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Anomalous Electron Transport Equations for Ion Sound and Related Turbulent Spectra

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IPP 6/154 March 1977

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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March 1977 (in English)

Abstract

A complete set of anomalous electron transport equations is derived for ion sound and related spectra.Turbulence causes a drastic modification of magnitude and structure of the transport effects. The self-consistent distribution function is also used to determine wave growth. The important role of the new contributions to wave growth, resistivity and heat conduction in experiments as well as their description by codes are discussed.

I. Introduction

Macroscopic manifestations of turbulence are of great interest to many problems in laboratory and space plasma physics. By now there have been so many observations of anomalous transport effects and of their correlation with the excitation of micro-instabilities that only very few examples can be cited here. Turbulent heating experiments show very strong dissipation of the current that cannot be attributed to Coulomb collisions but is well correlated with the excitation of current driven instabilities.[1] The transition in the earth's bow shock occurs over distances for which Coulomb collisions are negligible and (among other fluctuations) one observes low frequency electrostatic waves excited in the region of large magnetic field gradients (currents) and relaxing downstream.[2] In Z and theta pinches one finds broad profiles and rapid penetration of the magnetic field.[3, 4] Temperature profiles and energy balance show that besides anomalous resistivity there is also anomalous heat conduction.[4] Anomalous viscosity effects have also been suggested.[5] Turbulent spectra have been determined directly by laser scattering.[6, 7] If charge neutrality requires vanishing net current, the heat flux can be the source of instabilities[8] which in turn limit heat flux, as observed in the solar wind[9] and laser plasmas.[10] Static electric and magnetic fields, however, may also be important in these cases.[11, 12]

These examples for anomalous transport suggest that an analysis starts with a discussion of the relevant instabilities. There are usually several instabilities that can be excited for various plasma parameters. Some may be eliminated already on the basis of their growth rates, in fast pinches for example. The observation [4] that enhanced resistivity and reduced axial heat conduction in the Garching theta pinch correspond to roughly the same effective collision frequency $\nu_{eff} \approx 5.10^{-3} \omega_e$ gives strong evidence for the dominance of ion acoustic waves during the implosion phase. All other modes discussed in connection with resistive shocks are restricted to a narrow wedge about the plane perpendicular to the magnetic field, $(k_{\parallel}/k)^2 \approx 10^{-2}$, and thus cannot affect axial heat conduction.[13] Although the electron temperature is much smaller than in the toroidal Belt pinch [4, 14] the condition $T_e >> T_i$ is still satisfied in the current sheath. From the (linear) dispersion relation one also obtains important information about the nature of particle scattering by the wave spectrum.

Particle simulation has been most successful in the next stage of the investigation, the quasilinear and nonlinear evolution of a particular microinstability under idealized macroscopic conditions, current driven ion sound turbulence in a spatially homogeneous plasma for example.[15-17] The resulting theory, which, in principle at least, could be formulated in terms of coupled kinetic equations for particles and waves must then be applied to macroscopic transport phenomena. The macroscopic plasma dynamics will be found from anomalous transport equations, usually a numerical code. Clearly, such a code should be the synthesis of a detailed study of all linear and nonlinear aspects of the relevant instabilities. Feasibility on the other hand requires considerable simplifications and codes which incorporate a few gross features of the instabilities have been quite successful in predicting macroscopic behavior of turbulent plasmas. The most obvious way of arriving at anomalous transport equations is the introduction of effective collision frequencies in the classical transport relations.[3, 18] However, the transport relation between perturbing forces and fluxes may change in structure as well. For example, codes that use a limitation of drift velocities[3] and heat fluxes [10] to critical values determined by instability thresholds rather than the conventional transport relations for resistivity and heat flux may give good or better agreement with some observations.[19, 9] These two methods can be combined by using switch on-off conditions for anomalous transport which are deduced from instability criteria.[4, 20]

The basic problem is that unlike collision dominated plasmas the state of a turbulent plasma is not specified by the local fluid parameters density, mean velocity, mean energy (temperature). The wave spectrum which determines the effective collision frequency evolves usually in a dynamic rather than quasi-static fashion with the macroscopic (and microscopic) plasma parameters.[16] In a generalized sense macroscopic anomalous transport equations are obtained if it is possible to specify particle distributions and wave spectrum by a few parameters that will include the fluid variables and the fluctuation level. It is far from obvious, however, that the kinetic equations can be reduced in this manner, particularly for resonant micro-instabilities for which the evolution depends on details of the distribution functions and the spectrum. Hybrid codes which combine microscopic and macroscopic description have been used successfully. The spectrum may be determined by following the evolution of a number of sample wave modes[21, 22] and such phenomena as ion reflection in shocks have been described by particle simulation for the ions and fluid equations for the electrons.[18, 20]

Fluid equations which have the same formal structure as in the classical case may be obtained by taking moments of the kinetic equation. Closure of this set of equations, however, requires information about the distribution functions. Customarily Maxwellian distributions are assumed. Other models like bi-Maxwellian distributions have also been used to describe anisotropies and heat conduction.[13] The parameters of these model distributions are then specified by an extended set of momentum equations. However, even in classical theory where Coulomb collisions maintain distributions close to a Maxwellian, such models as the thirteen moment approximation have only been modestly successful, particularly in describing heat flux. Instead one uses self-consistent distributions determined by the Chapman-Enskog method.[23] The particle distributions will usually no longer be close to a Maxwellian if turbulent scattering dominates. Thus the need for a self-consistent determination of the distribution functions is even more obvious. The methods evidently depend on the nature of the turbulent spectra.

In the examples of anomalous transport we have mentioned above a prominent role is played by ion sound turbulence and related spectra which are characterized by small phase velocities $(\omega/kv_e) \ll 1$ and short wavelengths $(kv_e/\Omega) \gg 1$. In a companion paper [24], henceforth referred to as I, we have shown how the electron distribution function can be determined self-consistently for these spectra. Dominance of elastic scattering in angle, much as for electron-ion collisions, was the basis for a reduction of the kinetic equation to a system of equations for the isotropic distribution $F(w)$ and the small anisotropic part $\hat{f}(\underline{w})$. The selfconsistent solution for $F(w)$ was discussed. We have shown that quasilinear flattening of F leads to a significant modification of the dispersion relation and the momentum and energy transfer rates by the time turbulent heating raises the electron temperature by a few percent. In this paper we determine the anisotropic distribution $\hat{f}(\underline{w})$ as a functional of F and show that the transition to a non-Maxwellian $F(w)$ changes magnitude and structure of the gradient related transport effects as well. The Onsager relations for the transport coefficients are no longer satisfied and there are now transport terms directly connected with the density gradient. For the same effective collision frequency the numerical coefficient in the heat conductivity is reduced considerably for a flat topped distribution F which has weaker tails than the Maxwellian. The magnetic field dependence of the transport coefficients is also altered. We demonstrate these features in Section II by evaluating the transport coefficients for distribution functions of the form

$$F(w) = n(C_s/v_o^3)\exp\{-(w/v_o)^s\} \qquad (1)$$

and nearly isotropic turbulence.[25] This facilitates the comparison with classical transport theory for which s = 2. If, as usual, turbulent heating dominates a selfsimilar distribution, s = 5, is approached rapidly [I]. An application of the resulting transport equations to "collisional" damping of a test wave in an isotropic turbulent spectrum has also been reported.[26]

Turbulent spectra are hardly ever isotropic. The solution of the kinetic equation in this case is much more difficult. As is characteristic of an anisotropic medium, transport processes of different tensorial character are coupled by the anisotropic spectrum. The problem is simplified considerably if gradients and drifts are either predominantly along the magnetic field (axial symmetry) or across a sufficiently strong magnetic field, conditions which are usually satisfied. Approximations for other cases will also be discussed in Section III and Appendix C. The resulting transport equations depend on a few characteristics of the turbulent spectrum which could be found for example by following the evolution of a few sample wave modes.

The solution for the anisotropic part of the distribution function is also used in Section IV to determine important contributions to the dielectric constant and to complete the kinetic equation for the selfconsistent determination of F(w). Growth rates are significantly increased by temperature and density gradients. Their angle dependence is also rotated with respect to the drift velocity. The important role of this modification of wave growth, heat conduction and other experimental results as well as their description by codes are discussed in Section V which also summarizes the conclusions from this work.[27]

II. Transport Equations for Isotropic Turbulence

Just as in classical transport theory it is necessary to determine the anisotropic part of the distribution function in order to obtain the transport coefficients which connect the perturbing forces and gradients with the various fluxes. The reduction of the kinetic equation to a system of equations for the isotropic part F(w) and the anisotropic part $\hat{f}(w)$ as well as the selfconsistent solution for F(w) have been discussed in I. The equation for the anisotropic part of the electron distribution has the form

$$\underline{w} \cdot \frac{\partial F}{\partial \underline{x}} + (\underline{a} \cdot \underline{w} - \underline{\underline{U}}:\underline{\underline{W}}) \frac{1}{w} \frac{\partial F}{\partial w} - \hat{C}^1 F = (\underline{w} \times \underline{\Omega}) \cdot \frac{\partial \hat{f}}{\partial \underline{w}} + C^o \hat{f} - \langle C^o \hat{f} \rangle \qquad (2)$$

where $\underline{a} = (e/m) \left[\underline{E} + (\underline{u}/c) \times \underline{B} \right] - d\underline{u}/dt$ is the acceleration in frame $\underline{u}$ (usually the electron rest frame $\underline{u}_e$), $\underline{w} = \underline{v} - \underline{u}$, $\underline{\Omega} = -(e\underline{B}/mc)$ $U_{ik} = (\partial u_i/\partial x_k + \partial u_k/\partial x_i)/2 - \delta_{ik}(\partial/\partial \underline{x})\cdot\underline{u}/3$, $\hat{C}^1$ is the small anisotropic collision operator acting on F and C^o the dominant collision operator acting on $\hat{f}(\underline{w})$. Both C^o and the magnetic field term tend to isotropize the distribution function against the perturbing factors on the left hand side of (2). In classical transport theory an equation of the same form is obtained[28]. In this case F is a local Maxwellian, C^o consists of the electron-electron and the lowest order electron-ion collision operator and $\hat{C}^1$ is due to the relative e-i drift $(\underline{u} - \underline{u}_i)/v_e \ll 1$,

$$\hat{C}^1_{ei} F = - \underline{r}_{ei}(w) \cdot \underline{w} \frac{1}{w} \frac{\partial F}{\partial w} \qquad (3)$$

where $\underline{r}_{ei}(w) = - \nu_{ei}(w) \, (\underline{u} - \underline{u}_i)$. If the wave spectrum has only first order anisotropies then C^o_{ei} and C^o_{ew} may be combined to an isotropic Lorentz collision term

$$\nu(w)C_L f = \frac{\nu(w)}{2} \frac{\partial}{\partial \underline{w}} \cdot (\underline{\underline{I}} - \underline{w}\,\underline{w}) . \frac{\partial}{\partial \underline{w}} f \tag{4}$$

with collision frequency $\nu(w) = \nu_{ei} + \nu_{ew}$, and $\hat{C}^1_{ew}F$ is also of the form (3) describing friction between electrons and waves in the wave rest frame $\underline{u}_w$, cf. I.19. Combining e-i and e-w terms gives

$$\underline{r}(w) = \underline{r}_{ei}(w) + \underline{r}_{ew}(w) \equiv -\nu(w) \; (\underline{u} - \underline{u}_o) \quad . \tag{5}$$

The solution of (2) is found by expansion of $F(\underline{w})$ in spherical harmonics

$$\hat{f}(\underline{w}) = \underline{w} \cdot \underline{f}_1(w) + \underline{\underline{W}}:\underline{\underline{f}}_2(w) + \underline{\underline{W}}_3 : \underline{\underline{f}}_3(w) + \ldots \tag{6}$$

where $\underline{W}_3$ is the symmetric l=3 tensor $\underline{\underline{W}}_3 = \underline{w}\,\underline{w}\,\underline{w} - w^2 \,(\underline{w}\,\underline{\underline{I}})_s/5$ etc. For an isotropic collision operator C^o spherical harmonics are eigenfunctions[29]

$$C^o\hat{f}(\underline{w}) = -\underline{w} \cdot \nu_1(w)\underline{f}_1(w) - 3\underline{\underline{W}}:\nu_2(w)\underline{\underline{f}}_2(w) - \ldots \frac{l(l+1)}{2} \underline{\underline{W}}_l : \nu_l \underline{\underline{f}}_l - \ldots \tag{7}$$

For the pitch angle scattering term (4) which describes e-i and isotropic e-w collisions, $\nu_l = \nu(w)$, but for the e-e term $C^o_{ee}\hat{f} = C_{ee}(F)\hat{f} + C_{ee}(\hat{f})F_l$ the ν_l are integrodifferential operators in w. Only the l = 1,2 components are needed in the transport equations to determine the mean velocity

$$\underline{u}_e = \underline{u} + \frac{1}{n}\int d\underline{w}\, \frac{w^2}{3}\, \underline{f}_1(w) \tag{8}$$

the heat flux

$$\underline{q} = \int d\underline{w} \, \frac{mw^2}{2} \frac{w^2}{3} \underline{f}_1(w) \qquad (9)$$

and the viscous stress

$$\underline{\underline{\Pi}} = \underline{\underline{P}} - p \, \underline{\underline{I}} = \int d\underline{w} \, \frac{2}{15} \, mw^4 \, \underline{\underline{f}}_2(w) \quad . \qquad (10)$$

The other two moments, n and $p = nT_e = nm\langle w^2/3\rangle$, are determined from F(w). It is also convenient to combine the eigenfunctions of the magnetic field term in (2),

$$\underline{w} \cdot \underline{f}_1 = w_o f_{1,o} + \frac{1}{2} \left(w_{-1} f_{1,1} + w_1 f_{1,1} \right) \qquad (11)$$

$$\underline{\underline{W}}:\underline{\underline{f}}_2 = \frac{3}{2} W_o f_{2,o} + W_{-1} f_{2,1} + W_1 f_{2,-1} + W_{-2} f_{2,2} + W_2 f_{2,-2} \qquad (12)$$

where, choosing orthogonal unit vectors $\underline{e}_o = \underline{B}/B$, $\underline{e}_1$, $\underline{e}_2$,

$w_o = \underline{w} \cdot \underline{e}_o$, $w_{\pm 1} = \underline{w} \cdot (\underline{e}_1 \pm i\underline{e}_2)$, $W_o = \underline{\underline{W}}:\underline{e}_o\underline{e}_o$, $W_{\pm 1} = \underline{\underline{W}}:\underline{e}_o(\underline{e}_1 \pm i\underline{e}_2)$, $W_{\pm 2} = \underline{\underline{W}}:(\underline{e}_1 \pm i\underline{e}_2)(\underline{e}_1 \pm i\underline{e}_2)/2$ and $i = \sqrt{-1}$.

