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Dissipative MHD Stability +

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IPP 6/157 H. Tasso Dissipative MHD Stability

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Abstract

A short survey of the literature on dissipative magnetohydrodynamic instabilities is given as introduction. A mathematical technique allowing "energy principles" is developed and applied to toroidal equilibria and to cylindrical tokamak-like equilibria taking into account resistivity and the full macroscopic tensor, i.e. F.L.R. effects and viscosity. This allows to make general statements about MHD stability in the presence of viscosity and F.L.R., and permits, without much computation , a qualitative and comparative study of resistive perturbations under the influence of F.L.R. and viscosity. Applications to tokamak observations are also sketched. Finally, it is proved that the stability of dissipative time-dependent force-free fields can be analyzed by a simple functional containing only the perturbed vector potential. This proof is valid even if all non-ideal effects of the two-fluid theory are considered. The conclusion contains a discussion of the open problems and suggestions are given for their solution.

Introduction

Dissipative instabilities in hydrodynamics have been considered since the beginning of this century in several well-known works, described in, for example, the book by C.C. Lin[1]. Dungey[2] (1958) is apparently the initiator of an instability mechanism due to resistivity which seems to play in MHD the role of viscosity in hydrodynamics. Several papers dealing with particular resistive instabilities appeared between 1958 and 1963. They are referred to in the paper of Furth, Killeen, Rosenbluth[3] (1963) dealing with the sheet pinch finite-resistivity stability. These authors discussed qualitatively the influence of other non-ideal effects such as viscosity and thermal conductivity on the eigenvalues, (see also Coppi's paper[4] (1963)). Several later papers based on [3] tried to introduce more sophistication in geometry and physics by using scaling and expansion techniques. Most of the contributions came from Coppi, Furth, Frieman, Greene, Johnson, Rosenbluth and Rutherford, and many references can be found in, for example, the Book by Cap[5]. Some references are also given by Glasser, Greene and Johnson[6] which is a sort of culmination point of expansion techniques and scalings involving geometry and physics.

Besides this progress in the physics of resistive instabilities, some mathematical progress has been achieved by Barston[7] (1969), who was the first to prove for the sheet pinch a necessary and sufficient condition similar to an energy principle. He also gave exact estimates for the growth rates. In fact, this energy principle was already used by Furth[8] but without proof (as far as the reviewer is aware). An extension of Barston's work to two-dimensional plasmas was done by Tasso[9] (1975).

It was noticed [3,8] early that gyroviscosity and viscosity could play an important role in the tyny resistive sheath of the modes. Generally, non-ideal effects and realistic geometry should be taken into account. This is too much for an eigenmode analysis, but other methods such as energy principles, if they exist, could at least give qualitative answers to such problems. A recent paper by Tasso[10] shows how to obtain a quasi energy principle for realistic geometries taking into account resistivity, gyroviscosity, and viscosity.

This talk closely follows reference [10]. It describes the equations and the geometries for which energy principles can be given as well as the results and applications which can be expected. In addition, a sufficient condition [11] for the stability of general dissipative force free fields is given, and its relation to Taylor [12]'s invariant is discussed.

I Stability Equation Allowing "Energy Principle"

Let us consider the following equation :

$$N\ddot{\underline{\zeta}} + (F+M)\dot{\underline{\zeta}} + Q\underline{\zeta} = 0, \qquad (1)$$

where $\underline{\zeta}$ is a complex multidimensional representation vector in a functional space, N and M are hermitic and positive operators and Q is a hermitic and F an antihermitic operator.

