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Diffusive tokamak equilibria,
generalized bootstrap currents and
the relation between particle diffusion
and magnetic field diffusion

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ABSTRACT

A condition for general axisymmetric diffusive equilibria
which relates the outward diffusion flux with the toroidal
current density is derived. In an approximate version it
requires that some effective diffusion velocity V_D^* must not
exceed the poloidal magnetic diffusion velocity V_m. Relevant
consequences follow in the regime of anomalous diffusion if
diffusion is caused by an anomalous parallel electron
viscosity instead of an anomalous perpendicular resistivity.
In the former case V_D^* equals the real diffusion velocity V_D
and an anomalous bootstrap current arises which leads to
rather low upper limits for β_p. If the usual trapped ion or
Bohm diffusion is assumed to be caused by an enhanced
viscosity, no stationary equilibria would be possible in a
system governed by the respective diffusion law.

1. INTRODUCTION

This paper deals with the relation between particle diffusion and magnetic field diffusion in axisymmetric systems. It has its origin in the following heuristic consideration. In a magnetically confined MHD plasma in equilibrium, ambipolar diffusion results in a flow of matter perpendicularly to the magnetic field with some velocity V_D. If the plasma had zero resistivity, the magnetic field would be taken out of the plasma with the same velocity. In the nonideal case, finite resistivity makes the magnetic field penetrate into the plasma with some velocity V_m. Stationary conditions can only be achieved if $V_D \stackrel{<}{\sim} V_m$, i.e. if the particle diffusion velocity is smaller than the penetration velocity of the magnetic field.

Table 1) gives roughly calculated values of V_D/V_m for a small (such as ST or PULSATOR), a medium size (such as PLT or ASDEX) and a reactor-like tokamak plasma.

Table 1)

	a [cm]	R [cm]	B [kG]	T [keV]	n [cm^{-3}]	V_D/V_m	τ_p [sec]
I	10	70	40	1	$2 \cdot 10^{13}$	~ 1	$\sim 5 \cdot 10^{-2}$
II	45	120	45	2,5	$5 \cdot 10^{13}$	$\sim 5 \cdot 10^1$	$\sim 10^{-1}$
III	400	1000	50	15	$5 \cdot 10^{13}$	$\sim 5 \cdot 10^3$	~ 2

In all cases, $q = 3$ and $Z_{eff} = 2$ are chosen. V_D and τ_p are calculated as $V_D \sim D/a$ and $\tau_p \sim a^2/D$, where D is the diffusion coefficient. In case I, pseudoclassical diffusion was assumed.

Cases II and III are based on trapped ion diffusion. The values of the temperatures are in agreement with results of present day transport codes.

Thus, for parameters expected in larger tokamaks V_D would exceed V_m by orders of magnitude in obvious contradiction to the relation $V_D \lesssim V_m$.

In present day transport codes $\frac{1}{c} \cdot \vec{V} \times \vec{B}$ terms in Ohm's law are neglected compared with terms $\eta_{\parallel} \cdot \dot{j}_\varphi$ [1] [2] ($\dot{j}_\varphi$ = the toroidal current density). As will be shown later, their ratio is approximately V_D/V_m. It is clear therefore from Table 1) that this procedure is rather inconsistent for large tokamaks with anomalous diffusion.

Another aspect of the relation $V_D \lesssim V_m$ concerns the plasma ß that can be reached in diffusive equilibria. Indeed, taking into account $V_m \sim V_c/\beta$, where V_c is the classical diffusion velocity in plane geometry, the condition $V_D \lesssim V_m$ immediately results in a ß-limiting relation: $\beta \lesssim V_c / V_D$. This condition, if applied to neoclassical banana diffusion $V_D \sim V_c \cdot q^2 A^{3/2}$, gives $\beta \lesssim 1/q^2 A^{3/2}$ or $\beta_P \lesssim A^{1/2}$. This is the well-known bootstrap condition [3], indicating that the relation $V_D \lesssim V_m$ should be related to a generalized bootstrap current.

