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Inclusion of Electrostatic Trapping and Detrapping in
the Nonlinear Fluid Description of the Dissipative
Trapped-Ion Instability

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Abstract: The trapped-fluid equations of KADOMTSEV and POGUTSE (K.P.)
describing the dissipative trapped-ion instability are corrected nonlinearly to
include adiabatic, electrostatic detrapping of trapped particles and trapping
of circulating particles.   In the course of this derivation a new nonlinear
effect is found, viz. variation of the effective collision frequencies of the
trapped particles owing to the electrostatic redistribution of trapped and
untrapped particles.   We anticipate that the numerical solution of the new
equations might lead to a different saturation level and, hence, to a
modified anomalous transport coefficient as compared with the unaltered
K.P. equations.   The new fluid equations can also serve as a device for
inserting microscopic effects, e.g. Landau damping and finite-banana-width
effects.

1. Introduction

Numerical results on anomalous diffusion induced by the dissipative trapped-ion instability have already been obtained (SAISON and WIMMEL, 1976; SAISON, WIMMEL, and SARDEI, 1977). There, the trapped-fluid equations of KADOMTSEV and POGUTSE (1970, 1971) were solved numerically in two spatial dimensions as an initial-value problem in time. On the other hand, it was realized earlier that the K.P. equations neglect, among other things, electrostatic trapping and detrapping by the parallel electric field of the instability, and preliminary corrections, linear in the perturbation amplitude, had been given (WIMMEL, 1976).

The aim of this paper is to develop nonlinear trapped-fluid equations that include electrostatic trapping and detrapping in a quasistatic, nonlinear fashion. These equations are obtained from a simplified version of the drift-kinetic equation that holds in the usual slab model, with $\underset{\sim}{B}$ = const. Contrary to KADOMTSEV and POGUTSE (1970, 1971) and to later work by other authors, we take into account that the boundary in velocity space between trapped particles and circulating particles is modified by the parallel $\underset{\sim}{E}$ field of the instability. An especially interesting effect is the ensuing change of the effective collision frequencies of trapped particles with the time-varying electric potential.

Other microscopic effects, such as Landau damping and finite-banana-width effect, are not considered here. However, the new fluid equations can be used as a starting point for adding terms representing these effects. COHEN et al. (1976) have taken such microscopic effects into account but only on the basis of the original K.P. equations, thus neglecting electrostatic trapping and detrapping altogether.

The case in which the unperturbed equilibrium fraction δ_0 of trapped particles vanishes, viz. $\delta_0 = 0$, must be excluded from consideration throughout. The reason for this is that the effective collision frequencies of the trapped particles become divergent for $\delta_0 \to 0$. The original K.P. theory is subject to the same restriction.

The paper is organized as follows: Untrapped-particle densities and quasineutrality are treated in Secs. 2 and 3. Section 4 derives the corrected trapped-fluid equations, with the important correction terms evaluated in Secs. 5 and 6. A necessary transformation of the equations is carried out in Sec. 7, while the appropriate boundary conditions are given in Sec. 8. Section 9 proves ambipolarity of the anomalous diffusion. Section 10 deals with the equilibrium solution and the linear dispersion equation. Finally, Sec. 11 presents the conclusion.

2. Untrapped-Particle Densities with Electrostatic Trapping and Detrapping Included

As in the original K.P. theory, the $\underset{\sim}{E} \times \underset{\sim}{B}$ motion of the untrapped particles is neglected. This approximation should be good for mode-irrational surfaces (where $K_{\parallel} \neq 0$) and for a time scale long relative to relevant transit times of the untrapped particles. The relevant transit times are defined with respect to the unstable mode considered and are of the order of

$$\tau \sim \left(2\pi q R / v_{te} \, m_o \right) , \qquad (2.1)$$

where m_o is the poloidal mode number, q the safety factor, and R the large radius of the torus. It follows that $\omega\tau \ll 1$ is required, where ω is the mode (angular) frequency. The distribution functions of the untrapped particles can then be assumed to be approximately Maxwell-Boltzmann, i.e.

$$\hat{F}_j = e^{-\varphi_j - \xi} \cdot N_p \left(m_j / 2\pi T_j \right)^{3/2} , \qquad (2.2)$$

with $\varphi_j = q_j \Phi / T_j$, $\xi = m_j v_j^2 / 2 T_j$, $j = i, e$,

Φ = electric potential, N_p = equilibrium plasma density, T_j = temperatures (in energy units), and q_j, m_j, v_j = particle masses, charges, and velocities. The untrapped particle densities are then given by

$$N_j = \frac{2}{\sqrt{\pi}} N_p \, e^{-\varphi_j} \int_0^\infty d\xi \sqrt{\xi} \; e^{-\xi} \left(1 - \delta_j\right) , \qquad (2.3)$$

where

$$\delta_j \equiv \left| \frac{v_\parallel}{v} \right|_{crit} = \delta_j \left(\xi, \varphi_j, \delta_o \right) , \qquad (2.4)$$

i.e. δ_j is the value of $\left| v_\parallel / v \right|$ that separates trapped and untrapped particles in velocity space, and δ_o is the unperturbed equilibrium fraction of trapped particles, $\delta_o = n_o / N_p$. The electrostatic trapping and detrapping effects are represented by the deviation of δ_j from its unperturbed value δ_o.