It can be seen that this equation contains several limiting cases: If $F = M = 0$, it is the ideal MHD case [13,14] for static equilibria. If $M = 0$, $F \neq 0$ it is the case of linearized conservative systems such as the linearized Vlasov equation [15]. If $F = 0$, $M \neq 0$ it is the case of a resistive plasma in 1 [7] and 2 [9] dimensional geometries.

a) Sufficient Condition for Stability:

Let us first recall the definition of the scalar product for 2 vectors $\underline{\xi}$ and $\underline{\eta}$ denoted by (ξ, η):

$$(\xi,\eta) = \int d\tau\, \underline{\xi}^* \cdot \underline{\eta}\,, \qquad (2)$$

the Hilbert space to which $\underline{\xi}$ and $\underline{\eta}$ belong being restricted to functions fulfilling specific boundary conditions suggested by physics.

Let us now consider the scalar product of $\dot{\underline{\zeta}}$ with the left-hand side of eq. (1) and add to the expression its complex conjugate. The $(\dot{\zeta},F\dot{\zeta})$ terms cancel because of the anti hermiticity of F, and one obtains

$$\frac{1}{2}\left[(\dot{\zeta},N\dot{\zeta}) + (\zeta,Q\zeta)\right]^{\cdot} = -\,(\dot{\zeta},M\dot{\zeta}) \qquad (3)$$

If Q is positive, i.e. $(\zeta,Q\zeta) > 0$ for all $\underline{\zeta}$, the system is stable owing to the positivity of N and M.

b) Necessary and Sufficient Condition for Stability:

If $M \not\equiv 0$ and if $(\zeta,Q\zeta) < 0$ for any $\underline{\zeta} = \underline{\eta}$, the system is unstable. Together with the previous result this leads to a necessary and sufficient condition.

Proof:

The proof is done by demonstrating incompatibility of stable $\underline{\zeta}$ and negative values of $(\zeta,Q\zeta)$. Indeed, it is then possible to choose $\underline{\zeta} = \underline{\eta}$ at a particular time with $(\zeta,Q\zeta) < 0$, and then, integrating eq. (3), we obtain $(\zeta,Q\zeta)$

at later times:

$$(\zeta,Q\zeta) = -2\int_{t_o}^{t}(\dot{\zeta},M\dot{\zeta})dt' - (\dot{\zeta},N\dot{\zeta}) + (\eta,Q\eta) \tag{4}$$

From eq. (4) it follows that $(\zeta,Q\zeta)$ remains negative and at least finite for all later $t > t_o$. This excludes the possibility that $\underline{\zeta} \to 0$ as $t \to \infty$. An oscillation of $\underline{\zeta}$ around a finite value at $t \to \infty$ is also in contradiction with eq. (4), the integral becoming infinite because $\dot{\underline{\zeta}}$ vanishes only on a countable set. The last possibility for a stable $\underline{\zeta}$ would be to tend to a constant in time, but this is in contradiction with eq. (1) itself since $Q\underline{\zeta}$ cannot vanish because of eq. (4).

It is appealing to conjecture that the growth will be exponential because any power growth is incompatible with eq. (4). A rigorous proof of exponential growth cannot be done in the same way [7,16] as for $F \equiv 0$, in which case overstability is forbidden. Let us conclude this section by saying that for eq. (1) with $M \not\equiv 0$

$$(\zeta,Q\zeta) \geq 0 \tag{5}$$

is necessary and sufficient for stability.

II Simplest Model and Time Scales

Let us consider a second-order differential equation with constant coefficients which is a particular case of eq. (1):

$$\frac{1}{2}\ddot{y} + (a + i\,b)\dot{y} + c\,y = 0, \tag{6}$$

with $a > 0$.
The solution is $y = e^{\omega t}$, with

$$\omega = -(a+ib) \pm \sqrt{(a+ib)^2 - 2c}\,, \tag{7}$$

and $c > 0$ is necessary and sufficient for stability.

a) If $a^2+b^2 \ll |c|$, then $\omega \simeq -(a+ib) \pm i\sqrt{2|c|}\left(1 - \frac{(a+ib)^2}{4|c|}\right)$ (7a)

In the unstable case the growth rate is given by $\sqrt{|c|}$ as expected.

b) If $a^2+b^2 \gg |c|$, then $\omega \simeq -(a+ib) \pm (a+ib)\left(1 - \frac{c}{(a+ib)^2}\right)$