In this paper, we study these topics in the case of general axisymmetric systems. We proceed by analogy with [4]: Starting with the momentum equations for a hydrogen-like plasma and Maxwell's equations (section 2) we calculate the rate equation for the total angular momentum of the electrons in equilibrium. This equation relates the particle flux, the Ware effect and a

generalized bootstrap current to each other. From it the
relation $V_D \lesssim V_m$ follows in a preliminary way by some
rough arguments (section 3). A detailed discussion is per-
formed in the following chapters. First we derive the toroidal
component of Ohm's law for a general axisymmetric system in
equilibrium (section 4). We then derive a general expression
for the outward particle flux in terms of the resistivity, the
electron viscosity, the pressure gradient and the external
electric field. This relation, together with Ohm's law gives
a necessary condition for stationary diffusive equilibria
coupling diffusion with the toroidal current density (section 6).
Another form of this condition, relating the particle diffusion
and magnetic diffusion velocity, is given which replaces the
heuristic relation $V_D \lesssim V_m$. Some implications are studied
on the basis of current diffusion formulae (section 6).

2. BASIC EQUATIONS

We start with the equations of motion for the two particle
species of a hydrogen-like plasma, which follow quite generally
from momentum conservation and read [5]

$$(1) \quad m^e n \frac{d\vec{V^e}}{dt} = -\nabla p^e - \nabla \cdot \overleftrightarrow{\Pi}^e - en\left(\vec{E} + \frac{1}{c}\vec{V^e}\times\vec{B}\right) + \vec{R},$$

$$(2) \quad m^i n \frac{d\vec{V^i}}{dt} = -\nabla p^i - \nabla \cdot \overleftrightarrow{\Pi}^i + en\left(\vec{E} + \frac{1}{c}\vec{V^i}\times\vec{B}\right) - \vec{R}.$$

In (1) and (2) $\overleftrightarrow{\Pi}^{e/i}$ is the anisotropic part of the pressure
tensors of electrons and ions. $\vec{R}$ is the rate of electron-ion

momentum exchange. (The full pressure tensor can be assumed to contain all kinds of momentum exchange resulting from fluctuations, the other quantities appearing in the equation being then averages.)

We consider equations (1) and (2) together with Maxwell's equations

$$(3) \quad \nabla \times \vec{E} = - \frac{1}{c} \frac{\partial}{\partial t} \vec{B} \ ,$$

$$(4) \quad \nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} \qquad \vec{j} = -en(\vec{V}^e - \vec{V}^i) \ ,$$

$$(5) \quad \nabla \cdot \vec{B} = 0 \ .$$

If the plasma is in equilibrium, the force balance equation for the total plasma must be satisfied. It results from adding (1) and (2) and neglecting the inertial terms.

$$(6) \quad \vec{F} = \nabla \cdot (p + \overleftrightarrow{\Pi}) = \frac{1}{c} \vec{j} \times \vec{B} \ , \quad p = p^e + p^i \ ,$$

$$\overleftrightarrow{\Pi} = \overleftrightarrow{\Pi}^e + \overleftrightarrow{\Pi}^i \ .$$

In (6) $\overleftrightarrow{\Pi}$ can be neglected compared with p in most cases. This will be done in the following.

In (1) the inertial term is small compared with the friction force $\vec{R}$. We shall therefore use (1) with the left-hand side set equal to zero [5].

Since we consider axisymmetric configurations, the magnetic field can be expressed by

$$\vec{B}_T = f(\Psi) \cdot \nabla \varphi = B_\varphi \cdot \vec{e}_\varphi \ ,$$

$$\vec{B}_P = \nabla \Psi \times \nabla \varphi = B_P \cdot \vec{e}_P \ ,$$

defining the rectangular coordinates ψ, ℓ, φ, where $2\pi\psi$ is the poloidal magnetic flux, ℓ the distance in the poloidal direction and φ the large azimuth. $\vec{e_\varphi}$ and $\vec{e_p}$ are unit vectors in the φ and ℓ directions respectively. Since $\vec{e_\varphi} = R \cdot \nabla\varphi$, where R is the distance from the axis of symmetry, we have

(7) $\qquad R \cdot B_\varphi = f(\psi)$.

3. GLOBAL RELATIONS

Multiplying the φ-component of (1) by R, we obtain the rate equation for the toroidal angular momentum density in the direction of the axis of symmetry:

$$\text{(8)} \quad 0 = -B_p \frac{\partial}{\partial\psi}\left(\pi^e_{\varphi\psi} R^2\right) - B_p \frac{\partial}{\partial\ell}\left(\pi^e_{\varphi p} \frac{R}{B_p}\right)$$

$$-enRE_\varphi - \frac{en}{c} RV_\psi B_p + R R_\varphi$$

Integration of (8) over the whole plasma yields the rate equation for the total angular momentum of the electrons.