If the effective potential energy, $q_j \Phi^* + \mu_j B^*$, with μ_j = magnetic moment, does not have a maximum larger than $\mu_j B_{max}$, where B_{max} is the maximum B-field along the magnetic line considered, Φ^*, B^* are the values of Φ and B anywhere along the magnetic line, and Φ, B are the values at the point of observation, i.e. where the relevant fluid quantities are derived, then conservation of energy and magnetic moment, with the assumption $\Phi = 0$ at $B = B_{max}$, yields

$$\delta_j = \begin{cases} 0 & \text{for } 0 \le \xi \le \xi_j , \\[2mm] \delta_o \left(1 - \dfrac{\xi_j}{\xi}\right)^{1/2} & \text{for } \xi \ge \max\left(\xi_j, -\varphi_j\right), \\[2mm] 1 & \text{for } 0 \le \xi \le -\varphi_j , \end{cases} \qquad (2.5)$$

with ξ_j defined by

$$\xi_j = \left[(1 - \delta_o^2) / \delta_o^2 \right] \varphi_j . \tag{2.6}$$

The assumption $\Phi = 0$ at $B = B_{max}$ agrees with linear, microscopic

theory (COPPI and REWOLDT, 1976). Whether this remains a good

approximation in the nonlinear regime could only be decided by

actually solving the nonlinear drift-kinetic equations together with

the condition of quasineutrality in a toroidal geometry, in order to

determine the true electric potential $\Phi(\underset{\sim}{x}, t)$. This is, obviously,

impossible at present, but could become feasible when bigger and

faster computers become available. (See also the discussion at the

end of this section.) On substituting eqs. (2.5) and (2.6) in (2.3),

the integral can be performed to yield

$$N_j = \begin{cases} N_p \left\{ \exp(-\varphi_j) - \delta_o \exp(-\varphi_j / \delta_o^2) \right\} \\ \qquad\qquad \text{for } \varphi_j \geq 0, \\ N_p \left\{ \exp(-\varphi_j) \, \text{erfc}(\sqrt{-\varphi_j}) \right. \\ \qquad\qquad \left. - \delta_o \exp(-\varphi_j / \delta_o^2) \, \text{erfc}(\sqrt{-\varphi_j} / \delta_o) \right\} \\ \qquad\qquad \text{for } \varphi_j \leq 0. \end{cases} \tag{2.7}$$

The normalized function $N_j(\varphi_j)$ is plotted in Fig. 1 for several

values of δ_o. Unlike a density proportional to a Boltzmann factor,

N_j <u>increases</u> with φ_j for $|\varphi_j|$ not too large. Such behavior

("anti-shielding") is also known from free particle streams in 1-D

electric fields with zero magnetic field. In <u>linear</u> approximation

N_j agrees with an expression given earlier (WIMMEL, 1976).

The values of N_j used by LAQUEY et al. (1975) and COHEN et al.

(1976), viz.

$$N_j^{K.P.} = N_p \left(1 - \delta_o \right) e^{-\varphi_j} \, , \tag{2.8}$$

would be recovered from eq. (2.3) if, rather than using eq. (2.5),

the <u>unperturbed</u> values $\delta_j = \delta_o$ were used, together with a Maxwell-

Boltzmann distribution. The <u>linearized</u> version of eq. (2.8) was

originally used by KADOMTSEV and POGUTSE (1970, 1971).

As mentioned above, to be valid eq. (2.5) requires that the effective

potentials, $V_j = q_j \Phi^* + \mu_j B^*$, with μ_j = magnetic moment, should

not have maxima larger than $\mu_j B_{max}$. Otherwise, additional particle

reflections may occur that are not taken into account in eq. (2.5).

Because this condition may be violated for small $(\mu_j B^*)$ and large

$\left(q_j \Phi^* \right) > 0$, the number of untrapped particles may be over-

estimated and that of trapped particles underestimated, especially for

large-amplitude perturbations $q_j \Phi^* > 0$. Again, this could only

be remedied by actually solving the complete nonlinear drift-kinetic

equations, including quasi-neutrality, for a toroidal geometry, thus

finding the true $\Phi (\underset{\sim}{x}, t)$. It is impossible to accomplish this at the

present time. Still, we expect that, by comparing the numerical results

obtained from this new theory and the old K.P. theory, one obtains an estimate of the importance of electrostatic trapping and detrapping effects for anomalous diffusion.

3. Quasineutrality

Knowing the untrapped-particle densities N_j from eq. (2.3), one can write down the quasi-neutrality condition, viz.

$$\rho =: n_i - n_e = N_e - N_i , \tag{3.1}$$

with n_j = trapped-particle densities, ρ = trapped-charge density (divided by e). Because the trapped-particle densities n_j are determined from trapped-fluid equations (see Sec. 4), eq. (3.1) is expected to determine the potential Φ. This is possible if the right-hand side is a monotonic function of Φ. If not, appropriate transformations of the fluid equations are necessary, and eq. (3.1) will determine one of the densities n_j rather than Φ (see Sec. 7).