The unstable case gives a growth rate

$$\mathrm{Re}(\omega) = \frac{-a\,c}{a^2+b^2}\,. \tag{7b}$$

c) If $a^2 \ll b^2 \approx |c|$, then $\omega \approx -(a+ib) \pm i\sqrt{b^2+2c}\left(1 - \frac{iab}{b^2+2c}\right)$

for $b^2+2c \approx |c|$

and $\omega \approx -(a+ib) \pm \sqrt{i2a\,b}$

for $b^2+2c = 0$,

so that $\mathrm{Re}(\omega) \approx a\left(\frac{b}{\sqrt{|c|}} - 1\right)$ for $b^2+2c = |c|$ (7c)

$$\mathrm{Re}(\omega) \approx \sqrt{ab} \qquad \text{for } b^2+2c = 0 \tag{7c}$$

d) If $a^2 \approx b^2 \gg |c|$, then

$$\mathrm{Re}(\omega) \approx \sqrt{|c|} \quad . \tag{7d}$$

This particular example shows that despite the fact that the sign of c governs stability independently of the values of a and b, the growth rates are strongly dependent on the relative magnitudes of a,b and c.

III Three-dimensional Plasmas with the Full Pressure Tensor

The macroscopic equations are of the following form:

$$\begin{aligned}
&\rho \frac{d\underline{v}}{dt} = \underline{j} \times \underline{B} - \nabla p - \nabla\cdot\underline{\underline{\Pi}} \ , \\
&\underline{E} + \underline{v} \times \underline{B} = 0, \\
&\dot{\rho} + \nabla\cdot\rho\underline{v} = 0 \ , \\
&p = f(\rho) \ , \\
&\nabla \times \underline{B} = j, \\
&\nabla \cdot \underline{B} = 0, \\
&\nabla \times \underline{E} = - \dot{\underline{B}} \quad .
\end{aligned} \tag{8}$$

The pressure tensor [17] Π is given by

$$\begin{aligned}
- \Pi_{xx} &= \alpha(\Gamma_{xx} + \Gamma_{yy}) + \beta\, \Gamma_{xy} + \frac{\beta^2}{4\alpha} (\Gamma_{xx} - \Gamma_{yy}) \ , \\
- \Pi_{yy} &= \alpha(\Gamma_{xx} + \Gamma_{yy}) - \beta\, \Gamma_{xy} - \frac{\beta^2}{4\alpha} (\Gamma_{xx} - \Gamma_{yy}) \ , \\
- \Pi_{zz} &= 2\, \alpha\, \Gamma_{zz} \ , \\
- \Pi_{xy} &= - \Pi_{yx} = \frac{\beta}{2} (\Gamma_{yy} - \Gamma_{xx}) + \frac{\beta^2}{2\alpha} \Gamma_{xy} \ , \\
- \Pi_{xz} &= - \Pi_{zx} = 2\, \beta\, \Gamma_{yz} + 2\, \Gamma_{xz} \frac{\beta^2}{\alpha} \ , \\
- \Pi_{yz} &= - \Pi_{2z} = 2 \frac{\beta^2}{\alpha} \Gamma_{yz} - 2\, \beta\, \Gamma_{xz} \ ,
\end{aligned} \tag{9}$$

$$\Gamma_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3} \frac{\partial v_n}{\partial x_n} \delta_{ij} \ ,$$

$$\beta = p/\omega_{ci} \ ,$$

$$\alpha = \frac{2}{3} \beta\, \omega_{ci}\, \tau_{ii} \quad .$$

x,y,z are a local system of Cartesian coordinates, z being along the magnetic field, ω_{ci} is the ion cyclotron frequency and τ_{ii} the ion-ion collision time.