It is $\int d^3x \cdots = 2\pi \oint\oint \frac{R}{|\nabla\psi|} d\ell\, d\psi \cdots = 2\pi \oint\oint \frac{\sigma}{B_p} d\ell\, d\psi \cdots$

where $\sigma = B_p/|B_p|$. σ is a function of ψ only. The change of sign of B_p would mean $B_p = 0$ on a whole magnetic surface ψ = const. We exclude this case and assume B_p not to change sign. Then volume integration yields

$$\text{(9)} \quad \int d\psi \left\{ \Gamma_\psi + \oint 2\pi\, d\ell\, \frac{Rnc E_\varphi}{B_p} - \int 2\pi\, d\ell\, \frac{RC\, R_\varphi}{e\, B_p} \right\} = 0$$

where $\Gamma_\psi = \oint 2\pi \, d\ell \, R \, n \, V_\psi$ is the particle flux through a magnetic surface.

Relation (9) holds, except for the condition $B_p \neq 0$, without any restriction. The second term is essentially the Ware effect and the last term is related to the toroidal current.

For stationary situations with no particle sinks in the plasma Γ_ψ does not change sign as a function of Ψ. Hence, if the Ware effect can be neglected, anomalous diffusion leads automatically to a large value of R_φ. If the concept of resistivity is applicable, possibly with some enhanced value of it, $\vec{R}$ can be assumed to have the form

$$(10) \quad \vec{R} = e \, n \left(\eta_{\shortparallel} \, \vec{J}_{\shortparallel} + \eta_\perp \, \vec{J}_\perp \right) ,$$

and therefore

$$(11) \quad R_\varphi = e \, n \left(\eta_{\shortparallel} \, J_{\shortparallel \varphi} + \eta_\perp \, J_{\perp \varphi} \right) .$$

To draw conclusions from this, we note that experiments indicate that in the usual low electron drift case $\eta_{\shortparallel}$ is close to its classical value. $\vec{J}_\perp$ is given by the equilibrium condition (6) as

$$\vec{J}_\perp = - c \, \frac{\nabla p \times \vec{B}}{B^2} .$$

Diffusion being larger than classical Pfirsch-Schlüter diffusion therefore requires either an enhanced perpendicular resistivity $\eta_\perp$ or a large bootstrap current $\vec{J}_{\shortparallel}$. If the latter is true, roughly the following holds owing to (9) and (11):

(12) $\quad \Gamma_\psi \approx 2\pi \oint d\ell \, R \dfrac{c \cdot n \cdot \eta_\| \cdot \dot{j}_{\|\psi}}{B_p}$

or even more roughly

(13) $\quad V_\psi \approx \dfrac{c \cdot \eta_\| \cdot \dot{j}_{\|\psi}}{B_p}$

From the ψ-component of (4) we get for the large aspect ratio case (r=minor radius)

$$\frac{1}{r} \frac{\partial}{\partial r} (r \cdot B_p) \approx \frac{4\pi}{c} \dot{j}_{\|\psi} \approx \frac{4\pi}{c^2} \frac{B_p \cdot V_\psi}{\eta_\|} \; .$$

An order of magnitude estimate gives

(14) $\quad V_\psi \sim \dfrac{c^2 \, \eta_\|}{4\pi \, a} = \dfrac{2 c^2 \, \eta_\| \, p}{a \, B^2} \dfrac{B^2}{8\pi \, p} \; .$

Here $2 c^2 \eta_\| p / a B^2 \sim V_c'$ is the classical diffusion velocity, but computed with $2\eta_\|$ instead of $\eta_\perp$.

From (14) we now find

(15) $\quad \beta \approx \dfrac{8\pi \, p}{B^2} \sim \dfrac{V_c'}{V_\psi}$

or $\quad \beta_p \sim \dfrac{V_c'}{V_\psi} A^2 q^2$

as stated in the introduction.

The following rigorous considerations follow essentially the same line in deriving expressions corresponding to 12) or 13). In addition, a general relation for Γ_ψ is derived which admits some further conclusions.