For simplicity, we consider the special case $T_i = T_e = T$. From eq. (2.7) one then obtains

$$\rho = f_0 \left(\Phi , N_p , T , \delta_0 \right) \equiv g_0 \left(\varphi , N_p , \delta_0 \right) , \tag{3.2}$$

with

$$g_0 = -N_p \operatorname{sign}(\varphi) \left\{ \exp(-|\varphi|) - \exp(|\varphi|) \operatorname{erfc}(\sqrt{|\varphi|}) \right.$$
$$\left. - \delta_0 \left[\exp(-|\varphi|/\delta_0^2) - \exp(|\varphi|/\delta_0^2) \operatorname{erfc}(\sqrt{|\varphi|}/\delta_0) \right] \right\} , \tag{3.3}$$

where $\varphi = \varphi_i = -\varphi_e$. It turns out that, in general, $\rho(\varphi) = g_0(\varphi)$ is non-monotonic (Fig. 2). Hence φ is not

uniquely determined from ρ . Section 7 gives the transformations

of the fluid equations necessary for overcoming this situation.

In the K.P. approximation ($\delta_j \equiv \delta_o$) this situation did not arise.

There $\rho(\varphi)$ is monotonic:

$$\rho^{K.P.} = N_p \left(1 - \delta_o\right)\left[\exp\left(\varphi\right) - \exp\left(-\varphi\right)\right] \tag{3.4}$$

for $T_i = T_e$. Comparing eqs. (3.3) and (3.4) shows that for

sufficiently small $|\varphi|$ the two expressions for $\rho(\varphi)$ differ in sign

as well as absolute magnitude. In fact, $g_o(\varphi)$ decreases with

growing φ, unlike $\rho^{K.P.}(\varphi)$. This behavior is connected with

the "anti-shielding" of the untrapped particle streams mentioned

in Sec. 2. It agrees with results obtained earlier (WIMMEL, 1976).

4. Trapped-Fluid Equations with Electrostatic Trapping and Detrapping Included

We derive the new trapped-fluid equations for a 2-D slab geometry (coordinates x, y), with $\underset{\sim}{B} = B\hat{z}$, $B = $ const. In this case it suffices to start with the following simplified version of the 3-D drift-kinetic equations:

$$\frac{\partial F_j}{\partial t} + \left(v_{\parallel}\hat{z} + \underset{\sim}{v}_E\right)\cdot\nabla F_j + \frac{2q}{m}E_{\parallel}v_{\parallel}\frac{\partial F_j}{\partial W} = C\left(F_j, F_{j'}\right), \qquad (4.1)$$

with $F_j = F_j(t, \underset{\sim}{x}, W, \mu, \sigma)$, $j = i, e$, $W = v_j^2$, $\mu_i = $ magnetic moment $= $ const, $\sigma = $ sign$(v_{\parallel})$, $\underset{\sim}{v}_E = (c/B)\hat{z} \times \nabla\Phi$, the remaining notation being standard. Magnetic drifts and drifts involving $d\underset{\sim}{v}_E/dt$ are omitted, and $\underset{\sim}{v}_E$ is assumed to be small, $v_E \ll v_{th}$. Because of $\underset{\sim}{B} = $ const, conservation of μ is equivalent to conserving $v_{\perp}^2$.

The new trapped-fluid equations are obtained by integrating eq. (4.1) over that part of velocity space occupied by trapped particles, i.e. by applying the operator

$$\underset{\sim}{I} =: \sum_{\sigma} \frac{\pi}{2}\int_0^{\infty} dW \int_0^{\delta_j^2 + 0} d\kappa \, \left(W/\kappa\right)^{1/2}, \qquad (4.2)$$

with $\kappa = \left(v_{\parallel}/v\right)^2$. Note that, according to eq. (2.5), δ_j depends on W, Φ, and δ_o. As with the untrapped particles

(Sec. 2), here, too, the electrostatic trapping and detrapping effects are represented by the deviation of δ_j from its unperturbed value δ_0^j. By observing the relations

$$\underset{\sim}{I}\, F_j = n_j ,$$

(4.3)

with n_j = trapped-particle densities, and

$$\underset{\sim}{I}\, \frac{\partial F_j}{\partial t} = \frac{\partial}{\partial t}\left(\underset{\sim}{I}\, F_j\right) - \underset{\sim}{I}\left[F_j \frac{\partial \delta_j^2}{\partial t}\, \delta(\kappa - \delta_j^2)\right]$$

$$= \frac{\partial n_j}{\partial t} - 2\pi \int_0^\infty dW \sqrt{W}\, \frac{\partial \delta_j}{\partial t}\, F_j \bigg|_{\kappa = \delta_j^2}$$

(4.4)

with $\delta(x)$ = Dirac δ-function, as well as a relation similar to eq. (4.4), but involving ∇F_j instead of $\partial F_j /\partial t$, one obtains the trapped-fluid equations in the form:

$$\frac{\partial n_j}{\partial t} - \sum_{\nu=1}^3 \left(\frac{\partial n_j}{\partial t}\right)_\nu + \underset{\sim}{v}_E \cdot \left\{\nabla n_j - (\nabla n_j)_1\right\} = \hat{C}\,(n_j, n_{j'}),$$

(4.5)

with the following definitions:

$$\left(\frac{\partial n_j}{\partial t}\right)_1 = 2\pi \int_0^\infty dW \sqrt{W}\, \frac{\partial \delta_j}{\partial t}\, F_j \bigg|_{\kappa = \delta_j^2} ,$$

(4.6)

$$\left(\nabla n_j\right)_1 = 2\pi \int_0^\infty dW \sqrt{W}\, \nabla \delta_j\, F_j \bigg|_{\kappa = \delta_j^2} ,$$

(4.7)

$$\left(\frac{\partial n_j}{\partial t}\right)_2 = -\underset{\sim}{I}\left(v_\parallel \frac{\partial F_j}{\partial z}\right),$$