The components of $\underline{f}_1$ and $\underline{f}_2$ are defined analogously. The solution of (2) becomes then

$$f_{1,j} = -(\nu_1 - ij\Omega)^{-1} \left[\frac{\partial}{\partial \underline{x}} + (\underline{a} + \underline{r}) \frac{1}{w} \frac{\partial}{\partial w} \right]_j F \; ; \; j = 0, \pm 1 \qquad (13)$$

$$f_{2,j} = (3\nu_2 - ij\Omega)^{-1} U_j \frac{1}{w} \frac{\partial F}{\partial w} \; ; \; j = 0, \pm 1, \pm 2 \quad . \qquad (14)$$

If e-e collisions are important as in the classical case, (13-14) are only formal solutions of integrodifferential equations. In the classical case these equations have been solved by direct numerical integration, variational methods or an expansion in orthogonal Laguerre polynomials[23]. For distributions of the form (1) this expansion can be generalized to

$$\underline{f}_1(w) = \frac{F(w)}{v_e} \sum_{k=o}^{N} \underline{b}_k \, L_k^{(5/s)-1}(t) \qquad (15)$$

$$\underline{\underline{f}}_2(w) = \frac{F(w)}{v_e^2} \sum_{k=o}^{N-1} \underline{\underline{c}}_k \, L_k^{(7/s)-1}(t) \qquad (16)$$

where $t = (w/v_o)^s$, reducing the integrodifferential equations to a linear algebraic system of order N. The electron mean velocity (8) becomes $\underline{u}_e = \underline{u} + \underline{b}_o v_e$, thus $\underline{b}_o = 0$ if as usual the electron rest frame $\underline{u} = \underline{u}_e$ is choosen. The heat flux (9) involves all b_k, $k \geq 1$, except for a Maxwellian, s = 2. The viscous stress (10) is determined by the coefficient $\underline{\underline{c}}_o$. We may note that the truncation of the expansion at N = 1 is equivalent to a solution of (2) by the thirteen moment method in which the simplest w dependence, $L_o^m = 1$ $L_1^m = m + 1 - t$ is used.
A larger number of polynomials, usually four to six, is required, however, for reasonable accuracy,[23] particularly for transport connected with the temperature gradient and for intermediate magnetic fields $\Omega / \nu^* \sim 10 - 100$. An expansion to order N implies that the magnetic field dependence is approximated by a rational function of order N in $(\Omega/\nu^*)^2$.

Since (2) is linear in the perturbing factors, they may be treated separately. (Note that there may be a complicated implicit dependence through the wave spectrum). We evaluate the transport terms as usual in the electron rest frame $\underline{u} = \underline{u}_e$ in which

$$\underline{a}^1 = \frac{\nabla p - \underline{R}}{nm} \qquad (17)$$

and the rate of momentum transfer $\underline{R}$ is determined from the condition $\int d\underline{w}\, w^2 \underline{f}_1 = 0$, cf. (8), for each perturbing factor. The relative drift $\underline{u}_e - \underline{u}_o$ leads, using (5), (8), (13) and (17) to

$$R_{u,j} = - \frac{nm}{\tau_e} (1 - \rho_{u,j}) (u_e - u_o)_j \quad ; \quad j = 0, \pm 1 \quad , \qquad (18)$$

where $1 - \rho_{u,j} = \tau_e \int d\underline{w} \frac{w^2}{3} (\nu_1 - ij\Omega)^{-1} \nu(w) \frac{1}{w} \frac{\partial F}{\partial w} \Big/ \int d\underline{w} \frac{w^2}{3} (\nu_1 - ij\Omega)^{-1} (\frac{1}{w} \frac{\partial F}{\partial w})$.

The collision time τ_e is normalized such that $\underline{R}_u^o = - nm(\underline{u}_e - \underline{u}_o)\tau_e$ is the rate of momentum transfer for a drifting isotropic distribution $F(|\underline{v} - \underline{u}|)$ and ρ_u describes the effect of the distortion of f by the speed dependence of the collision frequency $\nu = \nu_{ei} + \nu_{ew}$ for elastic scattering [I]

$$\nu(w) = \frac{1}{\tau_e} (\frac{\pi}{2})^{1/2} \frac{3}{a_{-3}} (\frac{v_e}{w})^3 \quad , \qquad (19)$$

where the form factor a_{-3} is normalized to unity for Maxwellian $F(w)$. For the heat flow connected with the drift we obtain from (9)

$$q_{u,j} = \int d\underline{w} \frac{mw^2}{2} \frac{w^2}{3} f_{u,j} = nT_e \kappa_{u,j} (u_e - u_o)_j \quad ; \quad j = 0, \pm 1 \qquad (20)$$

where $\kappa_{u,j} = - \frac{m}{6nT_e\tau_e} \int d\underline{w}\, w^4 (\nu_1 - ij\Omega)^{-1} [1-\rho_{u,j} - \nu(w)\tau_e] \frac{1}{w} \frac{\partial F}{\partial w}$.

The viscosity coefficients are obtained from (14) and (16)

$$\Pi_j = \int d\underline{w} \frac{2}{15} mw^4 f_{2,j} = - nT_e\tau_e \eta_j U_j \quad ; \quad j = 0, \pm 1, \pm 2 \quad , \qquad (21)$$

where $\eta_j = - \frac{2m}{15nT_e\tau_e} \int d\underline{w}\, w^4 (3\nu_2 - ij\Omega)^{-1} (\frac{1}{w} \frac{\partial F}{\partial w})$.

To obtain the transport connected with the gradients we set

$$F(\underline{x}, w, t) = \frac{n}{v_e^3} \hat{F}(\frac{w}{v_e}, \underline{x}, t) \quad , \qquad (22)$$

thus

$$\left[\nabla + \frac{\nabla p}{nm}\frac{1}{w}\frac{\partial}{\partial w}\right] F = (1 + \frac{T_e}{m}\frac{1}{w}\frac{\partial}{\partial w}) F \nabla \ln n + \left[-\frac{3}{2} + (1 - \frac{mw^2}{2T_e})\frac{T_e}{m}\frac{1}{w}\frac{\partial}{\partial w}\right] F$$

$$\cdot \nabla \ln T_e + \frac{n}{v_e^3} \nabla \hat{F} \quad (\frac{w}{v_e}, \underline{x}, t) \quad . \tag{23}$$

Using (23) in (13) we obtain in the same manner, cf. Appendix A, as for the drift connected transport the transport relations

$$R_{n,j} = T_e \rho_{n,j} \nabla_j n \tag{24}$$

$$q_{n,j} = \frac{T_e^2 \tau_e}{m} \kappa_{n,j} \nabla_j n \tag{25}$$

$$R_{T,j} = - n\rho_{T,j} \nabla_j T_e \tag{26}$$

$$q_{T,j} = - \frac{nT_e\tau_e}{m} \kappa_{T,j} \nabla_j T_e \tag{27}$$

,

$j = 0, \pm 1$. The last term in (23) leads to transport connected with the nonlocal change in the shape $\hat{F}$ of the distribution function. In the classical case we have only terms proportional to ∇T_e. We see from (23) that this is the case only for a local Maxwellian. There remains ofcourse an implicit dependence of transport on ∇n through $\underline{u}$ (diamagnetic drift). We may go from rotating coordinates to the tensor form e.g.

$$\underline{q}_T = - \frac{nT_e\tau_e}{m} \left[\kappa_{T,o} \nabla_{\parallel} T_e + \kappa_{T,\perp} \nabla_{\perp} T_e + \kappa_{T,\wedge}(\underline{e}_o \times \nabla T_e)\right] \quad , \tag{28}$$

where $\kappa_{T,1} = \kappa_{T,\perp} + i\kappa_{T,\wedge}$ and

$$\underline{\underline{\Pi}} = - nT_e\tau_e \left[\eta_o \underline{\underline{U}}_o + \eta_{1,\perp} \underline{\underline{U}}_1 + \eta_{1,\wedge} \underline{\underline{U}}_{1,\wedge} + \eta_{2,\perp} \underline{\underline{U}}_2 + \eta_{2,\wedge} \underline{\underline{U}}_{2,\wedge} \right] , \qquad (29)$$

where $\eta_1 = \eta_{1,\perp} + i\,\eta_{1,\wedge}$, $\eta_2 = \eta_{2,\perp} + \eta_{2,\wedge}$

$$\underline{\underline{U}}_o = \frac{3}{2} U_o \left(\underline{e}_o \underline{e}_o - \underline{\underline{I}}/3 \right) = \frac{3}{2} U_o \underline{\underline{I}}_{2,o}$$

$$\underline{\underline{U}}_1 + i\underline{\underline{U}}_{1,\wedge} = U_{-1} \left[\underline{e}_o (\underline{e}_1 + i\underline{e}_2) + (\underline{e}_1 + i\underline{e}_2) \underline{e}_o \right] \equiv U_{-1}\underline{\underline{I}}_{2,1}$$

$$\underline{\underline{U}}_2 + i\underline{\underline{U}}_{2,\wedge} = U_{-2} (\underline{e}_1 + i\underline{e}_2)(\underline{e}_1 + i\underline{e}_2) \equiv U_{-2}\underline{\underline{I}}_{2,2}$$

$$\underline{\underline{U}} = \underline{\underline{U}}_1 + \underline{\underline{U}}_2 + \underline{\underline{U}}_3 = (\nabla \underline{u})_s - \frac{1}{3} \underline{\underline{I}}\, \nabla \cdot \underline{u} .$$

If as discussed in I, the turbulent fluctuation level is so low that e-e collisions are able to maintain a Maxwellian F(w), the transport relations may be found from classical theory by an appropriate substitution for Z_{eff}

$$\frac{1}{\tau_{eff}} = \frac{a_{-3}}{3} \left(\frac{2}{\pi}\right)^{1/2} \left[3Z \frac{\ln\Lambda}{\Lambda} + \pi \frac{W}{nT_e} \left< \frac{\omega_e}{kv_e} \right> \right] \omega_e \equiv a_{-3} \left(\frac{2}{\pi}\right)^{1/2} Z_{eff} \frac{\ln\Lambda}{\Lambda} \omega_e =$$

$$= Z_{eff} \nu_{ee}^{*} \qquad (30)$$

($a_{-3} = 1$ for a Maxwellian), which also measures the relative importance of e-e collision in $C^o\hat{f}(\underline{w})$. Since the turbulence must also be isotropic the conditions for such an adaption of the classical transport equations are rather restrictive, however. Usually turbulence dominates not only elastic scattering , but inelastic scattering as well. The e-e term in (2) may be neglected for $Z_{eff} \gg 1$. Using the speed dependence w^{-3} of $\nu = \nu_1 = \nu_2$, the integrals required in the transport coefficients (18), (20) etc. may be performed. In order to demonstrate the dependence of the transport coefficients on F(w) we choose (1) with s = 2, 5. Typical results are shown in Fig. 1 for the resistivity and in Fig. 2 for heat conduction. The quasilinear distribution s = 5 has much weaker tails than s = 2 (Lorentz gas) which reduces the dimensionless transport coefficients and also shifts the magnetic field dependence to larger $\Omega\tau_e$. For the use in transport codes the exact $\Omega\tau_e$ dependence obtained by numerical integration has been approximated by rational functions of the form $(d_1 + d_2 x)/(d_3 + d_4 x + d_5 x^2)$, $x = (\Omega\tau_e)^2$ generated by the N = 1 ($d_2 = d_5 = 0$) and N = 2 Laguerre expansion, but determining now the coefficients d_j by a least square approximation in a given interval of $\Omega\tau$. The transverse coefficients, which vanish for $\Omega\tau_e \to 0$, were divided by $\Omega\tau_e$ and then approximated in this manner. In agreement with our discussion of the Laguerre expansion (15-16) and the thirteen moment method, reasonable accuracy requires subdivision of the $\Omega\tau_e$ range into several intervals. We see from (13) that the longitudinal transport coefficients are obtained from the perpendicular transport coefficients for $\Omega\tau_e \to 0$. From (14) and (21) it follows that $\eta_o = \eta_1(0)$, $\eta_2 = \eta_1(2\Omega\tau_e)$. We have already pointed out from (23) that transport terms connected with ∇n vanish only for a Maxwellian F(w). The Onsager relation[28] $\rho_T = \kappa_u$ also requires F to be a Maxwellian. The symmetry relations $\rho_n = \rho_u$, $\kappa_n = \kappa_u$ determine the distribution (1), s = 5, as can be shown from (13), cf. Appendix A.

For any $F(w,\underline{x},t)$, the longitudinal transport coefficients (or $\Omega\tau_e \to 0$) are expressed by various form factors of F which we normalize to unity for a Maxwellian. For $\Omega\tau_e = 0$ the transport relations are

$$\underline{R} = -\frac{nm}{\tau_e}(1-\rho_{u,o})(\underline{u}_e - \underline{u}_o), + \rho_{n,o} T_e \nabla n - \rho_{T,o} n\nabla T_e - nT_e \frac{\nabla a_5}{a_3} \quad (31)$$

$$\underline{q}_e = nT_e \kappa_{u,o}(\underline{u}_e - \underline{u}_o) + \frac{T_e^2\tau_e}{m}\kappa_{n,o}\nabla n - \frac{nT_e\tau_e}{m}\kappa_{T,o}\nabla T_e$$

$$+ \frac{nT_e^2\tau_e}{m}\frac{128a_{-3}}{3\pi}\left(\frac{a_5}{a_3}\nabla a_5 - \nabla a_7\right) \quad (32)$$

$$\underline{\underline{\Pi}} = -nT_e\eta_o\underline{\underline{U}} \quad (33)$$

The dimensionless transport coefficients are given in Table I and the form factors are listed in Appendix B. The transport coefficients for (1) s = 2 correspond to a Lorentz gas $Z_{eff} \to \infty$.

For a weak magnetic field, $\Omega\tau_e \ll 1$, the perpendicular coefficients have corrections of order $(\Omega\tau_e)^2$ and the transverse coefficients are proportional to $\Omega\tau_e$ in lowest order. Expanding the exact coefficients in $\Omega\tau_e$ one obtains again expressions in the form factors (normalized moments) of F. For the transverse thermal force and heat conduction we have for example

$\rho_{T,\wedge} = 9.844\ (0.915)\ \Omega\tau_e$; $\kappa_{T,\wedge} = 172.68\ (2.98)\ \Omega\tau_e$ for (1), s = 2(5).

The transport coefficients for a strong magnetic field[30] may be obtained from an expansion in $(1/\Omega\tau_e)$, but will be found in Section III from a finite Larmor radius expansion which also holds for anisotropic spectra.

