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Connection between Wave Envelope and Explicit Solution of a Nonlinear Dispersive Wave Equation

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Abstract

The envelope properties of the explicitly solvable nonlinear dispersive equation: $U_t = (U^3)_x + \frac{3}{2}(U^2)_{xx} + U_{xxx}$ are studied by using a perturbation scheme. The equation describing the envelope of plane wave solutions of this equation is found to be linear and does not admit localized solutions, in agreement with the exact solutions of the above equation.

I. Introduction

There exists a class of nonlinear partial differential equations which can be solved by applying the well-known Cole-Hopf transformation [1]. An interesting example of this class is

$$U_t = (U^3)_x + \frac{3}{2} (U^2)_{xx} + U_{xxx} . \qquad (1)$$

This equation is important from the point of view of constructing solvable models for physical problems having nonlinearity, dissipation and dispersion. In this report the envelope characteristics of the plane wave solutions of eq. (1) are studied by using a perturbation scheme due originally to Krylov, Bogoliubov and Mitropolsky [2]. In section 2 the equation describing the envelope of the plane wave solutions of eq. (1) is obtained. The spatial behaviour is studied in section 3 by employing the procedure used in the study of wave collapse. The stability against long-wavelength perturbations is also discussed in section 3.

II. Perturbation Scheme

We consider an expansion of U in terms of a small parameter ε as

$$U = \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \varepsilon^3 U^{(3)} + \dots , \qquad (2)$$

where $U^{(1)}$ is a monochromatic plane wave given by

$$U^{(1)} = a \exp(i\psi) + c.c. , \ \psi = kx - \omega t , \qquad (3)$$

where k and ω are the wave number and frequency respectively.

The amplitude a of the plane waves varies slowly in space and time. These variations are expressed in terms of ε as

$$a_t = \varepsilon A^{(1)}(a,\bar{a}) + \varepsilon^2 A^{(2)}(a,\bar{a}) + \dots$$

and (4)

$$a_x = \varepsilon B^{(1)}(a,\bar{a}) + \varepsilon^2 B^{(2)}(a,\bar{a}) + \dots .$$

On substituting eqs. (2) - (4) into eq. (1), we get equations to different orders in ε. The ε order equation is

$$U_t^{(1)} - U_{xxx}^{(1)} = 0,$$

so that k and ω satisfy the dispersion relation

$$D(k,\omega) \equiv -\omega + k^3 = 0 . \qquad (5)$$

The ε^2 equation is

$$U_t^{(2)} - U_{xxx}^{(2)} = -6k^2a^2\exp(2i\psi) - (A^{(1)} + 3k^2B^{(1)})\exp(i\psi) + c.c. \qquad (6)$$

The term proportional to $\exp(i\psi)$ on the right hand side gives rise to secularity owing to the dispersion relation (5). This secularity is removed by the condition

$$A^{(1)} + V_g B^{(1)} = 0 , \qquad (7)$$

where $V_g = - D_k/D_\omega = 3k^2$ is the group velocity of the plane waves.

The secular-free solution of the ε^2 equations is

$$U^{(2)} = \frac{i}{k} a^2 \exp(2i\psi) + b(a,\bar{a}) \exp(i\psi) + c.c. + c_2(a,\bar{a}) \quad , \tag{8}$$

where $c_2(a,\bar{a})$ is a constant with respect to ψ and will be determined from the condition for removal of secularity in an appropriate higher order equation.

The next higher order, i.e. ε^3 order equation is

$$U_t^{(3)} - U_{xxx}^{(3)} = (U^{(1)3})_x + 3(U^{(1)}U^{(2)})_{xx} - \frac{1}{\varepsilon}(U_t^{(2)} - U_{xxx}^{(2)} - \frac{3}{2}(U^{(1)2})_{xx}) - \frac{1}{\varepsilon^2}(U_t^{(1)} - U_{xxx}^{(1)}). \tag{9}$$

The secularity due to the terms proportional to $\exp(i\psi)$ can, as before, be removed by the condition

$$A^{(2)} + 3k^2 B^{(2)} - 3ik(B^{(1)}B_a^{(1)} + \bar{B}^{(1)}B_{\bar{a}}^{(1)}) - 3k^2 c_2 a = 0 \tag{10}$$

There is another secularity in eq. (9) due to the constant term on the right side. The condition for the removal of this secularity is

$$A^{(1)}(c_2)_a + \bar{A}^{(1)}(c_2)_{\bar{a}} = 0 \ ,$$

so that c_2 is a constant independent of a and $\bar{a}$.