4. OHM'S LAW FOR THE TOROIDAL DIRECTION [+)]

Applying only the operator $\langle \cdots \rangle = 2\pi \oint \frac{d\ell}{B_P} \cdots$ on (8)

instead of performing the full volume integration, we have

$$(16) \quad 0 = -\left\langle B_P \frac{\partial}{\partial \psi} \left(\Pi^e_{\varphi\psi} R^2 \right) \right\rangle - e \langle n R E_\varphi \rangle$$

$$- \frac{e}{c} \Gamma_\psi + \langle R \cdot R_\varphi \rangle .$$

The first term in (16) is a viscosity force due to toroidal momentum transfer perpendicular to the magnetic surface. The transport of momentum in the ψ-direction is essentially determined by the diffusion velocity. Therefore, this term is small compared with the momentum exchange term and can be neglected. The result is identical with setting the integrand in (9) equal to zero. From (10) it follows that

$$(17) \quad R_\varphi = e n \left[\eta_{\shortparallel} \frac{B_\varphi}{B^2} \left(B_P \cdot \dot{f}_P + B_\varphi \dot{f}_\varphi \right) \right.$$

$$\left. + \eta_\perp \left(\dot{f}_\varphi - \frac{B_\varphi}{B^2} \left(B_P \cdot \dot{f}_P + B_\varphi \dot{f}_\varphi \right) \right) \right] .$$

Taking into account the ψ-component of (6) with $\overleftrightarrow{\Pi}$ neglected and therefore $p = p(\psi)$

$$c \cdot R B_P \frac{\partial p}{\partial \psi} = \dot{f}_P B_\varphi - \dot{f}_\varphi B_P ,$$

we can eliminate j_p from (17) and find

$$(18) \quad R_\varphi = e n \left[\eta_{\shortparallel} \dot{f}_\varphi + (\eta_{\shortparallel} - \eta_\perp) \frac{c R B_P^2}{B^2} \frac{\partial p}{\partial \psi} \right] .$$

[+)] The following considerations are to some extent similar to those in [4].

From (16) with the first term neglected and (18) we now
obtain

(19)
$$\langle R n E_\varphi \rangle = \langle \eta_{||} R n \dot{J}_\varphi \rangle - \frac{1}{c} \Gamma_\psi$$

$$+ \langle (\eta_{||} - \eta_\perp) n \frac{c R^2 B_p^2}{B^2} \frac{\partial p}{\partial \psi} \rangle ,$$

which is essentially the toroidal component of Ohm's law.

Owing to the rapid transport of particles and heat within a
magnetic surface it can be assumed that all densities and
temperatures are constant on such a surface. Hence they
depend on ψ only and are not affected by the $\langle\ \rangle$ operation.
The same holds for $\eta_{||}$ and $\eta_\perp$ if it is assumed that they are
some functions of density and temperature only. We can thus
write (19) as

(20)
$$n \langle R E_\varphi \rangle = \eta_{||} n \langle R \dot{J}_\varphi \rangle - \frac{1}{c} \Gamma_\psi$$

$$+ c \frac{\partial p}{\partial \psi} n (\eta_{||} - \eta_\perp) \langle \frac{R^2 B_p^2}{B^2} \rangle .$$

In the stationary case $\nabla \times \vec{E} = 0$, which implies that
$\vec{E} = \frac{V_{ol}}{2\pi R} \cdot \ell_\varphi - \nabla \phi$, where V_a is the external voltage and ϕ
the self-consistent potential.
We than get instead of (20)

(20')
$$n \frac{V_{ol}}{2\pi} \langle 1 \rangle = \eta_{||} n \langle R \dot{J}_\varphi \rangle - \frac{1}{c} \Gamma_\psi$$

$$+ c \frac{\partial p}{\partial \psi} n (\eta_{||} - \eta_\perp) \langle \frac{R^2 B_p^2}{B^2} \rangle .$$

5. DIFFUSION

In this section the particle flux will be expressed in terms
of the resistivity, the electron viscosity, the pressure
gradient and the external electric field. To do this
one more   relation is necessary in addition to Ohm's law.
It can be derived from (1) by multiplying this equation by
$\vec{B}$. This gives

$$(21) \quad 0 = -B_p \frac{\partial p_e}{\partial \ell} - en \frac{V_a}{2\pi R} B_\varphi + e B_p \frac{\partial \phi}{\partial \ell} n$$

$$- \vec{B} \cdot \nabla \overleftrightarrow{\pi^e} + \eta_{\shortparallel} en (j_\varphi B_\varphi + j_p B_p) .$$