For anisotropic turbulence the expansion of $\hat{f}$ in spherical harmonics loses much of its usefulness since different harmonics are now coupled by the collision term C^o. It is then particularly desirable to simplify the e-e collision term. Fortunately Coulomb collisions will be important in elastic scattering only at very low fluctuation levels where the distribution F is kept close to a Maxwellian by inelastic e-e collisions, cf. I.30 and I.32. Simplified e-e collision terms which have been devised for classical and neoclassical transport should then be adequate for our purposes. These terms are of the form

$$C^o_{ee}\,\hat{f} = \nu_{ee}\,C_L\,\hat{f} - \underline{w}\cdot\underline{r}_{ee}\,\frac{1}{w}\,\frac{\partial F}{\partial w} - \underline{\underline{W}}:\underline{\underline{C}}_{ee}\,\frac{1}{w}\,\frac{\partial F}{\partial w} - \ldots \quad , \tag{34}$$

where the Lorentz collision operator (4) with the collision frequency $\nu_{ee}(w) = (D_{ee} - D^{ww}_{ee})/w^2$ is the elastic part of $C_{ee}(F)\,\hat{f}$ and the restoring terms are derived from $C_{ee}(\hat{f})\,F$ by expansion of $\hat{f}$ in spherical harmonics and Laguerre polynomials. Approximations of this kind have been treated systematically be Hirshman and Sigmar.[31] In the simplest case take [32]

$$\underline{r}_{ee}(w) = -\,\nu_{ee}(w)\,(\underline{u} - \underline{u}_{ee}) \quad , \tag{35}$$

where $\underline{u}_{ee}$ is determined from momentum conservation

$$\underline{R}_{ee} = - \int d\underline{w} \, \frac{mw^2}{3} \left[\nu_{ee} \, \underline{f}_1 + \underline{r}_{ee} \frac{1}{w} \frac{\partial F}{\partial w} \right] = 0 \qquad (36)$$

Energy conservation is satisfied automatically by (34). More sophisticated approximations would be required for the terms describing inelastic scattering. The inclusion of (34) in (2) presents basic difficulty. The first term is combined with the isotropic e-i term to the collision operator (4) with the collision frequency $\nu_{ee} + \nu_{ei}$ and the turbulent collision term C^{o}_{ew} on the r.h.s. of (2). The restoring terms represent speed dependent corrections to $\underline{a}$ and $\underline{\underline{U}}$ on the l.h.s. Note that in the expansion of $\hat{C}^1 F = (\hat{C}^1_{ei} + \hat{C}^1_{ew})F$ the l = 2 term is of second order in the small parameters (u/w), (v_i/w) and (ω/kw), thus may be neglected, cf, Section III ,but for anisotropic spectra the e-w term has higher (odd) l components.

$$\hat{C}^1 F = - (\underline{r}_{ei} + \underline{r}_{ew}) \cdot \underline{w} \frac{1}{w} \frac{\partial F}{\partial w} + \underline{\underline{W}}_3 : \underline{\underline{C}}_{3,ew} \frac{1}{w} \frac{\partial F}{\partial w} + \dots \qquad (37)$$

III. Transport Equations for Anisotropic Spectra.

For the usual case, turbulent spectra with strong anisotropy, the solution of the kinetic equation becomes more difficult. The collision operator C^o becomes now a diffusion operator in the polar angle (θ, ϕ) with the diffusion coefficients dependent on these variables. It is still possible to expand the distribution function into spherical harmonics, but as is characteristic of an anisotropic medium, the components $\underline{f}_l$ in (6) are now coupled by C^o. Expanding also the collision terms in (2) into spherical harmonics generates an infinite system of coupled equations

$$\left[\underline{\nabla} + \left(\frac{\nabla p - \underline{R}}{nm} + \underline{r}\right) \frac{1}{w} \frac{\partial}{\partial w} \right] F = (\underline{\Omega} \times \underline{f}_1) + \underline{\phi}_1 (\hat{f}) \quad , \tag{38}$$

$$- (\underline{\underline{U}} + \underline{\underline{C}}_2) \frac{1}{w} \frac{\partial F}{\partial w} = (\underline{\Omega} \times \underline{\underline{f}}_2)^s + \underline{\underline{\phi}}_2 (\hat{f}) \quad , \tag{39}$$

$$- \underline{\underline{C}}_3 \frac{1}{w} \frac{\partial F}{\partial w} = (\underline{\Omega} \times \underline{\underline{f}}_3)^s + \underline{\underline{\phi}}_3 (\hat{f}) \tag{40}$$

etc.,where the superscript s indicates symmetrization, e.g.

$$\left[\underline{\Omega} \times \underline{f_2} \right]^s_{ij} = \varepsilon_{ikl} \Omega_k f_{lj} + \varepsilon_{jkl} \Omega_k f_{li} \tag{41}$$

and the components of the collision term are defined by

$$\frac{3}{w^2} \langle \underline{w}\ Cf \rangle = - \underline{r}(w) \frac{1}{w} \frac{\partial F}{\partial w} + \underline{\phi}_1 (\hat{f}) \tag{42}$$

$$\frac{15}{2w^4} \langle \underline{\underline{W}}\ Cf \rangle = \underline{\underline{C}}_2 (w) \frac{1}{w} \frac{\partial F}{\partial w} + \underline{\underline{\phi}}_2 (\hat{f}) \tag{43}$$

$$\frac{35}{2w^6} \langle \underline{\underline{W}}_3\ Cf \rangle = \underline{\underline{C}}_3 (w) \frac{1}{w} \frac{\partial F}{\partial w} + \underline{\underline{\phi}}_3 (\hat{f}) \tag{44}$$

etc. The bracket indicates a spherical average for which one may use the identity

$$\langle A(\underline{w}) \frac{\partial}{\partial \underline{w}} \underline{D} \rangle = \frac{1}{w^2} \frac{\partial}{\partial w} w \langle A(\underline{w})\underline{w} \cdot \underline{D}(\underline{w}) \rangle - \langle \underline{D} \cdot \frac{\partial}{\partial \underline{w}} A \rangle \quad . \tag{45}$$

In general $\underline{r}$, $\underline{\phi}_l, \underline{\underline{C}}_l$ are operators in w. For the quasilinear collision term I.6 we find that the assumption $\hat{\omega} = (\omega_{\underline{k}} - \underline{k} \cdot \underline{u})/kw < 1$ considerably simplifies these equations. Keeping only dominant terms in $\hat{\omega}$ we find for the $l = 1$ component of the collision term

$$\underline{r}_{ew}(w) = \frac{3}{w^3} \int d\underline{k} \frac{4\pi^2 e^2}{m^2 k^3} W(\underline{k}) \; (\omega_{\underline{k}} - \underline{k} \cdot \underline{u})\underline{k} = - \underline{\underline{\nu}}_1^{ew}(w) . (\underline{u} - \underline{u}_w) \quad . \tag{46}$$

The wave frequency has been Doppler shifted from the wave reference frame to the drift frame $\underline{u}$. The second relation in (46) defines the wave rest frame $\underline{u}_w$ and the collision frequency a tensor

$$\underline{\underline{\nu}}_1^{ew}(w) = 3\pi \frac{W}{nT_e} \omega_e \left(\frac{v_e}{w}\right)^3 \left\langle \frac{\omega_e}{kv_e} \hat{\underline{k}} \hat{\underline{k}} \right\rangle \tag{47}$$

where the average is over the spectrum of resonating waves $\hat{\omega} < 1$ and $\hat{k} = \underline{k}/k$. The contribution from $\hat{f}(\underline{w})$ to the $l = 1$ component of the collision term is

$$\underline{\phi}_1(\hat{f}) = - \underline{\underline{\nu}}_1 \cdot \underline{f}_1 - w^2 \underline{\underline{\nu}}_{13}^{ew} : \underline{\underline{f}}_3 - \dots \tag{48}$$

with

$$\underline{\underline{\nu}}_{13}^{ew}(w) = - \pi \frac{W}{nT_e} \omega_e \left(\frac{v_e}{w}\right)^3 \left\langle \frac{9}{2} \frac{\omega_e}{kv_e} \hat{\underline{k}} \hat{\underline{\underline{K}}}_3 \right\rangle \tag{49}$$

where $\hat{\underline{\underline{K}}}_3$ is the $l = 3$ tensor constructed from $\hat{\underline{k}}$, cf.(6). For the $l = 2$ component we find that $\underline{\underline{C}}_2 = O(\hat{\omega}^2 F)$, thus may be neglected, and

$$\underline{\underline{\Phi}}_2(\hat{f}) = -3\,\underline{\underline{\nu}}_2 : \underline{\underline{f}}_2 - 3\,w^2 \underline{\underline{\nu}}_{24} \vdots\, \underline{\underline{f}}_4 \quad , \tag{50}$$

where

$$\underline{\underline{\nu}}_2^{ew} : \underline{\underline{f}}_2 = \pi \frac{W}{nT_e} \omega_e \left(\frac{v_e}{w}\right)^3 \left\langle \frac{5}{2} \frac{\omega_e}{kv_e} \left(\hat{\underline{k}}\,\hat{\underline{k}} \cdot \underline{\underline{f}}_2 + \hat{\underline{k}} \cdot \underline{\underline{f}}_2\,\hat{\underline{k}} - 2\,\hat{\underline{k}}\,\hat{\underline{k}}\,\hat{\underline{k}}\,\hat{\underline{k}} : \underline{\underline{f}}_2\right)\right\rangle .$$

For $l = 3$

$$\underline{\underline{C}}_3^{ew} = \pi \frac{W}{nT_e} \omega_e \left(\frac{v_e}{w}\right)^3 \frac{1}{w^2} \left\langle \frac{105}{4} \frac{\omega_e}{kv_e} \frac{\omega_{\underline{k}} - \underline{k}\cdot\underline{u}}{k} \hat{\underline{\underline{K}}}_3 \right\rangle = \underline{\underline{C}}_3^o - \frac{35}{6w^2} \underline{u} \cdot \underline{\underline{\nu}}_{13} \tag{51}$$

etc. We find that to lowest order in $\hat{\omega}$ all even l and all odd l components $\underline{\underline{f}}_l$, respectively, are coupled by the even $l = 2, 4 \ldots$ components of the spectrum and that source terms $\underline{r}$, $\underline{\underline{C}}_l$ arise only for odd components of the spectrum. Even source terms and coupling between odd and even $\underline{\underline{f}}_l$ are reduced by the small factor $\hat{\omega} \ll 1$. If the spectrum had only odd l anisotropies then to lowest order in $\hat{\omega}$ the transport problem would be reduced to that of Section II for isotropic spectra. More generally, symmetries of the spectrum will be reflected by the transport relations.

For strong magnetic fields the equations (38-40) etc, become effectively uncoupled for the perpendicular components of $\underline{f}_l$. The natural small expansion parameters are $(v_e/\Omega L) \ll 1$ and $(1/\Omega\tau_e) \ll 1$. We obtain the expansion $f = F(w) + \bar{f}(w,\theta) + \tilde{f}(w,\theta,\phi)$

$$\tilde{f}(w,\theta,\phi) = \tilde{f}^{1,o} + \tilde{f}^{1,c} + \tilde{f}^{2} + \ldots \tag{52}$$

where for convenience we have separated out the collisionless solution $\tilde{f}^{1,0} = \underline{w} \cdot \underline{f}_1^o + \underline{\underline{W}} : \underline{\underline{f}}_2^o$

$$\underline{f}_1^o = - \frac{1}{\Omega} \underline{e}_o \times \left[\nabla + \frac{\nabla p}{nm} \frac{1}{w} \frac{\partial}{\partial w} \right] F \qquad (53)$$

$$\underline{\underline{f}}_2^o = \frac{1}{\Omega} \frac{1}{w} \frac{\partial F}{\partial w} \left[\underline{\underline{U}}_{1,\Lambda} + \frac{1}{2} \underline{\underline{U}}_{2,\Lambda} \right] \qquad (54)$$

without implying any relative ordering between (r_L/L) and $(1/\Omega\tau_e)$. The other terms satisfy

$$\Omega \frac{\partial \tilde{f}^{1,c}}{\partial \phi} = \frac{\underline{R}^o}{nm} \cdot \underline{w}_\perp \frac{1}{w} \frac{\partial F}{\partial w} + \tilde{C}^1 F + \frac{\underline{R}^{o,\parallel}}{nm} \cdot \underline{w}_\perp \frac{1}{w} \frac{\partial F}{\partial w} + \tilde{C}^o \bar{f} \qquad (55)$$

$$\Omega \frac{\partial \tilde{f}^2}{\partial \phi} = \frac{\underline{R}^1}{nm} \cdot \underline{w}_\perp \frac{1}{w} \frac{\partial F}{\partial w} + \tilde{C}^o \tilde{f}^1 - \overline{\tilde{C}^o \tilde{f}^1} \qquad (56)$$

etc. The solution is thus reduced to repeated integrations in ϕ. The collisionless solution $\tilde{f}^{1,o}$ is generated by gradients and shear, the collisional solution $\tilde{f}^{1,c}$ arises from drifts and the anisotropy of the spectrum, which also couples longitudinal and perpendicular perturbations. If $\tilde{f}^{1,0}$ is used in (56) the collision terms in (55-56) are additive, thus transport related to e-e and e-i collisions assumes the large $\Omega\tau_e$ limit of the transport equations in Section II.

In this limit the problem of perpendicular transport is reduced to the evaluation of collision integrals for the anisotropic turbulent collision term. By taking moments of the unexpanded equation (I.40) or the expanded equations for $\tilde{f}$ we find, using (17), that

$$\underline{R}_\perp = \int d\underline{w}\, \underline{w}_\perp (C^1 F + C^o \hat{f}) \qquad (57)$$

evaluated to order n in $(1/\Omega)$ is equivalent to requiring $\langle \underline{w}_\perp \rangle = 0$ to order n+1. Similarly we obtain

$$\underline{q}_\perp = \underline{q}^{1,o} + \underline{e}_o \times \frac{1}{\Omega} \left[\frac{\delta \underline{q}}{\delta t} - \frac{5}{2} \frac{T_e}{m} \underline{R} \right] \tag{58}$$

$$\Pi_j = - \frac{2ip}{j\Omega} U_j + \frac{1}{j\Omega} \frac{\delta \Pi_j}{\delta t} \qquad j = \pm 1, \pm 2 \tag{59}$$

where the first terms represent the collisionless heat flux and viscosity arising from $\tilde{f}^{1,0}$.

$$\underline{q}^{1,o} = - \underline{e}_o \times \frac{1}{\Omega} \int d\underline{w} \frac{mw^2}{2} \frac{w^2}{3} \left[\nabla + \frac{\nabla p}{nm} \frac{1}{w} \frac{\partial}{\partial w} \right] F \tag{60}$$

$$= \underline{e}_o \times \frac{v_e^2}{\Omega} nT_e \left[\overline{\kappa}_{n,\wedge} \nabla \ln n - \overline{\kappa}_{T,\wedge} \nabla \ln T_e - \frac{5}{2} \nabla a_4 \right] ,$$

where the transport coefficients are expressed by the normalized form factors of F, cf. Table II. The collision integrals are defined by

$$\frac{\delta \underline{q}}{dt} = \int d\underline{w} \frac{mw^2}{2} \underline{w} \, C \, f \tag{61}$$

$$\frac{\delta \underline{\underline{\Pi}}}{\delta t} = \int d\underline{w} \, m\underline{\underline{W}} \quad C \, f \tag{62}$$

In I we have expressed the collision integrals $\underline{R}$ and Q(heat transfer) in terms of the spectrum and the conductivity tensor. Similarly, for electrostatic fluctuations

$$\frac{\delta \underline{\underline{\Pi}}}{\delta t} = \int d\underline{k} \, W(\underline{k}) \frac{8\pi}{k^2} \left[\underline{k} \, R_e \, \underline{\underline{\kappa}} \cdot \underline{k} + R_e \, \underline{\underline{\kappa}} \cdot \underline{k} \, \underline{k} \right] \tag{63}$$

We must point out, however, that these relations become useful only when the distribution function is known, which determines the conductivity tensor $\underline{\underline{\kappa}}\ (\underline{k},\ \omega_{\underline{k}})$.