After linearizing the system (8) around a static equilibrium and expressing all physical quantities in the perturbed velocity, we obtain (see ref. [18])

$$\rho_o \frac{\partial^2 \underline{v}}{\partial t^2} + \frac{\partial}{\partial t} \nabla \cdot \underline{\underline{\Pi}} + Q \underline{v} = 0, \tag{10}$$

where Q is the MHD stability operator. In order to find out what the properties of the operator $\nabla \cdot \underline{\underline{\Pi}}$ are, let us consider $\underline{\underline{\Pi}}$ in a general coordinate system x^1, x^2, x^3 , then

$$\int \left[\underline{v}' \cdot (\nabla \cdot \underline{\underline{\Pi}}) \right] d\tau = \int v'_r \Pi^{rn}_{,n} \sqrt{g} \, dx^1 \, dx^2 \, dx^3 \tag{11}$$

$$= - \int \Pi^{rn} v'_{r,n} \sqrt{g} \, dx^1 \, dx^2 \, dx^3 + \int \frac{\partial}{\partial x^n} (\sqrt{g} \, v'_r \, \Pi^{rn}) dx^1 \, dx^2 \, dx^3 ,$$

where ,n is the covariant derivative with respect to x^n.
The last integral vanishes because of the boundary conditions on $\underline{v}$, and $\Pi^{rn} v'_{r,n}$ can be evaluated in the local coordinates system.

It turns out [18] that the terms in α and $\frac{\beta^2}{\alpha}$ are symmetric and positive definite, and the terms in β are antisymmetric. The pure β terms are due to the finite Larmor radius, the α terms to the magnetic-free viscosity, and the $\frac{\beta^2}{\alpha}$ terms to the magnetic viscosity.

Equation (10) is of the same type as eq. (1), the finite Larmor radius effects correspond to the operator F, and the viscosity to the operator M. The stability is decided by the MHD operator but the growth rates can be affected by F.L.R. effects and viscosity. Such viscous destabilization has already been proposed by Green and Coppi [19], but not in the general form done here. Let us make estimates of the reduction in MHD growth rates.

We know that the most dangerous MHD modes are nearly divergence - free. This means that

$$\Gamma_{xx} + \Gamma_{yy} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = - \frac{\partial v_z}{\partial z} ,$$

but because of $v_\perp = \frac{E \times B}{B^2}$ and $E \simeq \nabla \phi$, $\mathrm{div}\, v_\perp \simeq \frac{v_\perp}{r} \frac{r}{R}$,

where r is the radial extent of the mode, and R the large radius of the torus. On the other hand, one has $\Gamma_{xy} \simeq \frac{v_\perp}{r}$, so that the ion-ion term dominates the F.L.R. term if

$$\frac{\alpha}{\beta} > \left(\frac{R}{r}\right)^2 .$$

For an ion temperature of 1 kev, a magnetic field of 30 kg, and a density of 10^{14} one has $\omega_{ci} \tau_{ii} \simeq 5.10^4$.

Very roughly, we can associate the coefficient a of eq. (6) with the ion-ion term in eq. (10) by the relation

$$a \simeq \frac{\alpha}{n_i M R^2} \simeq \frac{kT \tau_{ii}}{M R^2} \simeq 3.10^7$$

if one takes $R \approx 100$ cm.

A full MHD growth rate $\gamma_{MHD} \approx \sqrt{|c|}$ is usually also of this order. If for particular reasons (geometry, near marginality condition) an MHD growth rate is not full, then $a^2 \gg |c|$ and eq. (7b) lead to the reduced growth rate

$$\mathrm{Re}(\omega) \approx \frac{|c|}{a} .$$

a) Tokamak case

The $m = 1$, $n = 1$ mode observed [20] in Tokamaks near the magnetic axis has a rather small growth rate [21] (resistive kink or ideal internal kink) of the order of 10^5 to 10^6 s^{-1} compared with typical MHD growth, but still too high to explain the experiment. Viscosity leads to a reduced $\gamma \approx \frac{|c|}{a} \approx 10^3$ to 10^4 which agrees with observation.

b) High β, $l = 1$ stellarator

The $m = 2$ mode has a typical MHD growth rate and for the previous plasma parameters we should obtain $a^2 \approx b^2 \approx |c|$, so that we would have the case (7d) with a small reduction in growth rate.

