pi G \sum_{\substack{i=1,2 \\ j=1,2}} m_i m_j \int dV \left(\underline{\xi}_i \cdot \nabla n_0 + n_0 \nabla \cdot \underline{\xi}_i \right) \int dV' K(\underline{n}, \underline{n}') \left(\underline{\xi}_j \cdot \nabla' n_0 + n_0 \nabla' \cdot \underline{\xi}_j \right)$$

The last expression is symmetric under the interchange between Y_2 and Y_1 .

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