To lowest order in $(1/\Omega)$ we use F(w) in the collision integrals and obtain transport related to anisotropy of the spectrum in the electron drift frame $\underline{u}$. From (57), (42) and (46)

$$\underline{R}^{o}_{ew} = nm\underline{r}^{*}_{ew} = - nm\ \underline{\underline{\nu}}_{1}^{ew,*} \cdot (\underline{u} - \underline{u}_{w}) \tag{64}$$

where for the w^{-3} dependence

$$\underline{\underline{\nu}}_{1}^{*} = - \int d\underline{w}\ \frac{w^2}{3}\ \underline{\underline{\nu}}_{1}(w)\ \frac{1}{w}\frac{\partial F}{\partial w} = \frac{a_{-3}}{3}\ (\frac{2}{\pi})^{1/2}\ \underline{\underline{\nu}}_{1}(v_e) \tag{65}$$

etc. and using (47)

$$\underline{\underline{\nu}}_{1}^{ew,*} = (2\pi)^{1/2}\ a_{-3}\ \omega_e\ \frac{W}{nT_e} \left\langle \frac{\omega_e}{kv_e}\ \frac{\underline{k}\ \underline{k}}{k^2} \right\rangle . \tag{66}$$

Relations (64) and (66) are also derived from momentum conservation and the dielectric constant as shown in I.
The corresponding heat flux is from (58), (61)

$$\underline{q}^{1,c} = - \underline{e}_{o} \times \frac{1}{\Omega} \int d\underline{w}\ (\frac{mw^2}{2} - \frac{5}{2}\ T_e)\ \frac{w^2}{3}\ \underline{r}(w)\ \frac{1}{w}\ \frac{\partial F}{\partial w} \tag{67}$$

$$= - nT_e\ \frac{\vec{\kappa}_{u,\Lambda}}{\Omega}\ (\underline{e}_{o} \times \underline{r}^{*}) ,$$

Transport connected with the gradients is obtained by using the collisionless solution $f^{1,0}$ in the collision integrals.

$$\underline{R}^{1,0} = nT_e\ \underline{\underline{\nu}}_{1}^{*} \cdot \underline{e}_{o} \times \frac{1}{\Omega}\ (\bar{\rho}_{n,\Lambda}\ \nabla \ln n - \bar{\rho}_{T,\Lambda}\ \nabla \ln T_e + \frac{\nabla a_{-1}}{a_{-3}}) \tag{68}$$

$$\underline{q}^{2,0} = - nT_e (\frac{v_e}{\Omega})^2 \; \underline{e}_o \times \underline{\underline{\nu}}_1^* \cdot \underline{e}_o \times \Big[\bar{\kappa}_{n,\perp} \nabla \ln n - \bar{\kappa}_{T,\perp} \nabla \ln T_e \tag{69}$$

$$- \frac{1}{a_{-3}} \nabla (a_1 - \frac{5}{2} a_{-1}) \Big] ,$$

The viscous stress to this order is

$$\Pi_j^{2,0} = - nT_e \sum_K \nu_{2,jk}^* \frac{\bar{\eta}_{1,\perp}}{jk\Omega^2} U_k \qquad j,k = \pm 1 \pm 2 \tag{70}$$

where the components of the tensors are defined as in Section II for rotating coordinates,

$$U_j = \underline{\underline{U}} : \frac{1}{2} \underline{\underline{I}}_{2,j} \qquad \nu_{2,jk}^* = \frac{1}{2} \underline{\underline{I}}_{2,j} : \underline{\underline{\nu}}_2^* : \frac{1}{2} \underline{\underline{I}}_{2,k} .$$

To this order the contribution from the e-i and e-e collision terms may be added. The e-i contribution has the same form as the e-w contribution, but the collision frequency becomes a scalar. We may combine e-i and e-w terms, defining $\underline{\underline{\nu}}_1 = \underline{\underline{\nu}}_1^{ew} + \nu_{ei} \underline{\underline{I}}$, $\underline{\underline{\nu}}_2 = \underline{\underline{\nu}}_2^{ew} + \nu_{ei} \underline{\underline{I}}_4$ where $\underline{\underline{I}}_4$ is the unit tensor of rank four etc, and also set $\underline{r} = \underline{r}_{ew} + \underline{r}_{ei} = - \underline{\underline{\nu}}_1 \cdot (\underline{u} - \underline{u}_o)$ defining $\underline{u}_o$, cf. (5). In the next order of $(1/\Omega)$ classical and turbulent collision terms are coupled. Such terms, which represent corrections to the transport terms already given, are usually not considered. From (55) we obtain the collisional solution connected with the anisotropy of the spectrum in frame $\underline{u}$. The $l = 1$ component is

$$\underline{f}_1^{1,c} = \underline{e}_o \times \frac{1}{\Omega} \Big[\underline{r}^* - \underline{r}(w) \Big] \frac{1}{w} \frac{\partial F}{\partial w} \equiv - \underline{v}_D(w) \frac{1}{w} \frac{\partial F}{\partial w} \tag{71}$$

$$,$$

where $\underline{v}_D(w)$ is the shift of the particle gyration center with respect to $\underline{u}$, $F + \tilde{f}^{1,c} \approx F(|\underline{w} - \underline{v}_D|)$, which results from the speed dependent friction force. Using (57), (42), (48) the contribution of (71) to momentum transfer is obtained

$$\underline{R}^{1,c} = - nm \underline{\underline{\nu}}_1^{*} \cdot (\underline{e}_o \times \underline{r}^{*}) \frac{\bar{\rho}_{u,\wedge}}{\Omega} \quad . \tag{72}$$

Similarly, using (58) and (71) we obtain the heat flux

$$\underline{q}^{2,c} = nT_e \, \underline{e}_o \times \underline{\underline{\nu}}_1^{*} \cdot (\underline{e}_o \times \underline{r}^{*}) \frac{\bar{\kappa}_{u,\perp}}{\Omega^2} \quad , \tag{73}$$

The coefficients in (72-73)

$$\bar{\rho}_{u,\wedge} = \frac{1}{n} \int d\underline{w} \, \frac{w^2}{3} \frac{\nu_1}{\nu_1^{*}} \left(1 - \frac{\nu_1}{\nu_1^{*}} \right) \frac{1}{w} \frac{\partial F}{\partial w} \tag{74}$$

$$\bar{\kappa}_{u,\perp} = -\frac{1}{n} \int d\underline{w} \, \frac{w^2}{3} \left(\frac{mw^2}{2T} - \frac{5}{2} \right) \frac{\nu_1}{\nu_1^{*}} \left(1 - \frac{\nu_1}{\nu_1^{*}} \right) \frac{1}{w} \frac{\partial F}{\partial w} \tag{75}$$

exist only for distributions F which are sufficiently flat in the low w region if the w^{-3} dependence of ν_1 is used. Divergence signifies a breakdown of the expansion in (ν_1/Ω) and thus an asymptotic dependence different from $(1/\Omega)$. In addition of course the w^{-3} dependence also breaks down at speeds below the phase velocity range. Table II expresses the transport coefficient in terms of the form factors of F and gives numerical values for (1), s =2,5. The transport coefficients not listed also vanish as $\Omega\tau_e \to \infty$ but again the expansion (55-56) holds only for a restricted class of F(w).

The l = 2 component of $\tilde{f}^{1,c}$ is $O(\hat{\omega}^2)$ and the l = 3 contribution to $\underline{R}^{1,c}$, $\underline{q}^{2,c}$ can usually also be neglected since it follows from (40) and (51) that $\underline{f}_3^{1,c}$ is orthogonal to $\underline{\underline{\nu}}_{13}$ except for the contribution from $\underline{\underline{C}}_3^o$ which gives an $O(\nu_{13}/\Omega)$ correction to $\underline{u}_w$. Only in second order of ν_{13}/Ω one obtains a correction to $\underline{r}(w)$. Note that usually $\underline{u} \gg \underline{u}_w \approx (\omega/k)$. A similar conclusion holds for the higher l contributions to $\tilde{f}^{1,c}$. As seen from (55) there remains an additional term which arises from the coupling to the longitudinal perturbation by the anisotropy and which could be determined without difficulty once $\bar{f}(w,\mu)$ has been found.

As a special case the strong magnetic field limit of the transport coefficients for isotropic spectra in Section II is contained in the transport relations given above. The transverse coefficients have the dependence

$$\kappa_{T,\wedge} = \frac{1}{\Omega\tau_e} \bar{\kappa}_{T,\wedge} \tag{76}$$

and the perpendicular coefficients are

$$\kappa_{T,\perp} = \left(\frac{1}{\Omega\tau_e}\right)^2 \bar{\kappa}_{T,\perp} \tag{77}$$

etc.

Truncating the expansion of $\hat{f}(\underline{w})$ into spherical harmonics after l = 2 and using the now uncoupled equations (38-39) with (48) and (50) would generalize the transport equations to a system which becomes correct in the strong magnetic field limit and retains some effects of anisotropy for arbitrary magnetic fields. Such a system may be adequate for shock waves where $\Omega\tau_e$ may go from large to small values but most of the turbulent plasma lies in the large $\Omega\tau e$ limit and the perturbations are essentially perpendicular to the magnetic field. A more accurate treatment would require an ad hoc truncation at higher l or a direct solution of the partial differential equation for $\hat{f}(\underline{w})$.

We now solve directly the kinetic equation (I.40) for the longitudinal perturbation $\bar{f}$. Coupling to the perpendicular perturbations $\tilde{f}$ is due to the anisotropic turbulent collision operator. This term is of order (ν_{ew}/Ω), thus may be neglected in the strong magnetic field limit. The same situation arises in the absence of a magnetic field for perturbations with axial symmetry. Taking this axis or the direction of the strong magnetic field as polar axis of a spherical coordinate system (w, cos $\theta = \mu$, $\emptyset$) we obtain from (2)

$$\mu w \left[\nabla_{\parallel} + \frac{\nabla_{\parallel} p - R_{\parallel}}{nm} \; \frac{1}{w}\frac{\partial}{\partial w} \; F + \frac{1-3\mu^2}{2} \; U_o w \frac{\partial F}{\partial w} - \frac{\partial}{\partial p} \bar{D}^{\mu w} \frac{1}{w} \; \frac{\partial F}{\partial w} \right.$$

$$= \frac{\partial}{\partial \mu} \; \frac{\bar{D}^{\mu\mu}}{w^2} \; \frac{\partial \bar{f}}{\partial \mu} \tag{78}$$

where the bar indicates a $\emptyset$ average.

For weak anisotropies we can use the expansion

$$\bar{D}^{\mu w} = \frac{1-\mu^2}{2} \, w r_{\parallel} - \frac{w^2}{2} \, \mu (1-\mu^2) \; C_{2,o} + \ldots \tag{79}$$

$$\bar{D}^{\mu\mu} = (1-\mu^2) \bar{D}^{\theta\theta} = \nu(w) \frac{w^2}{2} \; (1-\mu^2) + \ldots \tag{80}$$

Coulomb collisions may be included in this manner if the model collision term (34) is used for e-e collisions and $C_{2,0} \to 0$ for e-i collisions, cf. (3) . The effective collision frequency for pitch angle scattering in this case is given by

$$\bar{\nu}(w,\mu) = \frac{2\bar{D}^{\theta\theta}}{w^2} = \nu_{ew}(w,\mu) + \nu_{ei}(w) + \nu_{ee}(w) \quad , \qquad (81)$$

For an anisotropic spectrum, Coulomb collisions may be important in certain regions, even at modest fluctuation levels. An integration of (78) gives

$$\frac{\partial \bar{f}}{\partial \mu} = - \frac{w}{\bar{\nu}(w,\mu)} \left[\nabla_{\parallel} + \left(\frac{\nabla_{\parallel} p - R_{\parallel}}{nm} + \frac{2\bar{D}^{\mu w}}{w(1-\mu^2)} - \mu w U_o \right) \frac{1}{w} \frac{\partial}{\partial w} \right] F \; . \qquad (82)$$

The longitudinal transport terms are obtained by w integration of the l = 1,2 components of $\bar{f}$ as outlined in Section II. We have

$$f_{1,0} = \frac{3}{2w} \left\langle (1-\mu^2) \frac{\partial \bar{f}}{\partial \mu} \right\rangle = -\tau_1(w) \left[\nabla_{\parallel} + \left(\frac{\nabla_{\parallel} p - R_{\parallel}}{nm} + \langle r_{\parallel} \rangle \frac{1}{w} \right) \frac{\partial}{\partial w} \right] F \qquad (83)$$

$$f_{2,0} = \frac{5}{2w^2} \left\langle \mu(1-\mu^2) \frac{\partial \bar{f}}{\partial \mu} \right\rangle = \frac{1}{3} \tau_2(w) \, U_o \frac{1}{w} \frac{\partial F}{\partial w} \quad , \qquad (84)$$

where

$$\tau_1(w) = \frac{3}{2} \left\langle \frac{1-\mu^2}{\bar{\nu}(w,\mu)} \right\rangle \quad , \qquad (85)$$

$$\tau_2(w) = \frac{15}{2} \left\langle \frac{\mu^2(1-\mu^2)}{\bar{\nu}(w,\mu)} \right\rangle \quad , \qquad (86)$$

$$\tau_1 \langle r_{\parallel} \rangle = \frac{3}{w} \left\langle \frac{\bar{D}^{\mu w}}{\bar{\nu}(w,\mu)} \right\rangle \quad . \qquad (87)$$

Coupling between odd and even l components is negligible for $\tilde{\omega} = (\omega_{\underline{k}} - \underline{k} \cdot \underline{u})/kw \ll 1$, as discussed above. This is seen directly by evaluating the components of the quasilinear diffusion tensor.