We now introduce slow space and time scales defined as

$$x_2 = \varepsilon x_1 = \varepsilon^2 x \ , \text{ and } t_2 = \varepsilon t_1 = \varepsilon^2 t \ .$$

Then from eq. (4) we get,

$$A^{(1)} = a_{t_1} \quad , \quad B^{(1)} = a_{x_1} \quad ,$$

$$A^{(2)} = a_{t_2} - \frac{1}{\varepsilon} A^{(1)} , \ B^{(2)} = a_{x_2} - \frac{1}{\varepsilon} B^{(1)} , \ B^{(1)}B_a^{(1)} + \bar{B}^{(1)}B_{\bar{a}}^{(1)} = a_{x_1 x_1}$$

and condition (10) becomes

$$i(a_{t_2} + V_g a_{x_2}) + 3ka_{x_1x_1} - 3ik^2 c_2 a = 0 \ .$$

The coordinate transformation

$$\xi = \varepsilon (x-V_g t) = x_1 - V_g t_1 \ ,$$

$$\tau = \varepsilon^2 t = \varepsilon t_1 = t_2 \ ,$$

reduces this equation to

$$i a_\tau + p a_{\xi\xi} + r a = 0 \ , \qquad (11)$$

where

$$p = \frac{1}{2} (V_g)_k = 3k,$$

and

$$r = - 3ik^2 c_2 \ .$$

This equation is of the form of the Schroedinger equation and describes the envelope of the plane wave solutions of eq. (1). In general, the envelope of plane wave solutions of nonlinear dispersive wave equations is described by a nonlinear Schroedinger equation, i.e. eq. (11) with a cubic nonlinear term, which admits envelope solitons or hole solutions [3].

The linear Schroedinger equation (11) would correspond to a free particle in quantum mechanics because p is constant. It is then obvious that no bound states can exist, which means that a cannot be localized in space and stationary in time. This fact demonstrates the importance of the perturbation method [2] because the same fact is verified for the exact explicit solutions [1] of equation (1). Section 3 contains more trivial details about the solutions of eq. (11) and serves as an illustration of well-known techniques.

III. a) Behaviour of Solutions

The localized solutions of the nonlinear Schroedinger equation in multi-dimensions is known to collapse spatially in finite times [4]. In the 1-dimensional case the existence of localized solutions depends on the specific problem. Assuming localized solutions of eq. (11), their spatial behaviour may be studied as follows. By the transformation $a \to a \exp(i r \tau)$, eq. (11) reduces to

$$i a_\tau + p a_{\xi\xi} = 0 \; . \tag{12}$$

On defining the Hamiltonian

$$H = - p \frac{\partial^2}{\partial \xi^2} \equiv p \Pi^2 \; , \; \Pi = - i \frac{\partial}{\partial \xi} \; , \tag{13}$$

where Π is the "momentum" operator, eq. (12) can be written in the usual Schroedinger form

$$i a_\tau = H a \; . \tag{14}$$

The localized solutions satisfy two conservation laws

$$I_1 = \int |a|^2 \, d\xi = \text{const.}$$

and

$$I_2 = p \int |a_\xi|^2 \, d\xi = \text{const.}$$

The spatial variance of the solution of eq. (14) is

$$< |\xi|^2 > = \int |\xi|^2 \frac{|a|^2}{I_1} \, d\xi \tag{15}$$

From the above Hamiltonian we obtain the equations of motion

$$\xi_\tau = 2p \, \Pi \; , \quad \Pi_\tau = 0 \; .$$

Then,

$$< |\xi|^2 >_\tau = 2p < \xi\Pi + \Pi\xi >$$

$$< |\xi|^2 >_{\tau\tau} = 2p < \xi_\tau \Pi + \xi \Pi_\tau + \Pi_\tau \xi + \Pi \xi_\tau > = \frac{8p I_2}{I_1}$$

and on integration we get

$$< |\xi|^2 > = c_1 + c_2\tau + \frac{8pI_2}{I_1}\tau^2 .$$

Since

$$\frac{pI_2}{I_1} = \frac{p^2 \int |a_\xi|^2 d\xi}{\int |a|^2 d\xi}$$

is always positive, the solution expands in space with time. This is again in agreement with the fact that the original equation (1) does not have localized solutions [1].

III. b) Modulational Stability

The complex amplitude a can be expressed in terms of two real functions ρ and σ as [5]

$$a = \rho^{1/2}(\xi,\tau)\ \exp\left[i\,\sigma\,(\xi,\tau)\right] .$$

On substituting this into eq. (12) and then separating the real and imaginary parts we get

$$\rho_\tau + 2p\,(\rho\sigma_\xi)_\xi = 0 , \qquad (18)$$

$$\sigma_\tau + p\sigma_\xi^{\,2} + \frac{p}{4\rho^2}\,\rho_\xi^2 - \frac{p}{2\rho}\,\rho_{\xi\xi} = 0 . \qquad (19)$$

To study the stability of the wave envelope against long wavelength perturbations (k,Ω), we linearize eqs. (18) and (19) as

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix} = \begin{pmatrix}\rho_o\\ \sigma_o\end{pmatrix} + \begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}\ \exp\left[i(k\xi-\Omega\tau)\right] \qquad (20)$$

to yield the linear despersion relation

$$\Omega^2 = p^2k^4 .$$

Thus, the plane wave solutions of eq. (1) are stable against modulations.

IV. Conclusion

The properties of the envelope of the plane wave solutions of eq. (1) are studied here. In contrast to the general case, where it is a nonlinear Schroedinger equation, the equation describing the envelope is found to be a linear Schroedinger equation. This leads to the conclusion that the envelope is stable against modulations and cannot give rise to stationary localized solutions. These conclusions are in agreement with the properties deduced from the explicit solutions [1] of equation (1). This contribution illustrates quite well the potentialities of the perturbation technique of reference [2].

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