To eliminate p_e and ϕ from (21) we again apply the operation

$$\langle \cdots \rangle = 2\pi \oint \frac{d\ell}{B_p} \cdots , \text{ which yields}$$

$$(22) \quad 0 = -en \frac{V_a}{2\pi} \langle \frac{B_\varphi}{R} \rangle - \langle \vec{B} \cdot \nabla \overleftrightarrow{\pi^e} \rangle$$

$$+ \eta_{\shortparallel} en (\langle j_\varphi B_\varphi \rangle + \langle j_p B_p \rangle) .$$

(Note that $\langle \vec{B} \cdot \nabla \overleftrightarrow{\pi^e} \rangle$, contrary to (16), contains terms which
describe momentum transfer within a magnetic surface and there-
fore cannot be neglected.)
j_φ can be expressed by means of the ψ component of (6).

$$(23) \quad 0 = -en \frac{V_a}{2\pi} \langle \frac{B_\varphi}{R} \rangle - \langle \vec{B} \cdot \nabla \overleftrightarrow{\pi^e} \rangle$$

$$+ \eta_{\shortparallel} en \langle j_p \frac{B^2}{B_p} \rangle - \eta_{\shortparallel} enc \frac{\partial p}{\partial \psi} \langle R B_\varphi \rangle .$$

The same procedure of eliminating $\dot{j}_\varphi$ if applied to (20')

gives

$$(24) \quad 0 = -en \frac{V_a}{2\pi} \langle 1 \rangle - \frac{e}{c} \Gamma_\psi + \eta_{\shortparallel} en \left\langle \frac{R \, B_\varphi}{B_P} \dot{j}_P \right\rangle$$

$$- enc \frac{\partial p}{\partial \psi} (\eta_{\shortparallel} - \eta_{\perp}) \left\langle \frac{R^2 B_\varphi^2}{B^2} \right\rangle - \eta_{\perp} enc \frac{\partial p}{\partial \psi} \langle R^2 \rangle.$$

From $\vec{B}_T = f(\psi) \cdot \nabla \varphi$ and the poloidal component of (4) it

follows that

$$(25) \quad \vec{j}_P = \frac{4\pi}{c} f' \, \nabla \psi \times \nabla \varphi = \frac{4\pi}{c} f' \, \vec{B}_P \quad .$$

Hence

$$(26) \quad \left\langle \dot{j}_P \frac{B^2}{B_P} \right\rangle = \frac{\langle B^2 \rangle}{\langle R \cdot B_\varphi \rangle} \left\langle \dot{j}_P \frac{R \, B_\varphi}{B_P} \right\rangle.$$

Eliminating j_p by means of (26) between (24) and (23) results

in the intended expression for Γ_ψ.

$$(27) \quad \Gamma_\psi = cn \frac{V_a}{2\pi} \left(\left\langle \frac{B_\psi}{R} \right\rangle \frac{\langle R \cdot B_\varphi \rangle}{\langle B^2 \rangle} - \langle 1 \rangle \right) + \frac{c}{e} \frac{\langle R \cdot B_\varphi \rangle}{\langle B^2 \rangle} \langle \vec{B} \cdot \nabla \overleftrightarrow{\pi}_e \rangle$$

$$- \eta_{\shortparallel} c^2 n \frac{\partial p}{\partial \psi} \left\{ \left\langle \frac{R^2 B_\psi^2}{B^2} \right\rangle - \frac{\langle R \cdot B_\varphi \rangle^2}{\langle B^2 \rangle} \right\} - \eta_{\perp} c^2 n \frac{\partial p}{\partial \psi} \left\langle \frac{R^2 B_P^2}{B^2} \right\rangle.$$

This representation holds quite generally and independently

of the specific diffusion mechanism.