Introducing also polar coordinates $(k, \eta = \cos\alpha, \Psi)$ for the wavevector we obtain on integrating over the resonance in $\phi - \Psi$

$$\bar{D}^{\alpha\beta}_{ew} = \int_0^{2\pi} \frac{d\phi}{2\pi} \underline{e}_\alpha \cdot \underline{\underline{D}} \cdot \underline{e}_\beta = \int_0^\infty dk\, k^2 \int_0^{2\pi} d\psi \int_{\eta_1}^{\eta_2} d\eta \frac{8\pi e^2}{m^2 kw} \quad W(k,\eta,\Psi)\, \frac{1}{J}\, \hat{k}_\alpha\, \hat{k}_\beta \qquad (88)$$

where $J = \left[(\eta-\eta_1)\ (\eta_2-\eta)\right]^{1/2} \qquad \eta_{1,2} = \hat{\omega}\mu \pm \left[(1-\hat{\omega}^2)(1-\mu^2)\right]^{1/2}$

$$\hat{k}_w = \underline{k}.\underline{w}/kw = \hat{\omega}$$

$$\hat{k}_\mu = -\sin\theta\ \hat{k}_\theta = \eta - \hat{\omega}\mu \quad .$$

We have already made use of the assumption $\hat{\omega} \ll 1$ in writing down (78), thus to lowest significant order in $\hat{\omega}$ we use

$$J \approx (1-\mu^2 - \eta^2)^{1/2}\ , \quad \eta_{1,2} \approx \pm(1-\mu^2)^{1/2},\ \hat{k}_\mu \approx \eta \qquad (89)$$

It follows that $\bar{\nu}_{ew}(w,\mu)$, $\bar{D}^{\mu w}_{ew}$ are approximately even in μ, thus U_o may be neglected in (83) and $\bar{D}^{\mu w}_{ew}$ in (84). We also note the w^{-3} dependence of $\bar{\nu}_{ew}$ and $\bar{D}^{\mu w}_{ew}/w$.

If Coulomb collisions can be neglected (87) becomes, using (88-89),

$$\langle r_\mu \rangle = \frac{3w}{2} \langle \frac{\bar{D}^{\mu w}}{\bar{D}^{\mu\mu}} (1-\mu^2) \rangle \frac{1}{\tau_1} = -\frac{1}{\tau_1} (u_\parallel - \langle u_\parallel \rangle) \qquad (90)$$

which should be compared with (46). We see that (83-84) have the same form as the longitudinal components of (13-14) except that we now have two collision times. If we use the same normalization

$\tau_{1,2}(w) = (2/\pi)^{1/2}(a_{-3}/3)\tau^{*}_{1,2}\,(v_e/w)^3$ the longitudinal transport coefficients are the same as in the isotropic case. This is no longer true if e-e collisions are included since the speed dependence of $\tau_{1,2}$ depends than also on the degree of anisotropy of the turbulent collision frequency, cf. (81). For the model e-e term (34-35) it follows that $\langle r_{\parallel}\rangle$ retains the form (90) but with speed dependent $\langle u_{\parallel}\rangle$ and similarly U_o in (84) has a speed dependent correction term $C^{ee}_{2,o}$. Transport coefficients may be determined as before but the w integration becomes more difficult.

The collision times depend on the longitudinal component of the fluctuating electric field ($k_{\parallel} = \eta k$) as expected. They must satisfy $v_e\tau^{*}_{1,2}/L \ll 1$ for collisional theory to apply in the longitudinal direction. This amounts to requiring that $\bar{f} \ll F$ at least for velocity space regions that make the dominant contribution to transport.

IV. Effect of Anisotropy on Wave Growth and Heating

We may use the solution for the anisotropic distribution $\hat{f}(\underline{w})$ found in Sec. II and III to obtain corrections to the dielectric constant and to complete the kinetic equation for F(w). Expanding $\hat{f}(\underline{w})$ into spherical harmonics as in Sec. III we obtain from (I. 62)

$$\mathrm{Im}\ \varepsilon_e(\underline{k},\omega) = -\frac{4\pi^2 e^2}{m\,k^2}\int d\underline{w}\ \ \delta\,(\omega-\underline{k}\cdot\underline{u}-\underline{k}\cdot\underline{w})\ \underline{k}\cdot\frac{\partial f}{\partial \underline{w}}$$

$$= \frac{8\pi^3 e^2}{m\,k^2}\ \{\frac{\omega'}{k}\,F\,(\frac{\omega'}{k}) + (\frac{\omega'}{k})^2\,\underline{f}_1\ (\frac{\omega'}{k})\cdot\hat{\underline{k}}\ + \tag{91}$$

$$(\frac{\omega'}{k})^3\ \underline{\underline{f}}_2 : \hat{\underline{\underline{K}}}_2 - \int_{\omega'/k}^{\infty} dww\left[\underline{f}_1\cdot\hat{\underline{k}} + 3\hat{\omega}\,w\ \underline{\underline{f}}_2 : \hat{\underline{\underline{K}}}_2 - \frac{3w^2}{2}(1-5\hat{\omega}^2)\ \underline{\underline{f}}_3 : \hat{\underline{\underline{K}}}_3 + \ldots\right]\},$$

where $\omega' = \omega-\underline{k}.\underline{u}$ $\hat{\omega} = \omega'/kw$. The real part of the dielectric constant is dominated by the isotropic distribution F(w). Generally the anisotropy f_l of f produces an l anisotropy of the $\underline{k}$ dependence of $\varepsilon_e(\underline{k},\omega)$. The effect of even f_l on Im ε_e and that of odd f_l on Re ε_e is reduced, however, by the small factor $\hat{\omega}$. We see from (91) that to lowest order in $\hat{\omega}$ the effect of the $l=1$ distortion on wave growth is equivalent to an effective drift velocity

$$\underline{u}_{eff} = \underline{u} + \left[\int d\underline{w}\ \ \frac{1}{w}\,\underline{f}_1 \Big/ \int d\underline{w}\,(-\frac{1}{w^2}\,\frac{\partial F}{\partial w})\right] \tag{92}$$

in the relation (I. 76)

$$\mathrm{Im}\ \varepsilon_e(k\omega) = (\frac{\omega_e}{k\ v_e})^2\ (\frac{\pi}{2})^{1/2} a_{-3}\ \frac{\omega-\underline{k}.\underline{u}_{eff}}{kv_e} \tag{93}$$

Using (93) and momentum conservation (I. 67) one obtains

$$\underline{R}_{ew} = -\ nm\underline{\underline{\nu}}_1^{ew*}\cdot\ (\underline{u}_{eff}-\underline{u}_w) \tag{94}$$

which should be compared with (64) and (66).

The same relation between effective drift and $\underline{R}_e$ is obtained from the collision integral (57) using (42), (48) and the w^{-3}

dependence of $\underline{\underline{\nu}}_1$. The effective drift velocity for wave growth is thus obtained immediately from the transport relations for the rate of momentum transfer. Since e-i collisions have the same speed dependence w^{-3} and e-e collisions don't contribute to $\underline{R}_e$ the relation remains valid if Coulomb collisions are included in the transport relations ($\underline{\underline{\nu}}_1^{ew*} \rightarrow \underline{\underline{\nu}}_1^*$, $\underline{u}_w \rightarrow \underline{u}_o$). The effective drift velocity differs from the mean velocity $\underline{u}$ since low speed particles make a larger contribution to resistivity and wave growth. We see from (94) and (18) that the distortion $\hat{f}(\underline{w})$ reduces the effective drift along the magnetic field to $(1-\rho_{u,o})u_{\parallel}$ and the effective drift perpendicular to $\underline{B}$ is rotated by the angle

$$\tan\ \theta = \frac{-\rho_{u,\wedge}}{1-\rho_{u,\perp}} \tag{95}$$

from the direction of $\underline{u}_{\perp}$, i.e. opposite the sense of electron gyration if $\rho_{u,\wedge} > 0$. Additional effective drifts due to density and temperature gradients may be determined from (94), (24) and (26). In the limit $\Omega\tau_e \gg 1$ and anisotropic spectra the effective drift can be read off from the relations given above for

$\underline{R} = \underline{R}^o + \underline{R}^{1,c} + \underline{R}^{1,o}$,

$$\underline{u}_{\perp,eff} = \underline{u}_{\perp} + \frac{\bar{\rho}_{u,\wedge}}{\Omega}\ \underline{e}_o \times \underline{r}^* - \frac{v_e^2}{\Omega}\ \underline{e}_o \times (\bar{\rho}_{n,\wedge} \nabla \ln n - \bar{\rho}_{T,\wedge} \nabla \ln\ T_e + \frac{\nabla a_{-1}}{a_{-3}}) \tag{96}$$

The second term arises from the distortion $\tilde{f}^{1,c}$ due to the drift $\underline{u}$ and the speed dependence of $\underline{\underline{\nu}}_1$. The effect of gradients is obtained from the collisionless solution $\tilde{f}^{1,0}$. Transport coefficients and thus $\underline{u}_{eff}$ depend strongly on $F(w)$. For collisionless Maxwellian electrons the effective drift due to ∇T_e has been obtained previously [33]. The rotation of $\underline{u}_{eff}$ with respect to $\underline{u}$ was obtained[17] from perturbed orbit theory for the dielectric constant of a turbulent plasma in a magnetic field[34]. If, however, as discussed in I the correlation between wave and particles is limited by linear and turbulence effects to a straight line section of the orbit, $\Omega\, \tau_{corr} << 1$, than the turbulent or collisional shift of the gyration centers (71) enters Im ε_e only through the distortion $\tilde{f}^{1,c}$ of the distribution function in the linear, unmagnetized dielectric constant (91).

For anisotropic spectra we have $l = 3, 5 \ldots$ components f_l which also contribute to (91). For the longitudinal distortion $\bar{f}(w,\mu)$ we obtain these contributions directly by integrating Im ε_e by parts in w and over the resonance in $\phi-\Psi$ as for the diffusion coefficients (88). The result is

$$\mathrm{Im}\, \varepsilon_e(k,\omega) = \frac{8\pi^3 e^2}{m\, k^2} \quad \frac{\omega'}{k}\, f\left(\frac{\omega'}{k}, \eta\right) - \int_{\omega'/k}^{\infty} dw\, \frac{1}{\pi} \int_{\mu_1}^{\mu_2} d\mu\, \frac{1}{J}\, \frac{\eta - \hat{\omega}\mu}{1-\hat{\omega}^2}\, \frac{\partial \bar{f}}{\partial \mu} \tag{97}$$

$$\text{where } J \equiv \left[(\mu-\mu_1)\,(\mu_2-\mu)\right]^{1/2} \qquad \mu_{1,2} = \hat{\omega}\,\eta \pm \left[(1-\hat{\omega}^2)(1-\eta^2)\right]^{1/2} .$$

Again, we need to consider only dominant terms in $\hat{\omega} = \omega'/kw$. For an $l=1$ distortion $(\partial f/\partial \mu) = w f_{1,0}$ equ. (97) reduces then to (93) with the effective drift given by the longitudinal component of (92). In general however the $\hat{k}$ dependence of Im ε_e $(\underline{k}, \omega)$ differs from the $\cos \alpha = \eta = \frac{k_{\parallel}}{k}$ law. Using (82) and (97) the angle dependent effective drift velocity becomes

$$u_{\parallel,\mathrm{eff}}(\eta) = u_{\parallel} - \frac{1}{F(o)} \int_o^\infty dw\, w\tau_1(w,\eta)\{\nabla_{\parallel} + [\frac{\nabla_{\parallel} p - R_{\parallel}}{nm} + r_{\parallel}(w,\eta)] \frac{1}{w}\frac{\partial}{\partial w}\}F ,$$

where

$$\tau_1(w,\eta) = \frac{1}{\pi} \int_{-\sqrt{1-\eta^2}}^{\sqrt{1-\eta^2}} d\mu \frac{1}{(1-\mu^2-\eta^2)^{1/2}} \frac{1}{\bar{\nu}(w,\mu)} \tag{99}$$

,

$$\tau_1(w,\eta) r_{\parallel}(w,\eta) = \frac{1}{\pi} \int_{-\sqrt{1-\eta^2}}^{\sqrt{1-\eta^2}} d\mu \frac{1}{(1-\mu^2-\eta^2)^{1/2}} \frac{2\bar{D}\mu w}{w(1-\mu^2)\bar{\nu}(w,\mu)} = -[u_{\parallel} - u_{\parallel}^{o}(\eta)] \tag{100}$$

which should be compared to (85) and (90).

If e-e collisions are neglected $\tau_1(w,\eta)$ has a w^3 dependence and $u_{\parallel}^{o}(\eta)$ is independent of w. Performing the w integration gives then

$$u_{\parallel,\mathrm{eff}} = -\tau_1^{*}(\eta)\frac{R_{\parallel}}{nm} + u_{\parallel}^{o}(\eta) , \tag{101}$$

where $R_{\parallel}$ is given by (31), $\tau_e \rightarrow \tau_1^{*}$.

This relation generalizes (94) for the longitudinal component to anisotropic spectra. Unlike (94), relation (101) no longer holds in this simple form if e-e collisions are included because the μ averages required for R_e and Im ε_e are different and this leads to a different speed dependence in the case of anisotropic spectra.

We now use the solution for the anisotropic distribution $f(\underline{w})$ to complete the kinetic equation (I. 35) for the isotropic distribution $F(w)$. We begin by considering the collision term $\langle Cf\rangle = \langle C\rangle F + \langle \hat{C}\hat{f}\rangle$.

The first term takes the form

$$<C>F = \frac{1}{w^2} \frac{\partial}{\partial w} w^2 \left[-<A^w> + <D^{ww}> \frac{\partial}{\partial w} \right] F(w) \qquad (102)$$

,

where the drag force $<\underline{A}(w)>$ is due mainly to e-e collisions and the e-i and e-w diffusion terms must be transformed to the $\underline{u}$ frame as discussed in I. Doppler shifting the frequency from the wave rest frame $\underline{u}_w$ to $\underline{u}$ gives, cf. (I. 22)

$$<D^{ww}_{ew}> = <D^{ww}_{ew}>^w + \frac{1}{3} (\underline{u}-\underline{u}_w)\cdot \underline{\underline{\nu}}_1^{ew} \cdot (\underline{u}-\underline{u}_w) \qquad (103)$$

,

where $\underline{u}_w$ is defined by (46). In the anisotropic collision term

$$<\hat{C}_{ew} \hat{f}> = \frac{1}{w^2} \frac{\partial}{\partial w} w^2 <\underline{D}^{ww}_{ew} \cdot \frac{\partial \hat{f}}{\partial \underline{w}}> \qquad (104)$$

we expand $\hat{f}(w)$ into spherical harmonics, using (42-45)

$$<\underline{D}^{ww} \cdot \frac{\partial \hat{f}}{\partial \underline{w}}> = \frac{w}{3} \underline{r}\cdot\underline{f}_1 - \frac{2w^5}{35} \underline{\underline{C}}_3 : \underline{\underline{f}}_3 + \qquad (105)$$

,

again retaining only dominant terms in $\hat{\omega} << 1$. We find from (55) that $\tilde{f}^{1,c}$ does not contribute to (105) except for the term arising from coupling to the longitudinal perturbation $\tilde{f}$. To lowest order in $(1/\Omega)$ we may thus insert the collision-less solution $\tilde{f}^{1,0}$ in (105) for the perpendicular part. In the longitudinal part, however, we must retain all higher l terms and thus use the unexpanded form of (82) and (105).

$$<\bar{D}^{w\mu}_{ew} \frac{1}{w} \frac{\partial \bar{f}}{\partial \mu}> = - \frac{\tau_1 \langle r_{\parallel} \rangle_{ew}}{3} \{ w \nabla_{\parallel} + \left[a_{\parallel} + (r_{ei} + r_{ee})_{\parallel} \right] \frac{\partial}{\partial w} \} F$$

$$- \left\langle \frac{2(\bar{D}^{\mu w}_{ew})^2}{\bar{\nu}(1-\mu^2)w^2} \right\rangle \frac{\partial F}{\partial w} \quad (106)$$