(4.8)

$$\left(\frac{\partial n_j}{\partial t}\right)_3 = - \underset{\sim}{I} \left(\frac{2 q_j}{m_j} v_{\parallel} E_{\parallel} \frac{\partial F_j}{\partial W}\right). \tag{4.9}$$

Like KADOMTSEV and POGUTSE (1970, 1971), we approximate the collision term by a relaxation term of the form

$$\hat{C}(n_j) = -\hat{\nu}_j (n_j - n_{jo}), \tag{4.10}$$

with $\hat{\nu}_j$ = effective collision frequency, n_{jo} = instantaneous quasi-equilibrium density of trapped particles. Unlike K.P., however, we evaluate $\hat{\nu}_j$ and n_{jo} by taking into account that δ_j deviates from δ_0, thus considering electrostatic trapping and detrapping of particles.

Approximate expressions for the right-hand sides of eqs. (4.6) to (4.9) can be obtained by assuming F_j to be Maxwell-Boltzmann distributions. It then follows that

$$\left(\frac{\partial n_j}{\partial t}\right)_2 + \left(\frac{\partial n_j}{\partial t}\right)_3 = 0, \tag{4.11}$$

and the remaining quantities can be expressed in closed analytic form, to be listed in Secs. 5 and 6. The trapped-fluid equations assume the simpler form

$$\frac{\partial n_j}{\partial t} - \left(\frac{\partial n_j}{\partial t}\right)_1 + \underset{\sim}{v}_E \cdot \left\{ \nabla n_j - (\nabla n_j)_1 \right\} = -\hat{\nu}_j (n_j - n_{jo}). \tag{4.12}$$

It is seen that the assumptions of slab geometry and Maxwell-Boltzmann distributions together have led to a form of the fluid equations that can

be investigated in two spatial dimensions rather than three. This will be done throughout the rest of this paper. These 2-D fluid equations must, of course, be supplemented by the quasi-neutrality condition, eq. (3.1), in order to obtain a closed set of equations.

Trapped ions and electrons have been treated alike by assuming Maxwell-Boltzmann distributions F_j . The Maxwell-Boltzmann distribution may, in fact, be a good approximation for the trapped electrons, with $\hat{\nu}_e > \omega$, but not necessarily for the trapped ions, with $\hat{\nu}_i < \omega$, if collisions are essential in order to restore the M.-B. distribution. However, cases are known in which collision-free particles conserve their M.-B. distribution while moving through a static potential field. Hence we expect that the approximation used may be fairly good for the trapped ions, too.

5. Evaluation of the Instantaneous Trapping-Detrapping Terms

By using for F_j the Maxwell-Boltzmann distributions $\hat{F}_j$ [eq. (2.2)],
eqs. (4.6) and (4.7) can be evaluated approximately. One obtains

$$\left(\frac{\partial n_j}{\partial t}\right)_1 = \frac{2}{\sqrt{\pi}} N_p \, e^{-\varphi_j} \int_0^\infty d\xi \sqrt{\xi} \, e^{-\xi} \frac{\partial \delta_j}{\partial t} \, , \qquad (5.1)$$

$$\left(\nabla n_j\right)_1 = \frac{2}{\sqrt{\pi}} N_p \, e^{-\varphi_j} \int_0^\infty d\xi \sqrt{\xi} \, e^{-\xi} \, \nabla \delta_j \, , \qquad (5.2)$$

with δ_j given by eq. (2.5). Evaluation of eq. (5.1) is straight-
forward and yields

$$\left(\frac{\partial n_j}{\partial t}\right)_1 = \begin{cases} -N_p \left[(1-\delta_o^2)/\delta_o\right] \exp\left(-\varphi_j/\delta_o^2\right) \frac{\partial \varphi_j}{\partial t} \, , \\ \qquad\qquad \text{for } \varphi_j \geq 0, \\ \\ -N_p \left[(1-\delta_o^2)/\delta_o\right] \exp\left(-\varphi_j/\delta_o^2\right) \, \text{erfc}\left(\sqrt{-\varphi_j}/\delta_o\right) \\ \qquad \cdot \frac{\partial \varphi_j}{\partial t} \, , \quad \text{for } \varphi_j \leq 0. \end{cases} \qquad (5.3)$$

While the evaluation of $\left(\partial n_j/\partial t\right)_1$ holds for the general case,
i.e. with $\nabla \delta_o \not\equiv 0$, and to all orders in φ , we give the
result for $\underset{\sim}{v_E} \cdot \left(\nabla n_j\right)_1$ for only two special cases. Firstly, for

$\nabla \delta_0 \not\equiv 0$ and for linear approximation in ϕ we obtain

$$\underset{\sim}{v}_E \cdot (\nabla n_j)_1 \approx N_p \underset{\sim}{v}_E \cdot \nabla \delta_0 , \qquad (5.4)$$

which agrees with an earlier result (WIMMEL, 1976). Secondly, for $\nabla \delta_0 \equiv 0$ the nonlinear result to all orders in ϕ for $(\nabla n_j)_1$ is obtained by substituting $\nabla \varphi_j$ for $\partial \varphi_j / \partial t$ on the r.h.s. of eq. (5.3). Taking into account that $\underset{\sim}{v}_E \cdot \nabla \phi = 0$, the final result is

$$\underset{\sim}{v}_E \cdot (\nabla n_j)_1 = \begin{cases} N_p \left[(1-\delta_0^2)/\delta_0\right] \varphi_j \exp\left(-\varphi_j/\delta_0^2\right) \\ \cdot \underset{\sim}{v}_E \cdot \nabla T_j / T_j , \text{ for } \varphi_j \geq 0, \\ \\ N_p \left[(1-\delta_0^2)/\delta_0\right] \varphi_j \exp\left(-\varphi_j/\delta_0^2\right) \\ \cdot \operatorname{erfc}\left(\sqrt{-\varphi_j}/\delta_0\right) \underset{\sim}{v}_E \cdot \nabla T_j / T_j , \text{ for } \varphi_j \leq 0. \end{cases} \qquad (5.5)$$