It is instructive to evaluate (27) for the Pfirsch-Schlüter field. Using toroidal coordinates (r, ϑ, φ), we have

$$B_\vartheta = \Theta(r)\, B_\varphi, \quad B_\varphi = B_0/\mathcal{S}, \quad \mathcal{S} = 1 + \varepsilon \cos \vartheta, \quad \varepsilon = r/R,$$

$$R = R_0\, \mathcal{S}, \quad \varepsilon, \Theta \ll 1.$$

Retaining only lowest order term in ε and Θ, we obtain

$$(28) \quad \tilde{\Gamma}_r = -\frac{c\,n}{B_0} \frac{V_a}{2\pi R_0}\Theta + \frac{c}{e} \frac{1}{\Theta B_0^2} \frac{\langle \vec{B} \cdot \nabla \overleftrightarrow{\pi_e} \rangle}{\langle 1/\mathcal{S}^2 \rangle}$$

$$+ \eta_{\shortparallel} c^2 n \frac{|\nabla p|}{B_0^2} 2 \frac{\varepsilon^2}{\Theta^2} + \eta_\perp c^2 n \frac{|\nabla p|}{B_0^2}$$

($\tilde{\Gamma}_r = \Gamma_r / 4\pi^2 r R_0$ is the flux density).

The last term describes classical diffusion, and the last term but one Pfirsch-Schlüter diffusion in this geometry. The first term describes a pinch effect. The second term describes diffusion driven by the parallel electron viscosity.

We conclude that Γ_ψ can become anomalously high owing to an anomalously high parallel or perpendicular resistivity or by an anomalous parallel viscosity.

6. EQUILIBRIUM CONDITIONS

Inserting (27) in the steady state version of (20), we get

$$(29) \quad n \frac{V_a}{2\pi} \left\langle \frac{B_\varphi}{R} \right\rangle \frac{\langle R\, B_\varphi \rangle}{\langle B^2 \rangle} = \eta_{\shortparallel} n \langle R\, j_\varphi \rangle - \frac{1}{c} \Gamma_\psi^*,$$

with the abbreviation

$$\Gamma_\psi^* = \frac{c}{e} \frac{\langle R\, B_\varphi \rangle}{\langle B^2 \rangle} \langle \vec{B} \cdot \nabla \overleftrightarrow{\Pi^e} \rangle - \eta_{\shortparallel}\, c^2 n\, \frac{\partial p}{\partial \psi}$$

$$\left\{ \langle \frac{R^2 B_\varphi^2}{B^2} \rangle - \frac{\langle R\, B_\varphi \rangle^2}{\langle B^2 \rangle} \right\} - \eta_{\shortparallel} c^2 n\, \frac{\partial p}{\partial \psi} \langle \frac{R^2 B_p^2}{B^2} \rangle .$$

The left-hand side of (29) must always be positive.
This is obviously the case if the external electric field and
the poloidal magnetic field have the same sign. It now follows
from (29) that at the magnetic axis, where Γ_ψ and hence Γ_ψ^*
vanish, the external field and $\dot{j}_\varphi$ have the same sign. Taking into
account the coordinate orientation and Ampere's law, it then
follows that B_p and the external field have the same sign in
the neighbourhood of the axis. Since B_p should not change sign,
this then holds in the whole volume. Hence

(30) $\quad \eta_{\shortparallel}\, n \langle R\, \dot{j}_\varphi \rangle - \frac{1}{c}\, \Gamma_\psi^* \geq 0$.

This is a necessary condition for a diffusive plasma to be in
a stationary equilibrium. For a given Γ_ψ^*, equation (30) imposes
lower limitations on the toroidal current and hence on the
poloidal magnetic field, thus leading to β_p limitations. Before
discussing this point further, some general features of the
condition (30) will be stated.
A comparison of (27) and (29) shows that Γ_ψ^* results from Γ_ψ
by omitting the pinching part and substituting $\eta_{\shortparallel}$ for $\eta_\perp$ in
the classical part.

We are chiefly interested in the implications of (30) in the
case of anomalously high diffusion. Γ_ψ may become

anomalously high owing to an anomalously high parallel or
perpendicular resistivity or owing to a nonvanishing parallel
electron viscosity. It is easy to discuss (30) qualitatively
for various combinations of these possibilities, but for clarity
we shall confine ourselves to the case that $\eta_{\parallel}$ remains
essentially classic. As mentioned earlier, this is in agree-
ment with current experiments with low electron drift velocities.