The heating rate is

$$Q_{ew} = - \int d\underline{w} mw \left[<D^{ww}_{ew}> \frac{\partial F}{\partial w} + <D^{ww}_{ew} \cdot \frac{\partial \hat{f}}{\partial \underline{w}}> \right] = Q^{o}_{ew} + Q^{\perp}_{ew} + Q^{\parallel}_{ew} \quad (107)$$

Using the w^{-3} speed dependence of (103) and (64-66) we obtain the contribution (107)

$$Q^{o}_{ew} = (2\pi)^{1/2} a_{-3} \omega_e nT_e \frac{W}{nTe} \left\langle \frac{\omega_e}{kv_e} \left(\frac{\omega - \underline{k} \cdot \underline{u}_w}{kv_e} \right)^2 \right\rangle - (\underline{u} - \underline{u}_w) \cdot \underline{R}^{o}_{ew} \quad (108)$$

from F(w). We compare (105) with

$$\underline{R} = - \int d\underline{w} \frac{mw^2}{3} \left[\underline{r} \frac{1}{w} \frac{\partial F}{\partial w} + \underline{\underline{\nu}}_1 \cdot \underline{f}_1 + w^2 \underline{\underline{\nu}}_{13} : \underline{\underline{f}}_3 + .. \right] \quad (109)$$

and get

$$Q^{\perp}_{ew} = - (\underline{u} - \underline{u}_w) \cdot (\underline{R} - \underline{R}^{o})_{ew} \quad (110)$$

from the perturbation $\tilde{f}$ perpendicular to $\underline{B}$. The same relations are obtained from (93) and the conservation laws for momentum and energy (I. 67-68). From (106) we get by direct integration, noting $\underline{a} = (\nabla p - \underline{R})/nm$ and the speed dependence of the various terms,

$$Q_{ew}^{\parallel} = - \int d\underline{w}\ mw \left\langle \bar{D}_{ew}^{w\mu} \frac{1}{w} \frac{\partial \bar{f}}{\partial \mu} \right\rangle = \tau_1 \langle r_w \rangle_{ew} - 3nm \left\langle \frac{(\bar{D}_{ew}^{\mu w})^2}{\bar{D}^{\mu\mu}} \right\rangle^{*}, \qquad (111)$$

if e-e collisions are neglected in $\bar{f}$.

We can use the general relation between energy transfer in any two frames $\underline{u}$ and $\underline{u}_o$

$$Q = Q^{\Delta} - (\underline{u}-\underline{u}_o)\cdot\underline{R} \qquad (112)$$

and find that for $\underline{u}_o = \underline{u}_w$ the term Q_{ew}^{Δ} consists of the first term in (108) and a term due to the longitudinal perturbation $\bar{f}$,

$$Q_{ew}^{\Delta,\parallel} = Q_{ew}^{\parallel} + (u-u_w)_{\parallel} (R-R^o)_{ew}^{\parallel} \qquad (113)$$

which as follows from (111) and (79-80) vanishes for isotropic spectra.

It is evident from our earlier considerations, cf. also I, that e-i collisions can be included as a special case of fluctuations which are isotropic in the ion rest frame $\underline{u}_i$. For isotropic fluctuations we need to retain only the first term in (105). The e-e collision term $\langle C_{ee}(\hat{f})\hat{f}\rangle$ cannot be approximated in this form, however, since (105) was obtained by neglecting diffusion in w which is appropriate only for e-i and e-w collisions. The heating term due to e-i collisions is

$$Q_{ei} = 3n \frac{m}{M} \nu_{ei} \left(T_i - T_e \frac{a_{-1}}{a_{-3}}\right) - (\underline{u}-\underline{u}_i)\cdot\underline{R}_{ei} \qquad (114)$$

which for a Maxwellian F assumes its usual form[28]. The term proportional to T_e is due to the polarization force $\underline{A}_{ei}$. Energy conservation gives $Q_{ee} = 0$ but indirectly all scattering effects enter through the perturbed distribution $\hat{f} = \tilde{f} + \bar{f}$.

The kinetic equation (I. 35) for F(w) takes the form

$$\frac{\partial F}{\partial t} + \underline{u}\cdot\frac{\partial F}{\partial \underline{x}} - \frac{w}{3}\nabla\cdot\underline{u}\frac{\partial F}{\partial w} + \frac{\partial}{\partial \underline{x}}\cdot\frac{w^2}{3}\underline{f}_1 + \frac{1}{w^2}\frac{\partial}{\partial w}w^2\left[\frac{w}{3}\underline{a}\cdot\underline{f}_1 - \frac{2w^3}{15}\underline{\underline{U}}:\underline{\underline{f}}_2\right] = \langle Cf\rangle \quad (115)$$

Inserting the solutions for $\underline{f}_1$ and $\underline{\underline{f}}_2$ on the left hand side one obtains diffusion terms in ($\underline{x}$, w) which may be combined with the collision term <Cf>. We split the diffusion tensors into symmetric and antisymmetric parts and use the identities

$$\frac{\partial}{\partial x_i} D^{ik}\frac{\partial F}{\partial x_k} = -\frac{\partial D_a^{ik}}{\partial x_k}\frac{\partial F}{\partial x_i} + \frac{\partial}{\partial x_i} D_s^{ik}\frac{\partial F}{\partial x_k} \quad (116)$$

and

$$\nabla\cdot\left(\frac{\underline{e}_o}{\Omega}\times\underline{A}\right) = -\frac{\underline{e}_o}{\Omega}\cdot(\nabla\times\underline{A}) + \frac{\nabla\times\underline{B} + 2\underline{e}_o\times\nabla B}{B\,\Omega}\cdot\underline{A} \quad (117)$$

for the magnetic field terms, $\underline{e}_o = \underline{B}/B$.

Using the strong magnetic field limit of (38-39) for the perpendicular perturbation and (83-84) for the longitudinal part we obtain after some algebra finally

$$\frac{\partial F}{\partial t} + \{ \underline{u} + \langle \underline{r}_{\parallel} \rangle \tau_1 + \frac{\underline{e}_o}{\Omega} \times [\underline{a} + \underline{\underline{\nu}}_1 \cdot (\frac{\underline{e}_o}{\Omega} \times \underline{r})] - \frac{w^2}{3} \frac{\nabla \times \underline{B} + 2\underline{e}_o \times \nabla \underline{B}}{B\Omega} \} \cdot \frac{\partial F}{\partial \underline{x}}$$

$$- \nabla . \{ \underline{u} + \langle \underline{r}_{\parallel} \rangle \tau_1 + \frac{\underline{e}_o}{\Omega} \times [\underline{a} - \underline{\underline{\nu}}_1 \cdot (\frac{\underline{e}_o}{\Omega} \times \underline{r})] \} \; \frac{w}{3} \frac{\partial F}{\partial w} = C_{ee}(F) \; F +$$

$$(\frac{\partial}{\partial \underline{x}}, \frac{\partial}{\partial w}) \cdot \underline{\underline{D}} \cdot (\frac{\partial}{\partial \underline{x}}, \frac{\partial}{\partial w}) \quad F \qquad (118)$$

where

$$D^{ww} = \langle D^{ww} \rangle^{w} - \frac{1}{3}(\underline{u} - \underline{u}_w) \cdot \underline{r} - \frac{2}{3} \underline{r} \cdot (\frac{\underline{e}_o}{\Omega} \times \underline{a}) + \frac{1}{3} [\frac{\underline{e}_o}{\Omega} \times (\underline{a} - \underline{r}) \cdot \underline{\underline{\nu}}_1 \cdot$$

$$\frac{\underline{e}_o}{\Omega} \times (\underline{a} + \underline{r})] + \frac{1}{3} a_{\parallel}^2 \tau_1 - \langle \frac{(\underline{D}^{\mu w})^2}{\underline{D}^{\mu\mu}} \rangle + \frac{2w^2}{15} [\frac{\tau_2}{2} U_o^2 + \frac{3}{\Omega^2} (\underline{\underline{U}}_{1,\wedge} + \frac{1}{2} \underline{\underline{U}}_{2,\wedge})$$

$$: \underline{\underline{\nu}}_2 : (\underline{\underline{U}}_{1,\wedge} + \frac{1}{2} \underline{\underline{U}}_{2,\wedge}) - \frac{\tau_1}{\Omega} (\underline{\underline{U}}_{1,\wedge} + \frac{1}{2} \underline{\underline{U}}_{2,\wedge}) : \underline{\underline{\nu}}_2 : \underline{\underline{U}}_o]$$

$$D^{x_{\parallel} x_{\parallel}} = \frac{w^2}{3} \tau_1 \quad ; \quad D_s^{x_{\parallel} w} = \frac{w}{3} a_{\parallel} \tau_1$$

$$\underline{e}_i \cdot \underline{\underline{D}}_s^{\underline{x}_{\perp} \underline{x}_{\perp}} \cdot \underline{e}_k = \frac{w^2}{3\Omega^2} (\underline{e}_o \times \underline{e}_i) \cdot \underline{\underline{\nu}}_1 \cdot (\underline{e}_o \times \underline{e}_k)$$

$$\underline{D}_s^{\underline{x}_{\perp} w} = \frac{w}{3} \frac{\underline{e}_o}{\Omega} \times [\underline{r} - \underline{\underline{\nu}}_1 \cdot (\frac{\underline{e}_o}{\Omega} \times \underline{a})]$$

$\underline{a} = \frac{\nabla p - \underline{R}}{nm}$ in the frame $\underline{u} = \underline{u}_e$.

The difference in the signs of the collisional terms on the left hand side of (118) is not a misprint but comes from evaluating $(\partial/\partial \underline{w}) \cdot w \, \nu \underline{r}/3$, using the w^{-3} speed dependence. In the anisotropic distribution we have not included e-e collisions which have a different w dependence and have thus also omitted $\langle C_{ee}(\hat{f})\,\hat{f}\rangle$ on the right hand side. For turbulent fluctuations scattering in w is relatively slow compared to scattering in angle, thus $C_{ee}(F)\,F$ has been included. The noncollisional terms in the first square bracket of (118) will be recognized as the guiding center drift averaged over a distribution which is isotropic in frame $\underline{u}$. For isotropic turbulence one could of course retain the full magnetic field dependence $\hat{f}(\underline{w})$ which gives

$$\underline{D}^{x}_{s\perp}{}^{w} = \frac{w}{3}\,\frac{\nu_1}{\nu_1^2 + \Omega^2}\,\underline{a} + \frac{w}{3}\,\frac{\Omega^2}{\nu_1^2 + \Omega^2}\,\frac{\underline{e}_o}{\Omega} \times \underline{r} \qquad (119)$$

In $\underline{D}^{x}_{a}{}^{w}$, $\underline{a}$ and $\underline{r}$ are interchanged and the other terms are modified in a similar manner. In $\hat{f}$ we have also terms which arise from coupling of longitudinal and perpendicular perturbations by the anisotropy of the spectrum. They have been omitted for simplicity, although there is no basic difficulty in adding these terms. Usually, however, either perturbations along $\underline{B}$ or across $\underline{B}$ dominate. The methods of solving (118) have been discussed in I. For currents $\underline{u}_{\perp}$ across the magnetic field the dominant diffusion term has a w^{-3} dependence and thus leads to a selfsimilar distribution (1), s = 5. The increase in $\langle D^{ww}\rangle$ by a longitudinal drift $u_{\parallel}$, however, is essentially canceled by the term $\langle\!\langle (\bar{D}^{\mu w})^2/\bar{D}^{\mu\mu}\rangle\!\rangle$. For isotropic spectra this cancellation is complete. In addition, a current along $\underline{B}$ leads to the runaway term $a_{\parallel}^2\tau_1 \sim w^3$ discussed in I.

V. Applications and Conclusions

The relaxation of the energy distribution F(w) from a Maxwellian by quasilinear flattening and the resulting modification of the dispersion relation, energy and momentum transfer rates has been studied in a stochastic acceleration model[17] and in 2D simulation of ion sound turbulence excited by a current across the magnetic field[16]. They are found to be in excellent quantitative agreement with the theory presented in I. Although the magnetic field keeps the electron distribution essentially isotropic a small shift $\underline{v}_D(w)$ of the gyration centers with respect to the drift $\underline{u}$ was also observed. According to (71) this shift is due to the speed dependence of turbulent and collisional scattering. The small anisotropy of the distribution $f(\underline{w}) \approx F(|\underline{w} - \underline{v}_D(w)|)$ leads to a rotation with respect to $\underline{u}$ of the effective drift velocity (96) for wave growth and momentum transfer. The symmetry axis of the spectrum and the electric field $(e/m)\,\underline{E}_o = -(\underline{R}_e/nm) = \underline{\underline{\nu}}^* \cdot (\underline{u}_{eff} - \underline{u}_w)$ are observed to be rotated opposite the sense of electron gyration in excellent quantitative agreement[17] with the 2D version of (96), cf. Appendices B, C. This rotation of the spectrum, typically 20^o and independent of wavenumber is also observed in perpendicular shocks[6,7]. We see from (96) that in this case gradients in temperature and density lead to an additional term in $\underline{u}_{eff}$ which is in the same direction as the drift

$$\underline{u}_\perp = \frac{c\underline{E}x\underline{B}}{B^2} - v_e^2\,\frac{\underline{e}_o}{\Omega}\; x\; (\nabla \ln n + \nabla \ln T_e)$$

and comparable to it. In fact, for the plasma parameters measured in the current sheath of the Garching Belt Pinch[14] it follows that only this latter terms make ion sound instability over prolonged times possible. The effective drift is connected with a distortion of the distribution functions at low speeds. It follows from the discussion in I that for the diffusion coefficient $D^{ww} \sim w^{-3}$ the transport coefficients determining momentum transfer and effective drift should assume their values for the flat topped distribution (1), s = 5 well before there is substantial heating. The effective drift velocity (101) for wave growth due to drifts and gradients along the magnetic field or axis of symmetry depends on the angle of wave propagation if the spectrum responsible for diffusion is anisotropic. Simulation of current driven ion turbulence for B = 0 indeed shows peaking of the spectrum off axis due to the stronger quasi-linear effects in current direction[15]. There is also a substantial reduction of the effective drift, resistivity and heating by the runaway distortion while the bulk of the distribution remains fairly isotropic. These observations are in agreement with the theory although no detailed quantitative comparison has been carried out as in the case of a current across the magnetic field. Previous results on growth of waves $(\omega/kv_e)<<1$, $(kv_e/\Omega)>>1$ due to temperature gradients[8, 33] can be read off immediately as special cases from the classical transport relation for momentum transfer. Excitation of turbulence, however, leads to a substantial modification of this relation.