The general, nonlinear expression for $\underset{\sim}{v}_E \cdot (\nabla n_j)_1$, valid in the case $\nabla \delta_0 \not\equiv 0$, could also be derived, but is suppressed here for brevity. Again, $\underset{\sim}{v}_E = (c/B) \hat{z} \times \nabla \phi$.

In the K.P. theory the terms $(\partial n_j / \partial t)_1$ and $\underset{\sim}{v}_E \cdot (\nabla n_j)_1$ do not appear because it is assumed there $\delta_j = \delta_0$ and (tacitly) $\nabla \delta_0 \equiv 0$, That is, electrostatic trapping and detrapping effects are neglected, as well as $\nabla \delta_0$. Neglect of trapping/detrapping was also adopted by LAQUEY et al. (1975) and by COHEN et al. (1976).

6. Evaluation of the Collision Terms

We first deal with the instantaneous quasi-equilibrium densities n_{jo} . These densities are defined as those in equilibrium with the instantaneous electric potential Φ . Hence they derive from the instantaneous Maxwell-Boltzmann distributions $\hat{F}_j$ of eq. (2.2), viz.

$$n_{jo} = \frac{2}{\sqrt{\pi}} N_p \, e^{-\varphi_j} \int_0^\infty d\xi \, \sqrt{\xi} \; e^{-\xi} \, \delta_j \left(\xi, \varphi_j, \delta_o \right), \qquad (6.1)$$

a formula analogous to eq. (2.3). On substituting δ_j from eq. (2.5) the integration can again be performed to yield

$$n_{jo} = \begin{cases} N_p \, \delta_o \, \exp\left(-\varphi_j / \delta_o^2 \right), \quad \text{for } \varphi_j \geqslant 0, \\[2mm] N_p \left\{ \exp\left(-\varphi_j \right) \operatorname{erf}\left(\sqrt{-\varphi_j} \right) \right. \\[2mm] \left. + \delta_o \, \exp\left(-\varphi_j / \delta_o^2 \right) \operatorname{erfc}\left(\sqrt{-\varphi_j} / \delta_o \right) \right\}, \quad \text{for } \varphi_j \leq 0. \end{cases} \qquad (6.2)$$

Note that the <u>total</u> instantaneous quasi-equilibrium densities of ions or electrons (= sum of trapped and untrapped densities) obey a Boltzmann law, as they ought to:

$$\hat{N}_{pj} =: N_j + n_{jo} = N_p \, e^{-\varphi_j}. \qquad (6.3)$$

The normalized function $n_{jo}(\varphi_j)$ is plotted in Fig. 3 for several values of δ_o . Had one used the approximation $\delta_j = \delta_o$ in eq. (6.1),

then the following value of n_{jo},

$$n_{jo}^{K.P.} = N_p \delta_o \, e^{-\varphi_j},$$ (6.4)

would have resulted instead. This latter value, neglecting electro-static trapping and detrapping, was used by LAQUEY et al. (1975) and COHEN et al. (1976) in their fluid theories, while KADOMTSEV and POGUTSE (1970, 1971) use the zeroth-order approximation

$$n_{jo}^{K.P.} = N_p \delta_o \equiv n_o .$$

Turning now to the effective collision frequencies $\hat{\nu}_j$ of the trapped particles, we first note that the $\hat{\nu}_j$ are treated as constants in the K.P. theory. This is not a good approximation, however, because the effective collision frequencies considerably vary with the plasma perturbation. A more reasonable ansatz than the usual formula

$$\hat{\nu}_j = \nu_j^{COUL} / \delta_o^2 , \qquad \nu_j^{COUL} \quad \text{being the unperturbed}$$

frequency of $90°$ deflections, is the following:

$$\hat{\nu}_j = \left\langle \tilde{\nu}_j^{COUL} / \delta_j^2 \right\rangle ,$$ (6.5)

where $\tilde{\nu}_j^{COUL}$ is the perturbed $90°$ collision frequency and δ_o^2 has been replaced by the instantaneous quasi-equilibrium value δ_j^2. The pointed brackets designate the average over trapped particles in velocity space. In order to arrive at closed analytic expressions for eq. (6.5), we shall neglect the energy dependence of $\tilde{\nu}_j^{COUL}$,

but keep its proportionality to total density. Neglecting the energy

dependence is fully justified for like-particle collision frequencies,

i.e. for $\hat{\nu}_i$ and for ν_{ee}^{COUL} (see below). For ν_{ei}^{COUL} one has

instead a $\left(1/v_e^3\right)$ dependence, but because $\nu_{ei}^{COUL} \ll \nu_{ee}^{COUL}$