Then Γ_ψ may become anomalously high owing to an anomalously high
parallel viscosity or to an anomalously high perpendicular
resistivity. Since $\eta_\perp$ is no longer contained in Γ_ψ^* , any
anomaly of $\eta_\perp$, though it may enhance Γ_ψ , does not affect Γ_ψ^* and
hence the relation (30).

On the other hand, parallel viscosity enhances Γ_ψ and Γ_ψ^* in the
same way. Especially if Γ_ψ is anomalous chiefly owing to an
anomalous parallel viscosity, then $\Gamma_\psi \sim \Gamma_\psi^*$ and (30) is strongly
affected.

To give a quantitative idea of the implications of the condition
(30), we evaluate it by an order of magnitude estimate.

With the toroidal component of (4), it follows from (30) that

$$(31) \quad V_D^* \lesssim V_m$$

where $V_D^* = \Gamma_\psi^* / n \oint 2\pi \, d\ell \, R$. $V_m \sim \eta_{\parallel} c^2 / 4\pi L$ is again the poloidal
magnetic diffusion velocity and L is some average characteristic
length, being of the order a for typical circular cross-section
tokamaks. Equation (31) is in agreement with the introductory
considerations, apart from the fact that the real diffusion
velocity has been replaced by the effective velocity V_D^*.

With the relation (for definition of Vc' see after (14))

$$(32) \quad \beta \, V_m \sim V_c'$$

(31) can be brought into the equivalent form

$$(33) \quad \beta \lesssim \frac{V_c'}{V_D^*} \, .$$

For circular cross-sections it holds that $\beta = \beta_p / q^2 A^2$, so that (33) can be written as

$$(34) \quad \beta_p \lesssim \frac{V_c' \, q^2 A^2}{V_D^*} \, .$$

Various models have been proposed for particle transport in axisymmetric systems. In the following, we shall discuss (31) on the basis of the most unfavourable assumption $V_D^* \sim V_D$ for some of these models. We further assume $\eta_{\|}$ to be classical.

In the Pfirsch-Schlüter-regime it holds that $V_D \sim V_c' \, q^2$, giving

$$(35) \quad \beta \lesssim 1/q^2 \qquad\qquad \beta_p \lesssim A^2$$

In the neoclassical banana regime we have $V_D \sim V_c' \, q^2 A^{3/2}$. Hence

$$(36) \quad \beta \lesssim \frac{1}{q^2 A^{3/2}} \qquad\qquad \beta_p \lesssim \sqrt{A} \, .$$

This is the well known bootstrap condition up to a numerical factor of the order one [3] [6].

In the pseudoclassical regime we get $V_D \sim V_c' q^2 A^2$. Hence

$$(37) \quad \beta \lesssim \frac{1}{q^2 A^2} \qquad\qquad \beta_p \lesssim 1 \,.$$

Next we consider the regime of trapped ion diffusion which should be relevant for tokamak reactors. Here we use [7]

$$V_D \sim \frac{\tau_e}{A^{5/2} L^3} \left(\frac{T c}{e B_z} \right)^2 .$$

Inserting $\eta_{\shortparallel}^{-1} \sim 2 e^2 n \tau_e / m_e$ and $\tau_e \sim \frac{3 \sqrt{m_e}\, T_e^{3/2}}{4 \sqrt{2\pi}\, \ln\Lambda\, e^4 n}$ [5] directly into (31), we obtain

$$(38) \quad \frac{T^5}{n B_z^2 L^2 A^{5/2}} \lesssim \mu \qquad \mu = 5 \cdot 10^{-73} \,.$$

Eliminating n with $n = \frac{\beta B^2}{16\pi T}$ and assuming $B_z \sim B \sim B_a$ where B_a is the value of B at the plasma boundary, it follows that

$$(39) \quad T^6 \lesssim \frac{\mu}{16\pi} \beta L^2 A^{5/2} B_a^4$$

$$\text{or} \quad T \lesssim 2{,}9 \cdot 10^{-2}\, \beta^{1/6}\, L^{1/3}\, A^{5/12}\, B_a^{2/3}$$

$$(T\ [keV],\ L[cm],\ B[kG]) \qquad\qquad .$$

Hence, our condition results in a maximal temperature that can be attained in a stationary tokamak governed by trapped ion diffusion. For a typical reactor-size plasma with L = 400 cm, B_a = 50 KG, A = 2, 5, q = 3, ß = 0,10, this maximum temperature is $T_{max} \approx$ 3 keV. Even lower temperatures would result from the more refined theory of [8]