For the shock geometry $(\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0 \;;\; \underline{B} \parallel \underline{e}_z)$ we see from (64-69) that magnetic field penetration, Joule heating $(\underline{R}_e \cdot \underline{u})$ and heat flux (q_x) all depend on the same effective collision frequency (ν^*_{yy}), even for spectra which are strongly anisotropic, in agreement with experimental findings in the Garching Belt Pinch[4, 14], cf. Appendix C. The transport coefficients for perpendicular heat flow due to the drifts, temperature and density gradients depend again on lower order moments of the energy distribution, thus should rapidly assume their asymptotic values by quasilinear flattening of F(w). In a linear theta pinch axial heat flow is also important. The effective collision time (85) for this process depends on the electric field fluctuations along the magnetic field. Even if the spectrum is excited primarily by radial gradients, axial components still can be significant for ion sound waves, cf. Fig.4 of Ref. 15, but are negligible for the other modes discussed in connection with resistive shocks[13]. Direct excitation of ion sound turbulence by the axial heat flow has also been observed[35]. We note furthermore that quasilinear flattening leads to a very substantial reduction of the heat conductivity coefficients, Table I, and (inward) heat flux due to density gradients. These considerations give strong evidence for the dominance of ion sound turbulence during the implosion phase of the Garching theta pinch[4], as mentioned in Sect.I. For laser plasmas we conclude on comparing longitudinal and perpendicular transport that spontaneous generation of magnetic fields[36] should drastically modify not only transport[11] but wave growth ($\underline{u}_{eff}$) as well.

In high density, low temperature (stable) plasma regions, as in front of the magnetic piston of a theta pinch[20, 37], classical transport may be important. The magnetic field strength $\Omega\,\tau_{eff}$ may also vary considerably throughout the plasma. We conclude from Sects. II and III that it is not possible to simply add classical and anomalous transport since transport coefficients depend on the relative importance of electron-electron collisions and have a magnetic field dependence determined by the total (turbulent and Coulomb) effective collision time τ_{eff} for elastic scattering. As mentioned in Section III the complete magnetic field dependence may be determined from a truncated expansion in spherical harmonics if transport is predominantly across the magnetic field and anisotropic turbulence occurs mainly in the regions of strong magnetic field. An earlier hybrid code[18] has been modified by switching on anomalous transport relations with their proper magnetic field dependence which correspond to the flat topped distribution (1), s = 5, if an instability criterion based upon $\underline{u}_{eff}$, equ. (96) is satisfied. Classical transport is used elsewhere. The result is a substantially better agreement between hybrid code and pinch experiments, especially for the temperature profiles.[4, 14]

The major part of this work is restricted to a theory of the interaction of electrons with a given wave spectrum of which only a few integral characteristics like the effective collision frequencies are needed. These can be determined directly from the measured spectra as in the comparison between theory and simulation or experiments or they must be found from a wave kinetic equation which includes wave convection and generation. We have studied the excitation by the electron resonance and quenching by a (model) high energy ion tail [I] which as shown by simulation and experiments determine the evolution of ion sound in shocks and turbulent heating experiments. Electron relaxation and initial wave growth are very fast compared to ion tail build-up and the macroscopic time scales in a high density pinch. This may justify a local switch on-off condition $u_1 < u_{eff} < u_2$ in anomalous transport codes, where u_1 is determined by the initial (Maxwellian) ion distribution and u_2 by the high energy ion tail at later times and u_{eff} by the flattened electron distribution. In pinches the condition $u_{eff} \gtrsim u_2$ is indeed well satisfied at later times[14, 19, 38]. From heat flux limitation in the solar wind[9] and laser plasmas[10] one may deduce a similar condition.[39] Simulation experiments[15-17] show that ion sound is excited in a fairly wide cone ($\approx 50^\circ$) about $\underline{u}_{eff}$ and reaches levels of typically $(W/nT_e) \approx 10^{-2}$. Using effective collision frequencies corresponding to such spectra gives optimal agreement between code and experiments[4, 20]. Clearly, codes which follow the evolution of the spectrum from the thermal level[21, 22] are more desirable but as we have seen the use of self-consistent particle distributions is at least as important. We have presented an analytic theory for the electrons which can be used in hybrid codes. Such codes [4, 18, 20, 40] have been used most successfully for such complicated phenomena as ion reflection in shocks. The ion distribution in present day hybrid codes however does not account for tail formation since momentum[18, 20] and energy[40] transferred from the electrons by the waves are distributed equally among the ions and not primarily to high energy particles as required.

In summary, the theory of the interaction between electrons and ion sound and related spectra $(\omega/kv_e) \ll 1$, $(kv_e/\Omega_e) \gg 1$ presented in I has been extended to a complete transport theory which includes classical transport but differs from it in a number of important aspects. The electron fluid equations have the same formal structure as in the classical case. The transport relations, e.g. for momentum transfer and heat flux, which close this set of equations, however, are drastically altered in turbulent plasmas. The transport relations are obtained from a selfconsistent solution for the distribution function which is necessary due to the usually rapid transition to a non-Maxwellian distribution in plasmas that are no longer dominated by like particle collisions. The energy distribution F(w) is found from a kinetic equation which in addition to turbulent heating contains nonlocal speed dependent convection and diffusion terms. Our approach resembles quasi-linear theory rather than the Chapman-Enskog method[23, 28] in that the anisotropic part $\hat{f}(w)$ of the distribution is determined as a functional of F(w) linear in the perturbing forces and the kinetic equation for F has been completed (Sec. IV) by inserting the solution for $\hat{f}(\underline{w})$. For weak and strong magnetic fields the transport coefficients are expressed by form factors (moments) of F(w) which are normalized to unity for a Maxwellian. For anisotropic turbulent spectra the collision frequency is replaced by tensors which are obtained from spectral averages. The electron contribution $\varepsilon_e(\underline{k},\omega)$ to the dielectric constant also differs substantially from the conventional expression for drifting Maxwellians. The effective drift velocity for wave growth is obtained in a remarkably simple way from the transport relation for momentum transfer. We have discussed a number of applications to anomalous transport problems and their description by codes.

Some of the methods developed in this work should also be applicable to other turbulent spectra. We may conclude that the analysis of anomalous transport can and must go beyond an identification of relevant instabilities and perhaps some elementary transport model. Particularly important is a consideration of the speed and angle dependence of turbulent scattering and the self-consistent determination of particle distributions.

Acknowledgements

It is a pleasure to acknowledge many discussions with F. Soeldner about the Garching Pinch experiments. Helpful comments on the manuscript by M. Hellberg are very much appreciated.

"This work was performed under the terms of the agreement on association between the Max-Planck-Institut für Plasmaphysik and EURATOM".

Appendix A Symmetry Relations for the Transport Coefficients

The Onsager relation $\rho_T = \kappa_u$ between the thermal force and the drift related heat flux follows from general principles of irreversible thermodynamics. On this basis one would expect this relation to hold only near thermal equilibrium. We can show this more directly from the kinetic equation. The heat conductivity (20) may be written as

$$\kappa_{u,j} = - \frac{m}{nT_e \tau_e} \int d\underline{w} \frac{w^2}{3} \left(\frac{mw^2}{2T_e} - A\right) (\nu_1 - ij\Omega)^{-1} (1-\rho_{u,j} - \nu\tau_e) \frac{T_e}{m} \frac{1}{w} \frac{\partial F}{\partial w}, \quad \text{(A 1)}$$

where A is an arbitrary constant, expressing the condition $\langle \underline{w} \rangle = 0$ in the electron rest frame. Using (23) in (13) and (26) one obtains from the collision integral

$$\rho_{T,j} = - \frac{m}{nT_e \tau_e} \int d\underline{w} \frac{w^2}{3} (B - \nu\tau_e) (\nu_1 - ij\,\Omega)^{-1} \Big[\frac{3}{2} - (1+\rho_{T,j} - \frac{mw^2}{2\,T_e}) \frac{T_e}{m} \frac{1}{w} \frac{\partial}{\partial w} \Big] F \quad \text{(A 2)}$$

where B is also an arbitrary constant. We choose $A = 1 + \rho_{T,j} + \frac{3}{2}$ $B = 1 - \rho_{u,j}$ and see that $\rho_T = \kappa_u$ follows for the Maxwellian $- \frac{T_e}{m} \frac{1}{w} \frac{\partial F}{\partial w} = F$. If e-e collisions are included one must make use of the self adjointness of the collision operator to prove symmetry. The thermal force, actually, is not determined by (A2), an identity, but by the condition $\langle \underline{w} \rangle = 0$.

$$\rho_{T,j} = -\frac{m}{T_e C_j} \int d\underline{w} \frac{w^2}{3} (\nu_1 - ij\Omega)^{-1} \left[\frac{3}{2} - (1 - \frac{mw^2}{2T_e}) \frac{T_e}{m} \frac{1}{w} \frac{\partial}{\partial w}\right] F \quad , \tag{A 3}$$

where $$C_j = \int d\underline{w} \frac{w^2}{3} (\nu_1 - ij\,\Omega)^{-1} \frac{1}{w} \frac{\partial F}{\partial w} \quad . \tag{A 4}$$

The heat conductivity becomes, using (27)

$$\kappa_{T,j} = -\frac{m^2}{6nT_e^2\tau_e} \int d\underline{w}\, w^4 (\nu_1 - ij\Omega)^{-1} \left[\frac{3}{2} - (1 + \rho_{T,j} - \frac{mw^2}{2T_e}) \frac{T_e}{m} \frac{1}{w} \frac{\partial}{\partial w}\right] F \tag{A 5}$$

The transport connected with the density gradient is obtained from (23) in a similar manner

$$1 - \rho_{n,j} = -\frac{m}{T_e C_j} \int d\underline{w} \frac{w^2}{3} (\nu_1 - ij\,\Omega)^{-1} F \quad , \tag{A 6}$$

$$\kappa_{n,j} = -\frac{m^2}{\sigma n T_e^2 \tau_e} \int d\underline{w}\, w^4 (\nu_1 - ij\,\Omega)^{-1} \left[1 + (1 - \rho_{n,j}) \frac{T_e}{m} \frac{1}{w} \frac{\partial}{\partial w}\right] F \; . \tag{A 7}$$

Comparing (22) with (A6) we see that $\rho_n = \rho_u$ requires

$$\frac{\partial \ln F}{\partial w} = -\frac{w}{\nu(w)\,\tau_e} \frac{m}{T_e} \quad , \tag{A 8}$$

which for $\nu(w) \sim w^{-3}$ leads to (1), s = 5. For the same distribution we also obtain $\kappa_n = \kappa_u$ on comparing (A7) and (20).
For distributions (1) the transport coefficients can be expressed as a combination of integrals of the form

$$I_\beta(x) = \int_0^\infty dt e^{-t} t^\beta \frac{1 + ixt^{3/s}}{1 + x^2 t^{6/s}} \to \Gamma(\beta+1) \text{ for } x \sim \Omega\tau_e \to 0 \tag{A 9}$$

which have been evaluated numerically.

Appendix B Form factors of the energy distrubution

In addition to the form factors of F(w) already listed in IA we need the form factors listed below. They are normalized to unity for a Maxwellian and the numerical values are for the distribution (1), s = 5.

$$a_1 = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} < \frac{w}{v_e} > \qquad 1.0355 \qquad \text{(B 1)}$$

$$a_3 = \frac{1}{8}\left(\frac{\pi}{2}\right)^{1/2} < \left(\frac{w}{v_e}\right)^3 > \qquad 0.9121 \qquad \text{(B 2)}$$

$$a_4 = \frac{1}{15} < \left(\frac{w}{v_e}\right)^4 > \qquad 0.79278 \qquad \text{(B 3)}$$

$$a_5 = \frac{1}{48}\left(\frac{\pi}{2}\right)^{1/2} < \left(\frac{w}{v_e}\right)^5 > \qquad 0.6609 \qquad \text{(B 4)}$$

$$a_7 = \frac{1}{384}\left(\frac{\pi}{2}\right)^{1/2} < \left(\frac{w}{v_e}\right)^7 > \qquad 0.4131 \qquad \text{(B 5)}$$

The form factor

$$a_{-6} = - \frac{3\pi}{2n} v_e^6 \int d\underline{w} w^{-5} \frac{\partial F}{\partial w} \qquad \text{(B 6)}$$

enters $\bar{\rho}_{u,\wedge}$ and $\bar{\kappa}_{u,\perp}$ equ. (74-75) and Table II. For the distribution (1) we obtain

$$(a_{-6}/a_{-3}^2) = \Gamma(1+3/s)\ \Gamma(1-3/s)\ ;\ s \geq 3$$
$$= 1.982 \quad ; \quad s = 5\ . \qquad \text{(B 7)}$$

In two dimensions the relations (92-94) and $\bar{\rho}_{u,\Lambda} = (a_{-6}/a_{-3}^2)-1$ still hold but

$$a_{-6} = -\frac{4v_e^6}{\pi n} \int d\underline{w}_\perp \, w^{-5} \, \frac{\partial F}{\partial w_\perp} \quad , \qquad \text{(B 8)}$$

or for the 2D version of (1)

$$a_{-6}/a_{-3}^2 = \Gamma(1+2/s)\,\Gamma\,(1-4/s)\,/\,\Gamma^2\,(1-1/s) \quad ; \; s \geq 4 \qquad \text{(B 9)}$$

$$= \; 3.0052 \qquad ; \; s = 5 \; .$$

Appendix C Electron Transport in Perpendicular Shock and Anisotropic Spectra

We may assume quasi-neutrality $n = Zn_i$ and neglect the displacement current. In the slab geometry $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$, $\underline{B} \parallel \underline{e}_z$ we have then

$$u_{ex} = u_{ix} \ , \ j = - \frac{c}{4\pi} \frac{\partial B}{\partial x} = - n|e| \ (u_e - u_i)_y \qquad (C\ 1)$$

Magnetic field penetration is governed by

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x} (u_{ex} B) = \frac{\partial}{\partial x} \frac{c}{n|e|} (-R_{ey} + \frac{\partial \pi^e_{xy}}{\partial x}) \qquad (C\ 2)$$

using the electron equation of motion and neglecting electron inertia. For $\Omega\tau_{eff} \gg 1$ we get from (64-68) and (72)

$$- \frac{\underline{R}_e}{nm} = \underline{\underline{\nu}}^*_1 \cdot \underline{u}_{eff,o} - \underline{\underline{\nu}}^*_1 \cdot (\underline{e}_o \times \underline{\underline{\nu}}^* \cdot \underline{u}) \frac{\bar{\rho}_{u\Lambda}}{\Omega} = \underline{\underline{\nu}}^*_1 \cdot \underline{u}_{eff} \qquad (C\ 3)$$

where $\underline{u} = -\underline{j}/n|e|$, neglecting the difference between the wave and ion rest frame, $\underline{u}_w \approx \underline{u}_i$. The last term is orthogonal to $\underline{u} \parallel \underline{e}_y$ thus

$$- \frac{R_{ey}}{nm} = \nu^*_{yy} u_{eff,o} \ ; \qquad u_{eff,o} = u - \frac{v_e^2}{\Omega} (\bar{\rho}_{u\Lambda} \frac{\partial \ln n}{\partial x} - \bar{\rho}_{T\Lambda} \frac{\partial \ln T_e}{\partial x}) \qquad (C\ 4)$$

The various terms in $u_{eff,o}$ are usually of the same order as mentioned in Sect.V. The term due to the nonlocal change of the form factor a_{-1} has been omitted. The viscous term in (2) is reduced by $(r_L/L)^2 \ll 1$ from the resistive diffusion term, $r_L = (v_e/\Omega)$, $L = (\partial \ln B/\partial x)^{-1}$. The electron heat equation takes the form

$$\frac{3}{2}\, n \left[\frac{\partial}{\partial t} + u_{ex} \frac{\partial}{\partial x}\right] T_e + n\, T_e \frac{\partial u_{ex}}{\partial x} = - \frac{\partial q_{ex}}{\partial x} - Q_i^{\Delta} - \frac{R_{ey}\, j}{n|e|} \qquad (C\ 5)$$

,

where Q_i^{Δ} is the energy transfered to the ions and waves, cf. Sect. IV. and viscous dissipation $- \underline{\underline{\Pi}} : \underline{\underline{U}}$ has been neglected. The electron heat flux is from (67), (69)

$$q_{ex} = n\, T_e \frac{\nu^{*}_{yy}}{\Omega} \left[\bar{\kappa}_{u\Lambda} \frac{j}{n|e|} + \frac{v_e^2}{\Omega} \left(\bar{\kappa}_{n\perp} \frac{\partial \ln n}{\partial x} - \bar{\kappa}_{T\perp} \frac{\partial \ln T_e}{\partial x}\right)\right] \qquad (C\ 6)$$

,

where all terms are usually also of the same order, except for a Maxwellian for which the ∇n terms vanish. Magnetic field penetration, Joule heating and heat conduction depend thus on the same collision frequency ν^{*}_{yy}, even for strongly anisotropic spectra. The total effective drift velocity for wave growth $\underline{u}_{eff}$ is rotated with respect to $\underline{u}$ by

$$\text{tg}\ \Theta = \frac{\bar{\rho}_{u\Lambda}}{\Omega} \nu^{*}_{yy} \frac{u}{u_{eff,o} - (\bar{\rho}_{u\Lambda}/\Omega)\nu^{*}_{xy}\, u} \approx \frac{\bar{\rho}_{u\Lambda}}{\Omega} \nu^{*}_{yy} \frac{u}{u_{eff,o}} \qquad (C\ 7)$$

.