This mode has rarely been observed [22] in the present experiments, this being due to the fact that $\omega_{ci} \cdot \tau_{ii}$ is 2 to 3 orders of magnitude lower ($T_i \approx 100$ ev.) than calculated before so that $a^2 \ll b^2 \approx |c|$, which is the case of eq. (7c). The F.L.R. stabilization dominates in this case.

The derivation of the pressure tensor as given by eqs. (9) is done for $\frac{r_L}{L} \ll 1$ and $\Lambda_{MFP} \ll L$, with r_L = Larmor radius, L = inhomogeneity length, Λ_{MFP} = mean free path. For incompressible motion the validity of the pressure tensor can be extended to the domain $\Lambda_{MFP} \gg L_\perp$ but $\Lambda_{MFP} \ll L_\parallel$.

IV Two-dimensional Plasmas with Resistivity and the Full Pressure Tensor

It is well known that resistivity leads [3] to new modes and one can expect a much more difficult behaviour. At present only the 2-dimensional case for straight plasmas can be solved, as we shall see.

The equilibrium is characterized [9] by

$$\underline{j}_o = \underline{e}_z J_o(\psi) \quad , \tag{12}$$

$$\underline{E}_o = \underline{c}\underline{t} = \underline{e}_z \eta_o(\psi) J_o(\psi) ,$$

$$\underline{B}_o = \underline{e}_z \times \nabla\psi \; (\underline{e}_z \cdot \underline{B}_o) \; \underline{e}_z ,$$

$$\nabla^2\psi = J_o(\psi) = - \frac{dP_o}{d\psi} .$$

z is the coordinate along the straight plasma, ψ is the meridional magnetic flux, and η_o is the resistivity.

The meridional currents are assumed to be zero in order to have a static plasma in equilibrium, which is important for this kind of formulation but not necessarily for the physical results.

After linearizing the equation of motion we obtain

$$\rho_o \ddot{\underline{\xi}} + \nabla p_1 + \nabla \cdot \underline{\underline{\Pi}}(\underline{\xi}) - \underline{j}_1 \times \underline{B}_o - \underline{j}_o \times \underline{B}_1 = 0, \tag{13}$$

$$\nabla \cdot \underline{\xi} = 0, \tag{14}$$

$$\dot{\underline{A}}_1 + \eta_o \nabla \times \nabla \times \underline{A}_1 + \eta_1 \underline{j}_o - \dot{\underline{\xi}} \times \underline{B}_o = 0, \tag{15}$$

$$\underline{B}_1 = \nabla \times \underline{A}_1 , \tag{16}$$

$$\eta_1 = - \xi \cdot \nabla \eta_o \tag{17}$$

Apart from the pressure tensor term, these equations are the same as in ref. [9]. Equation (17) is valid as long as the heat conductivity is small enough. This will be discussed later.

Restriction to two-dimensional perturbations leads to

$$\underline{\xi} = \underline{e}_z \times \nabla U = - \nabla \times \underline{e}_z U , \quad \underline{B}_1 = - \underline{e}_z \times \nabla A = \nabla \times \underline{e}_z A ,$$

$$\underline{j}_1 = - \underline{e}_z \nabla^2 A , \qquad \nabla \times \underline{j}_1 = - \nabla(\nabla^2 A) \times \underline{e}_z ,$$

where U and A are two scalars: the stream function and the z component of the vector potential.