(see below) this dependence has been neglected. On approximating

the total density by eq. (6.3), we obtain the result:

$$\hat{\nu}_i \approx \nu_{ii}^{COUL} \, e^{-\varphi_i} \left< 1/\delta_i^2 \right> \tag{6.6}$$

and

$$\hat{\nu}_e \approx \left(\nu_{ei}^{COUL} \, e^{-\varphi_i} + \nu_{ee}^{COUL} \, e^{-\varphi_e} \right) \left< 1/\delta_e^2 \right> , \tag{6.7}$$

with $\nu_{ee}^{COUL} \approx 2\sqrt{2} \, \nu_{ei}^{COUL}$ according to ROSE and CLARK

(1961). The averages $\left< 1/\delta_j^2 \right>$ are again evaluated with the

Maxwell-Boltzmann distributions $\hat{F}_j$ of eq. (2.2). This gives

$$\left< \frac{1}{\delta_j^2} \right> = \frac{2 N_e}{\sqrt{\pi} \, n_{jo}} \, e^{-\varphi_j} \int\limits_{max(0, \, \xi_j)}^{\infty} d\xi \, \frac{\sqrt{\xi} \, e^{-\xi}}{\delta_j \left(\xi, \varphi_j, \delta_o \right)} \, . \tag{6.8}$$

Substituting δ_j from eq. (2.5) then yields

$$\left\langle \frac{1}{\delta_j^2} \right\rangle = \begin{cases} \dfrac{N_p}{\delta_o \, n_{jo}} \, exp\left(-\varphi_j / \delta_o^2\right)\left(1 + 2\xi_j\right), \\[2mm] \qquad\qquad\qquad for \ \varphi_j \geq 0, \\[3mm] \dfrac{N_p}{n_{jo}} \left\{ exp\left(-\varphi_j\right) \cdot erf\left(\sqrt{-\varphi_j}\right) \right. \\[2mm] \quad + \dfrac{2}{\sqrt{\pi}} \dfrac{1 - \delta_o^2}{\delta_o^2} \sqrt{-\varphi_j} \\[2mm] \quad \left. + \dfrac{1}{\delta_o} exp\left(-\varphi_j / \delta_o^2\right) erfc\left(\sqrt{-\varphi_j}/\delta_o\right)\left(1 + 2\xi_j\right) \right\}, \\[2mm] \qquad\qquad\qquad for \ \varphi_j \leq 0. \end{cases}$$

(6.9)

Figures 4 and 5 show plots of the normalized functions $\hat{\nu}_i(\varphi_i)$ and $\hat{\nu}_e(\varphi_e)$, the latter for $T_i = T_e = T$, i.e. with $\varphi_e = -\varphi_i = -\varphi$. The large values assumed by $\hat{\nu}_j$ for positive values of φ_j can be understood approximately by considering the φ_j—dependence of an average $\bar{\delta}_j$ (Fig. 6), defined by

$$\bar{\delta}_j(\varphi_j) = n_{jo} / (N_j + n_{jo})$$

(6.10)

and representing the quasi-equilibrium fraction of trapped particles given by n_{jo}. As mentioned in Sec. 2, the approximate evaluation of δ_j [eq. (2.5)] possibly provides for particle detrapping to be overestimated to some degree. As a consequence, the quantities n_{jo} and $\bar{\delta}_j$ may be underestimated, while the averages $\left\langle \delta_j^{-2} \right\rangle$ and the effective collision frequencies $\hat{\nu}_j$ may be overestimated.

7. Transformation of Equations

As shown in Sec. 3, $\varrho(\varphi)$ is, in general, non-monotonic. Hence, substituting the values of n_j in the quasi-neutrality condition, eq. (3.1), does not determine ϕ uniquely. This can be remedied by transforming the equations. In the new form, ϕ and n_ϱ will be the unknown functions, to be determined from the trapped-fluid equations, while n_i is determined from quasi-neutrality.

By forming the difference of the trapped-fluid equations, eq. (4.12), we arrive at a differential equation for the trapped-charge density ϱ. Putting $\varrho = f_o$ and using the relations

$$\frac{\partial f_o}{\partial t} = \frac{\partial f_o}{\partial \phi} \cdot \frac{\partial \phi}{\partial t}, \tag{7.1}$$

$$\nabla f_o = \frac{\partial f_o}{\partial \phi} \nabla \phi + \frac{\partial f_o}{\partial N_p} \nabla N_p + \frac{\partial f_o}{\partial T} \nabla T, \tag{7.2}$$

where, again, we consider the _special case_ of $T_i = T_e = T$, $\nabla d_o \equiv 0$, we arrive at the following transformed fluid equations:

$$f_2 \frac{\partial \phi}{\partial t} + \underset{\sim}{v}_E \cdot \left\{ \frac{f_o}{N_p} \nabla N_p - \frac{\phi f_2}{T} \nabla T \right\} =$$

$$\hat{\nu}_e \left(n_e - n_{eo} \right) - \hat{\nu}_i \left(n_e + f_o - n_{io} \right), \tag{7.3}$$

$$\left[\frac{\partial n_e}{\partial t} + \underset{\sim}{v}_E \cdot \nabla n_e \right] - \left[\left(\frac{\partial n_e}{\partial t}\right)_1 + \underset{\sim}{v}_E \cdot \left(\nabla n_e\right)_1 \right] =$$