However, it must be taken into account that the threshold temperature at which the trapped ion diffusion becomes dominant may be larger than the maximum value defined by (39). The threshold condition reads [9] [10]

$$(40) \quad T > 0.1 \; \beta^{1/3} \; A^{2/3} \; L^{1/3} \; B^{2/3} \; q^{1/3}.$$

$$(T \; [keV], \; L \; [cm], \; B \; [kG])$$

For the example given above the relation (40) results in a threshold temperature of $T_{thr} \approx 12$ keV.

Obviously, the larger of the values of T_{max} and T_{thr} is the effective limit temperature that would result from the equilibrium condition in the case of trapped ion diffusion. Lower limitations may result from diffusion laws prevailing in the regime $T < T_{thr}$, which will not be considered here.

The above statement, if applied to the example, implies that a reactor-like tokamak governed by trapped ion diffusion could not be in stationary equilibrium.

In the case of Bohm diffusion $V_D \sim \dfrac{c \, T}{16 e \, B \, L}$ and we again obtain an upper limit for the temperature:

$$(41) \quad T \lesssim 1.8 \cdot 10^{-2} \; B^{2/5} \quad (T \; [keV], \; B \; [kG]),$$

which relation imposes an even more severe restriction. However, the threshold temperature for Bohm diffusion is so high that Bohm diffusion is unlikely to be dominant in a tokamak reactor.

Up to now nothing can be said about the assumption $\Gamma_\psi \sim \Gamma_\psi^*$ i.e. about the role that viscosity plays in anomalous transport. But the above considerations show that these questions may have drastic consequences on the feasibility of tokamak reactors.

Finally, we return to the problem of Ohm's law within the framework of transport calculation. In transport codes, for reasons of simplicity, the so-called cylinder approximation is applied. Equation (29), if taken in this approximation, reads

$$(42) \quad E_\varphi = \eta_{\shortparallel} \dot{j}_\varphi - \frac{1}{c} V_D^* B_P$$

(all quantities depending on r only).

In usual transport calculations Ohm's law is taken in the form [1] [2]

$$(43) \quad E_\varphi = \eta_{\shortparallel} \dot{j}_\varphi .$$

Keeping in mind the considerations that lead to (31), it is obvious that the second term in (42) can be neglected if $V_D^* \ll V_m$, i.e. if the system is far from the limit given by (30).

SUMMARY

Global angular momentum considerations are used to obtain global relations between particle diffusion and bootstrap currents leading in an approximate way to low β_p limits in diffusive tokamak equilibria.

A detailed consideration confined to magnetic surfaces results
in a necessary condition for stationary diffusive tokamak
equilibria which couples the toroidal current density with
an effective outward particle flux Γ_ψ^* in such a way that a
given Γ_ψ^* defines a lower limit for the average toroidal
current density $\langle R j_\psi \rangle$. This condition, if evaluated by an
order of magnitude estimate, states that the effective
diffusion velocity v_D^* must not exceed the poloidal magnetic
diffusion velocity.

Most severe restrictions on the equilibrium result if diffusion
is chiefly caused by electron parallel viscosity. One then has
$\Gamma_\psi^* \sim \Gamma_\psi$ and an anomalous bootstrap current occurs.

For such situations the equilibrium condition is evaluated
for several current diffusion models. In the case of trapped
ion or Bohm diffusion, temperature limits exist which are
below the threshold temperature of the respective diffusion
mechanisms. Hence a system governed by trapped ion or Bohm
diffusion could not be in equilibrium.

Within the framework of our results the approximate form of
Ohm's law usually used in tokamak transport calculations can
be discussed. It is valid only in the limit $v_D^* \ll v_m$. Thus,
current transport calculations, if applied to the anomalous
regime, are implicitly based on the very specific assumption
that viscosity plays no role in anomalous diffusion.

The considerations of this paper are essentially based on
two assumptions, namely that the momentum transfer perpendicular
to a magnetic surface can be neglected within the momentum

equation of the electrons, and that the electron-ion momentum exchange is essentially proportional to the relative electron ion velocity and hence is of the form (10).

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