Taking the curl of eq. (13), we obtain

$$- \nabla \cdot \rho_o \nabla \ddot{U} + e_z \cdot \nabla \times \nabla \cdot \underline{\underline{\Pi}}(\nabla \times \underline{e}_z \dot{U}) - \underline{B}_o \cdot \nabla (\nabla^2 A) - \nabla \times j_o \cdot \nabla A = 0 , \tag{18}$$

$$\dot{A} + \underline{B}_o \cdot \nabla \dot{U} - \eta_o \nabla^2 A + J_o(\underline{e}_z \times \nabla \eta_o \cdot \nabla U) = 0 . \tag{19}$$

If $\nabla^2 A$ is taken from eq. (19) ($\eta_o \neq 0$) and inserted in eq. (18), we obtain the following system of equations in matrix operatorial form:

$$\begin{pmatrix} -\nabla \cdot \rho_o \nabla & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{U} \\ \ddot{A} \end{pmatrix} + \begin{pmatrix} \frac{-(\underline{B}_o \cdot \nabla)^2}{\eta_o} & \frac{-(\underline{B}_o \cdot \nabla)}{\eta_o} \\ \frac{\underline{B}_o \cdot \nabla}{\eta_o} & \frac{1}{\eta_o} \end{pmatrix} \begin{pmatrix} \dot{U} \\ \dot{A} \end{pmatrix} +$$

$$+ \begin{pmatrix} e_z \nabla \times \nabla \cdot \underline{\underline{\Pi}} (\nabla \times \underline{e}_z \ldots) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{U} \\ \dot{A} \end{pmatrix} + \tag{20}$$

$$+ \begin{pmatrix} -J_o(\psi)\underline{B}_o \cdot \nabla \; e_z \times \frac{\nabla \eta_o}{\eta_o} \cdot \nabla & - \nabla \times \underline{j}_o \cdot \nabla \\ \nabla \times \underline{j}_o \cdot \nabla & - \nabla^2 \end{pmatrix} \begin{pmatrix} U \\ A \end{pmatrix} = 0$$

Apart from the $\nabla \cdot \underline{\underline{\Pi}}$ term, eq. (20) is identical with eq. (17) of ref. [9]. Let us investigate the operator $e_z \cdot \nabla \times \nabla \cdot \underline{\underline{\Pi}}$ $(\nabla \times e_z \ldots)$. We know from the previous section and from ref. [18] that $\nabla \cdot \underline{\underline{\Pi}}$ $(\underline{\xi})$ contains α terms which are positive definite operators and pure β terms which are antisymmetric, but if $\underline{\xi}$ is a curl and if $\nabla \cdot \underline{\underline{\Pi}}$ $(\underline{\xi})$ is replaced by $\nabla \times \nabla \cdot \underline{\underline{\Pi}}$ $(\underline{\xi})$ the symmetry properties will not be changed, as the following equation shows:

$$\int d\tau \; v \, \underline{e}_z \cdot \nabla \times \nabla \cdot \Pi \; (\nabla \times \underline{e}_z \, u) = \int \nabla \times \underline{e}_z v \cdot \nabla \cdot \underline{\underline{\Pi}} \; (\nabla \times \underline{e}_z \, u) \quad d\tau$$

The symmetry properties of $\nabla \cdot \underline{\underline{\Pi}}$ are thus the same as $e_z \cdot \nabla \times \nabla \cdot \underline{\underline{\Pi}} (\nabla \times e_z \ldots)$.

Then eq. (20) has the same character as eq. (1) and the stability condition is given by the last matrix operator of eq. (20), which is the same as in ref. [9].

This leads to the energy principle derived in [9] for zero pressure tensor and extended here to F.L.R. and ion-ion collisions:

$$\begin{aligned} \delta W = & \int d\tau \; \left(-\frac{dJ_o}{d\psi}\right) (\underline{e}_z \times \nabla\psi \cdot \nabla U)^2 \\ & + 2 \int d\tau \; \left(-\frac{dJ_o}{d\psi}\right) A \; (\underline{e}_z \times \nabla\psi \cdot \nabla U) \\ & + \int d\tau \; |\nabla A|^2 . \end{aligned} \tag{21}$$

A simple application to tokamaks is the instability of skin currents. If one localizes a test function with ∇U finite at $dJ_o/d\psi > 0$ and $A \simeq 0$, then $\delta w < 0$. This mode is similar to the rippling mode in one dimensional geometry [3].