$$- \hat{\nu}_e \left(n_e - n_{eo}\right), \tag{7.4}$$

and at quasi-neutrality in the form

$$n_i = n_e + f_0 , \tag{7.5}$$

with the definitions,

$$f_2 =: f_1 + \frac{\partial f_0}{\partial \phi} , \tag{7.6}$$

$$f_1 =: \left(\frac{\partial n_e}{\partial t}\right)_1 - \left(\frac{\partial n_i}{\partial t}\right)_1 , \tag{7.7}$$

with f_0 again being given by eqs. (3.2), (3.3). For $T_i = T_e = T$, f_2 is explicitly given by

$$f_2 = \frac{e N_p}{T} \left\{ \left[\exp\left(-|\varphi|\right) + \exp\left(|\varphi|\right) \, erfc\left(\sqrt{|\varphi|}\right)\right] \right.$$

$$\left. - \delta_0 \left[\exp\left(-|\varphi|/\delta_0^2\right) + \exp\left(|\varphi|/\delta_0^2\right) \, erfc\left(\sqrt{|\varphi|}/\delta_0\right)\right]\right\}. $$

$$\tag{7.8}$$

The function $f_2(\varphi)$ is positive-definite for $|\varphi| < \infty$, $0 \leq \delta_0 < 1$. This is so because both $y = e^x$ and $z = e^x \, erfc\left(\sqrt{x}\right)$ are monotonically decreasing functions for $0 \leq x < \infty$. Hence the expression

$$E =: y(x) - \delta_o\, y\left(x/\delta_o^2\right)$$

$$+ z(x) - \delta_o\, z\left(x/\delta_o^2\right) \qquad\qquad (7.9)$$

is positive for $0 \leq x < \infty$, $0 \leq \delta_o < 1$, which proves the assertion. The fact that $z(x)$ decreases monotonically follows from the inequality $z(x) < 1/\sqrt{\pi x}$ (for $0 \leq x < \infty$) and the differential equation $z' = z - 1/\sqrt{\pi x}$. From $f_2 > 0$ it follows that eq. (7.3) always uniquely determines $\partial\phi/\partial t$. Numerical values of $f_2(\varphi)$ are shown in Fig. 7.

8. Boundary Conditions

In earlier work (SAISON et al. 1977) appropriate boundary conditions
for the original K.P. equations were derived. In the simplest case,
the boundary conditions at $x = 0$ and $x = a$ were $n_i = n_e = n_o$
(zero perturbation locally) and they implied that $\mathcal{V}_x \equiv 0$ at the
boundaries $x = 0$ and $x = a$ of the slab model. Consequently, plasma
loss to the walls was forbidden. Trapped particle diffusion was never-
theless allowed, the collision terms providing for sources and sinks in
the plasma volume. In the y direction one had, of course, period-
icity with a period $b \sim a$, because y represents, essentially, the
"small angle" in a torus.

For consistency, we use the same boundary conditions here. This can
be done because these conditions satisfy the new fluid equations,
eq. (4.12) together with eq. (3.1), n_o having been assumed not to
depend on the coordinate y.

9. Ambipolarity of Anomalous Diffusion

Like the original K.P. equations (WIMMEL, 1976), the new theory
provides for ambipolarity of anomalous diffusion. This is easily
proved. The y average (y = ignorable coordinate) over the trapped
charge flux density in the x direction (x = direction of equilibrium
gradients) is given by

$$\dot{j}_x = \frac{1}{b} \int_0^b dy \, \rho \, v_E^x = - \frac{c}{Bb} \int_0^b dy \, \rho \, \frac{\partial \phi}{\partial y} \, . \tag{9.1}$$

Quasi-neutrality, eq. (3.2), provides for

$$\rho = f_0\left(\phi, N_p, T, \delta_0\right) = : \frac{\partial}{\partial \phi} F_0\left(\phi, N_p, T, \delta_0\right). \tag{9.2}$$

Then

$$\dot{j}_x = - \frac{c}{Bb} \int_0^b dy \, \frac{\partial F_0}{\partial y} = 0 \, , \tag{9.3}$$

because ϕ and F_0 are periodic in y with the period b. This
result hinges on the fact that the quantities N_p, T, and δ_0
are independent of y. The result is also valid for $T_i \neq T_e$.

10. Dispersion Equation

The trapped-fluid equations, eqs. (4.12), together with quasi-neutrality, eq. (3.1), have the equilibrium solution $n_j \equiv \delta_o N_P$, $\phi \equiv 0$, $N_j \equiv (1 - \delta_o) N_P$, $\underset{\sim}{v}_E \equiv 0$. Linearizing the equations with respect to the perturbations of the equilibrium yields the equations,

$$\frac{\partial n_j}{\partial t} - \left(\frac{\partial n_j}{\partial t}\right)_{1L} + \underset{\sim}{v}_E \cdot \left[\nabla n_o - \left(\nabla n_j\right)_{1L}\right] = -\nu_j \left(n_j - n_{joL}\right), \tag{10.1}$$

$$g =: n_i - n_e = f_{oL}, \tag{10.2}$$

with the abbreviations $n_o = \delta_o N_P$, $\nu_j = \hat{\nu}_j \ (\varphi_j = 0)$, and the index L indicating linearization in ϕ. Explicitly one has $\underset{\sim}{v}_E = (c/B) \hat{z} \times \nabla \phi$,

$$\left(\frac{\partial n_j}{\partial t}\right)_{1L} = - N_P \left[(1 - \delta_o^2)/\delta_o\right] \frac{\partial \varphi_j}{\partial t}, \tag{10.3}$$

$$\left(\nabla n_j\right)_{1L} = N_P \nabla \delta_o, \tag{10.4}$$

$$n_{joL} = n_o \left(1 - \frac{\varphi_j}{\delta_o^2}\right), \tag{10.5}$$

$$f_{oL} = - \frac{2 e N_P}{T} \cdot \frac{1 - \delta_o}{\delta_o} \phi. \tag{10.6}$$