Since this angle varies in the course of wave growth one expects that $\underline{u}_{eff}$ coincides only approximately with the symmetry axis of the spectrum and that for larger ν^{*}/Ω the spectrum also becomes wider in angle about the symmetry axis. Computer simulation[16,17] of current driven ion sound ($u_{eff,o} = u$) confirms these conclusions. The spectrum and the electric field $\underline{E}_o = \underline{R}_e/n|e|$ necessary to maintain constant current were measured. Relation (C 4) is confirmed and the direction of $-\underline{R}_e$ coincides roughly with the symmetry axis of the spectrum and thus with $\underline{u}_{eff}$, i.e. $\underline{u}^{II}_{eff} \approx 0$ in the relations

$$- \frac{\underline{R}_e}{nm} = \nu_I^* \underline{u}^I_{eff} + \nu_{II}^* \underline{u}^{II}_{eff}$$

$$\underline{u}^I_{eff} = \underline{u}^I_{eff,o} + \frac{\rho_{u\Lambda}}{\Omega} \nu_{II}^* \underline{u}^{II} \qquad \text{(C 8)}$$

$$\underline{u}^{II}_{eff} = \underline{u}^{II}_{eff,o} - \frac{\rho_{u\Lambda}}{\Omega} \nu_I^* \underline{u}^I ,$$

using the principal axes of $\underline{\underline{\nu}}_1^*$. Relation (C 7) is also confirmed by varying the magnetic field.

The transport relations used here hold for anisotropic spectra but are restricted to $\Omega \tau_{eff} >> 1$. The complete magnetic field dependence has been derived only for isotropic fluctuations, cf. Sect. II. For $\Omega \tau_{eff} << 1$ one would expect e.g. that q_{ex} depends no longer on the y components of the spectrum but primarily on the x components. At the same time the x components of $\underline{u}_{eff}$ connected with the x gradients should become important. This behavior is reflected by the approximate transport relations derived from a truncated expansion in spherical harmonics. As mentioned at the end of Section III these equations are correct for isotropic spectra and become correct for anisotropic spectra as $\Omega \tau_{eff} >> 1$. From (38) and (48) we obtain, considering only the perpendicular components of $\underline{A} = \left[\nabla + \left(\frac{\nabla p - \underline{R}}{nm} + \underline{\dot{r}}\right) \frac{1}{w} \frac{\partial}{\partial w}\right] F$ and using the principal axes of $\underline{\underline{\nu}}_1^*$ as coordinate system,

$$f_1^I = - \frac{\nu_{II}^* A^I - \Omega A^{II}}{\nu_I^* \nu_{II}^* + \Omega^2} \quad ; \quad f_1^{II} = - \frac{\nu_I^* A^{II} + \Omega A^I}{\nu_I^* \nu_{II}^* + \Omega^2} . \qquad \text{(C 9)}$$

With the transformations $\hat{q} = (\alpha^{1/2} q^{I}, q^{II}/\alpha^{1/2})$ for $\underline{q}$ and $\underline{f}_1$ $\hat{R} = (R^{I}/\alpha^{1/2}, \alpha^{1/2} R^{II})$ for $\underline{R}$ and the perturbing gradients and drifts, $\alpha = (\nu_I^{*}/\nu_{II}^{*})^{1/2}$ one obtains the isotropic transport relations of Sect. II with $\nu^{*} = (\nu_I^{*} \nu_{II}^{*})^{1/2}$ replacing $1/\tau_e$ in the complete magnetic field dependence. The heat flux due to temperature gradients becomes

$$\underline{q}_T = - \frac{n T_e}{\nu^{*}} v_e^2 \left[\kappa_{T,\perp} \left(\frac{1}{\alpha} \nabla_I \ln T_e \underline{e}_I + \alpha \nabla_{II} \ln T_e \underline{e}_{II} \right) + \kappa_{T\Lambda} \underline{e}_o \times \nabla \ln T_e \right] . \qquad \text{(C 10)}$$

The same modification applies to the other terms in $\underline{q}$ and to the transport relation for the effective drift velocity for wave growth. Note that $-\underline{R} = -nm \underline{\underline{\nu}}_1^{*} \cdot \underline{u}_{eff}$ still holds and that

$\nu^{*} \alpha = \nu_I^{*}$, $\nu^{*}/\alpha = \nu_{II}^{*}$, $\nu_I^{*} \nu_{II}^{*} = \nu^{*2} = \nu_{xx}^{*} \nu_{yy}^{*} - (\nu_{xy}^{*})^2$.

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Table I Longitudinal transport coefficients in terms of the form factors of F (Appendix B) and numerical values for (1), s = 2,5

	F(w)	equ.(1), s=2	equ.(1), s=5
$\rho_{u,o}$	$1-(3\pi/32a_{-3}a_3)$	0.7055	0.2754
$\rho_{n,o}$	$1-a_5/a_3$	0	$\rho_{n,o} = \rho_{u,o}$
$\rho_{T,o}$	$(5\ a_5/2\ a_3)-1$	1.5000	0.8116
$\kappa_{u,o}$	$4(a_5/a_3)-5/2$	$\kappa_{u,o} = \rho_{T,o}$	0.3984
$\kappa_{n,o}$	$128(a_{-3}/3\pi)[(a_5^2/a_3) - a_7]$	0	$\kappa_{n,o}=\kappa_{u,o}$
$\kappa_{T,o}$	$128(a_{-3}/3\pi)[(7a_7/2)-(5a_5^2/2a_3)]$	13.581	1.5040
η_o	$512\ a_{-3}\ a_5/45\pi$	3.6220	1.0666

Table II Transport coefficients for $\Omega\tau_e \gg 1$ in terms of the form factors of F and numerical values for (1), s = 2,5; where

$$\kappa_{T,\Lambda} = \bar{\kappa}_{T,\Lambda}/\Omega\tau_e;\ \kappa_{T,\perp} = \bar{\kappa}_{T,\perp}/(\Omega\tau_e)^2 \text{ etc.}$$

	F(w)	(1), s=2	(1), s=5
$\bar{\rho}_{u,\Lambda}$	$(a_{-6}/a_{-3}^2)-1$	-	0.982
$\bar{\rho}_{n,\Lambda}$	$(a_{-1}/a_{-3})-1$	0	$\bar{\rho}_{n,\Lambda}=\bar{\rho}_{u,\Lambda}$
$\bar{\rho}_{T,\Lambda}$	$1 + (a_{-1}/2a_{-3})$	1.5	1.991
$\bar{\kappa}_{u,\Lambda}$	$(5/2)-(a_{-1}/a_{-3})$	$\bar{\kappa}_{u,\Lambda}=\bar{\rho}_{T,\Lambda}$	0.5180
$\bar{\kappa}_{n,\Lambda}$	$(5/2)\ (1-a_4)$	0	$\bar{\kappa}_{n,\Lambda}=\bar{\kappa}_{u,\Lambda}$
$\kappa_{T,\Lambda}$	$(5/2)\ (2a_4-1)$	2.5	1.46392
$\bar{\eta}_{1,\Lambda}$	2.0	2.0	2.0
$\bar{\kappa}_{u,\perp}$	$(a_{-1}/a_{-3}) - (3\pi a_{-4}/4a_{-3}^2)+(5/2)\bar{\rho}_{u,\Lambda}$	-	2.1131
$\bar{\kappa}_{n,\perp}$	$[(7/2)a_{-1}-a_1]\ (1/a_{-3})-(5/2)$	0	$\bar{\kappa}_{n,\perp}=\bar{\kappa}_{u,\perp}$
$\kappa_{T,\perp}$	$[a_1+(a_{-1}/2)]\ (1/2a_{-3})+(5/2)$	13.250	4.1569
$\bar{\eta}_{1,\perp}$	$12a_{-1}/5\ a_{-3}$	2.4	4.7568

Figure Captions

Fig. 1 Transport coefficients for momentum transfer

$\underline{R}_u = -(nm/\tau_e)\left[(1-\rho_{u,o})\underline{u}_{\parallel} + (1-\rho_{u,\perp})\underline{u}_{\perp} - \rho_{u,\Lambda}(\underline{e}_o \times \underline{u})\right]$

for Maxwellian (s = 2) and quasilinear distribution (s = 5).

Fig. 2 Transport coefficients for heat conduction

$\underline{q}_T = -(nT_e\tau_e/m)\left[\kappa_{T,o}\nabla_{\parallel}T_e + \kappa_{T,\perp}\nabla_{\perp}T_e + \kappa_{T,\Lambda}(\underline{e}_o \times \nabla T_e)\right]$

for Maxwellian (s = 2) and quasilinear distribution (s = 5)

Table I Longitudinal transport coefficients in terms of the form factors of F (Appendix B) and numerical values for (1), s = 2,5

Table II Transport coefficients for $\Omega\tau_e >> 1$ in terms of the form factors of F and numerical values for (1) s = 2,5; where $\kappa_{T,\Lambda} = \bar{\kappa}_{T,\Lambda}/\Omega\tau_e$; $\kappa_{T,\perp} = \bar{\kappa}_{T,\perp}/(\Omega\tau_e)^2$ etc.

Fig. 1

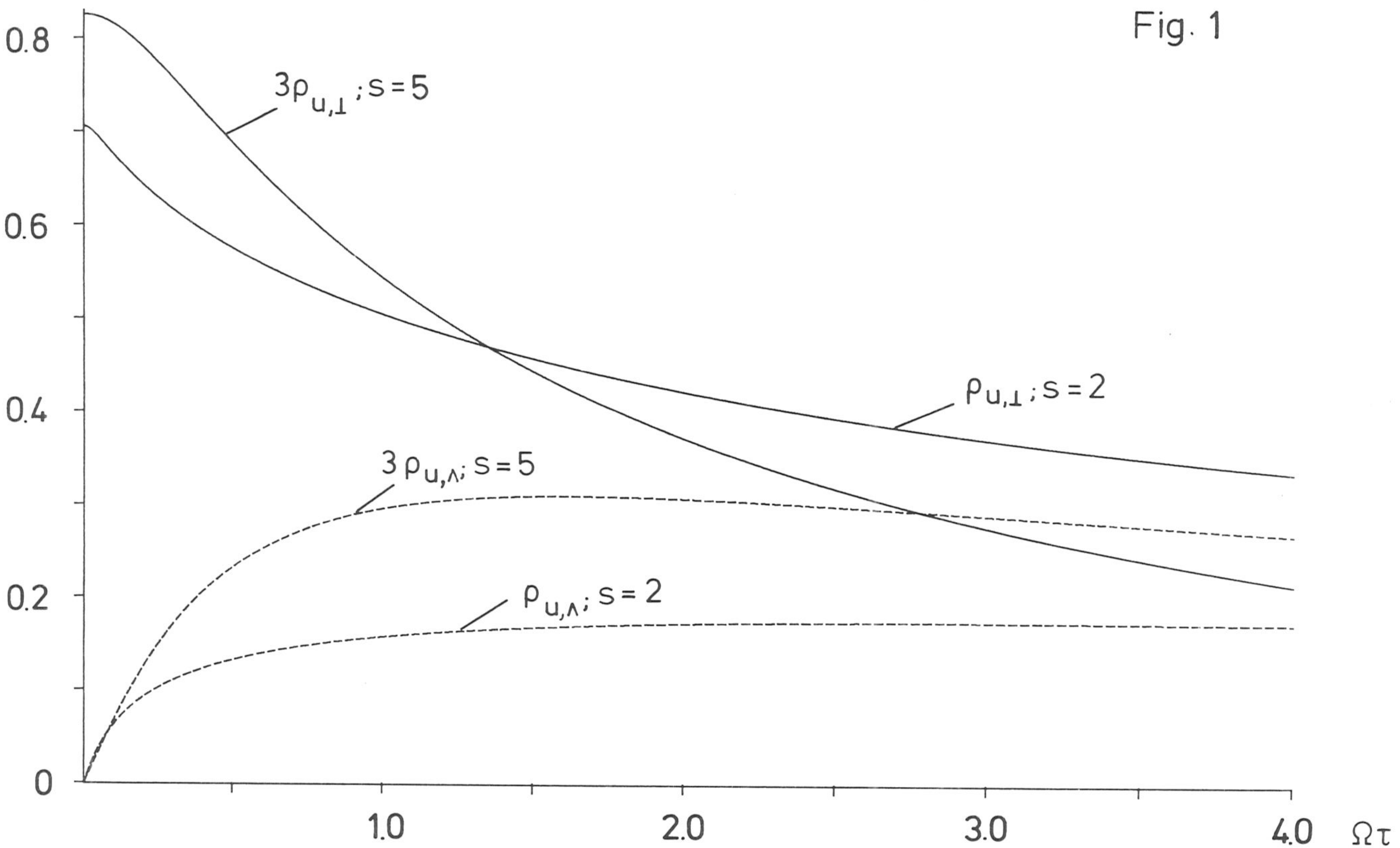

Fig. 2