Another application would be the stability of configurations with stagnation points (such as Doublet or for islands in tokamaks). This necessitates numerical calculations which are not easy to do because of the stagnation point, but eq. (21) at least allows the problem to be correctly formulated.

V All Perturbation with Resistivity, Viscosity, F.L.R. in the Tokamak Scaling

The energy principle of section IV can be extended to three-dimensional perturbations if one goes to helical coordinates for the representation of the equilibrium and perturbations and in the approximation of the tokamak scaling $Kr \simeq \frac{B_\theta}{B_z} \simeq \varepsilon$.

The proof and calculations are given in [23]. The necessary and sufficient criterion is given by:

$$\begin{aligned} \delta w = & \int \left(-\frac{dj_o}{dr}\right) (\underline{u} \times \underline{e}_r \cdot \nabla G) \; (\underline{u} \times \nabla F_o \cdot \nabla G) \; d\tau \\ & + 2 \int \left(-\frac{dj_o}{dr}\right) (\underline{u} \times \underline{e}_r \cdot \nabla G) \; F \quad d\tau \\ & - \int F \, L \, F \quad d\tau \quad , \end{aligned} \tag{22}$$

where $\underline{B}_o = f_o(r)\ \underline{u} + \underline{u} \times \nabla\ F_o(r)$

$\underline{B}_j = f(r,u,t)\ \underline{u} + \underline{u} \times \nabla\ F(r,u,t)$

$\underline{\xi} = g(r,u,t)\ \underline{u} + \underline{u} \times \nabla G\ (r,u,t)$

$$\underline{u} = \frac{l\ \underline{e}_z + hr\ \underline{e}_\theta}{l^2 + h^2\ r^2}\ , \qquad hr \simeq \varepsilon$$

$$u = l\ \theta - hz$$

A test function can always be found to make the system unstable as soon as $\frac{dj_o}{dr} \neq 0$. This test function can be taken as:

F $\simeq$0 and G concentrated on the side of the resonance ($\nabla F_o = 0$) at which the first integrand is negative which is always possible. An estimate of the growth rate can also be done as in section III.
This test function characterizes the rippling mode. A similar test function was used by Furth [8] in the sheet pinch geometry. The tearing mode test function is not localized but can, in principle, be found in the same way as in reference [8] and can be affected by cylindrical geometry (particularly the mode m =1, n = 1). The F.L.R. effects and viscosity do not stabilize the resistive modes but can appeciably reduce the growth rates (see estimate in section III).

Stabilization of the tearing mode alone seems possible [24]. To obtain it, one has to assume a non-fluctuating resistivity and shape the current density in a step-like form [24]. If the plasma has to be stable to all resistive modes (rippling included), then the current density as demonstrated by expression (22) has to be constant up to the boundary. B.B. Kadomtsev [25] came to this last result using Taylor's [12] invariant, which will be discussed in the next section.

VI Stability of Force-free Fields

Resistive force-free fields have to be time dependent and have to be restricted to the class [26] $j = \lambda\ \underline{B}$ with $\lambda = ct$ (see also ref. [11]) and $\underline{B} = \underline{B}_o\ e^{-\eta\lambda^2 t}$.

The linearized equations of motion around such solutions are:

$$\rho_o\ \ddot{\underline{\xi}} = \underline{j}_1 \times \underline{B}_o - \lambda \underline{B}_1 \times \underline{B}_o\ , \tag{23}$$

$$-\ \dot{\underline{A}}_1 + \dot{\underline{\xi}} \times \underline{B}_o = \eta \underline{j}_1 + \eta_1 \underline{j}_o\ , \tag{24}$$

$$\underline{B}_1 = \nabla \times \underline{A}_1\ ,\ \ \underline{j}_1 = \nabla \times \nabla \times \underline{A}_1\ . \tag{25}$$

The investigation is restricted to the case $\lambda = ct$, $\eta = ct$ consistent with $\eta_1 = 0$ and the gauge is chosen such that $\underline{E} = -\dot{A}$. The scalar product of equation (23) with $\dot{\underline{\xi}}$ yields

$$\rho_o \ddot{\underline{\xi}} = -(j_1 - \lambda \underline{B}_1) \cdot (\dot{\underline{A}}_1 + \eta j_1) \quad .$$

Integrating over the plasma volume limited by a perfectly conducting wall, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\left[(\rho_o \dot{\xi}, \dot{\xi}) + (\nabla \times \underline{A}_1, \nabla \times \underline{A}_1) - (\lambda A_1, \nabla \times \underline{A}_1) \right] =$$
$$- \eta \left[(j_1, j_1) - (\lambda j_1, B_1) \right] , \tag{26}$$

where $(\alpha,\zeta) = \int_V d\tau\, \underline{\alpha} \cdot \underline{\zeta}$.

Let δW_R be defined as

$$\delta W_R = (\nabla \times \underline{A}_1, \nabla \times \underline{A}_1) - \lambda (A_1, \nabla \times \underline{A}_1) \tag{27}$$

The variation of δW_R leads to the following Euler eigenvalue equation:

$$\nabla \times \nabla \times \underline{A}_1 - \lambda \nabla \times \underline{A}_1 = \alpha \underline{A}_1 \quad . \tag{28}$$

The variation of the right-hand side of equation (26) leads to

$$\nabla \times \nabla \times \underline{B}_1 - \lambda \nabla \times \underline{B}_1 = \beta \underline{B}_1 \quad . \tag{29}$$

The curl of equation (28) is identical with equation (29). This means that any solution of equation (29) verifying $\underline{n} \cdot \underline{B}_1 = 0$ at the boundary is also a solution of equation (28) with $\underline{n} \times \underline{A}_1 = 0$ at the boundary.

It follows that $\delta W_R > 0$ implies the negativeness of the right-hand side of equation (26). This means that $\delta W_R > 0$ is sufficient for stability with respect to MHD + resistive modes. This condition is found necessary and sufficient if one ignores resistivity and uses instead Taylor's hypothesis of a global invariant [12, 27]. This result is somewhat to be expected if one considers Woltjer's [28]

proof that λ = ct force-free fields represent the state of minimal energy in a closed system.

The important practical question for fusion plasmas is to know how much one can deviate [29] from λ = ct force-free fields without appeciably affecting the gross stability properties and without appeciably diminishing the confinement time. This question might require an understanding of the nonlinear problem, which would exceed the scope of this paper.

VII Discussion and Conclusion

The method pursued in this paper allows statements about stability without going to the solution of eigenmodes. This is only possible if the representation variable ζ in which the linearized equations of motion are of the same type as eq. (1) can be found. This depends, of course, on the physical equations used. Effects such as thermal conductivity and the Hall term affect the symmetry properties of the operators of eq. (1) and eq. (20). In fact, Ohm's law in hot plasmas is not known; it can be affected by trapped particles in toroidal geometry and generally by turbulence. Even cylindrical geometry presents difficulties: Energy principle (22) would not have been possible without making a tokamak expansion.

Answers to these questions can be found as follows:

1) One can restrict the investigation to a class of resistive modes (essentially the "tearing" modes) as done in, for example, references [6] and [24], which generally leads to optimistic results. But then it remains for us to understand the meaning of the restriction in the stable case and to know how the growth rates are affected by the ignored physical terms (F.L.R., Viscosity, etc ...) in the unstable case.

2) This dilemma may require general stability conditions to be found as in, for example, ref. [10] and patient searching for the representation ζ if it exists, for which the linearized physical equations become of the type of equation (1). This is demonstrated here for some non-ideal effects and geometries.

3) The hardest way, but the nearest to real plasmas, is to develop methods dealing with the stability of equations more general than equation (1), particularly those for which Q is not symmetic.

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