On substituting $\partial/\partial t \to (-i\omega + \gamma)$, $\partial/\partial y \to i K_y$, the following dispersion equation is obtained:

$$(-i\omega + \gamma)^2 + \nu_1 (-i\omega + \gamma) - i\omega_o \nu_2 + \nu_3^2 = 0, \tag{10.7}$$

with the definitions

$$\nu_1 = (\nu_e + \nu_i)\left(1 - \frac{\delta_0}{2}\right) / (1 - \delta_0) , \tag{10.8}$$

$$\nu_2 = (\nu_e - \nu_i) , \tag{10.9}$$

$$\nu_3^2 = \nu_e \nu_i / (1 - \delta_0) , \tag{10.10}$$

$$\omega_0 = K_y \frac{c T \partial_x N_p}{2 e B N_p} \cdot \frac{\delta_0}{1 - \delta_0} . \tag{10.11}$$

For small values of $|K_y|$ one obtains the approximate result

$$-i\omega + \gamma \approx \left[(1 - \frac{\delta_0}{2}) / (1 - \delta_0)\right] \left(i\omega_0 + \frac{\omega_0^2}{\nu_e}\right)$$

$$- \nu_i / \left(1 - \frac{\delta_0}{2}\right) , \tag{10.12}$$

and

$$-i\omega + \gamma \approx - \left[(1 - \frac{\delta_0}{2}) / (1 - \delta_0)\right] \left(i\omega_0 + \frac{\omega_0^2}{\nu_e} + \nu_e\right). \tag{10.13}$$

On the other hand, for large values of $|K_y|$ one gets

$$-i\omega + \gamma \approx \pm \left(\frac{1}{2} |\omega_0| \nu_e\right)^{1/2} \left(1 + i \, \text{sign} \, \omega_0\right). \tag{10.14}$$

One would expect that using the equilibrium value δ_0 instead of δ_j in the derivation of the fluid equations would not modify the linearized fluid equations, but only the nonlinear terms.

That this is, in fact, so has been confirmed for $T_i = T_e = T$ by comparing the dispersion equations of the two cases, which turn out to be identical. On the other hand, the nonlinear terms differ. This is most easily seen by realizing that expanding the new fluid equations beyond the linear approximation also yields odd half powers of ϕ for $\varphi_j \leq 0$, contrary to the situation with the original K.P. equations, where only integer powers of ϕ appear.

11.   Conclusion

It has been shown that adiabatic, electrostatic particle trapping and detrapping modifies the original K.P. equations (KADOMTSEV, POGUTSE 1970, 1971) in several respects.   Firstly, the convective trapped fluid terms are supplemented by terms representing instantaneous redistribution of trapped and untrapped particles, as shown in Sec. 5. Secondly, the collision terms are modified by this effect (see Sec. 6). It is an important result that, in particular, the effective collision frequencies are varied considerably by nonlinear electrostatic trapping and detrapping (see Figs. 4 and 5), in addition to the variation of the quasi-equilibrium densities n_{j_0} of the trapped particles.   Thirdly, the untrapped particle densities are modified considerably so as to yield a quasi-neutral, trapped-charge density, $\rho = f_0 = g_0$, quite different from the K.P. expression (Secs. 2 and 3).   Except for the variation of the collision frequencies and the terms containing $\underset{\sim}{v}_E$, the other effects already appear in linear approximation.

The equations derived are approximate in several respects.   Trapping and detrapping have been assumed to be quasi-static in view of $\omega < \omega_{\mathcal{B}}$, and within this approximation detrapping is possibly over-estimated (see Sec. 2).   Stochastic versions of trapping and detrapping by $E_{\parallel}$ have not been used (see JABLON, 1972, and SMITH, 1977). The neglect of such effects is supported by EHST (1977) who claims

that the contribution of these stochastic effects to saturation is small.

As for further approximations employed, the use of the Maxwell-Boltzmann distribution for the trapped ions is probably not quite optimal (Secs. 4 and 5). Other microscopic effects, such as Landau damping and finite-banana-width effects, have not yet been inserted in the new fluid equations. These equations are thought, however, to be a better medium for inserting such effects than the original K.P. equations exclusively used in the literature hitherto.

Because several different terms of the trapped-fluid equations are affected by electrostatic trapping and detrapping, it is difficult to forecast the consequences of trapping-detrapping for anomalous transport. However, the following heuristic consideration is in order. Figures 4 and 5 show that the collision frequencies $\hat{\nu}_j$, when averaged over one characteristic oscillation period, increase with growing perturbation amplitude. On the other hand, increasing the unperturbed collision frequencies ν_j in the linear dispersion equation would lead to lower growth rates γ in the case of small $|K_y|$ $\left[\text{see eq.(10.11)}\right]$. These small $|K_y|$ modes are the ones considered most effective in anomalous transport. Hence it does not seem implausible to expect that electrostatic trapping and detrapping furnish an additional mechanism for saturation by nonlinear increase of the effective collision frequencies. Of course, the only secure way of detecting the consequences of trapping-detrapping is solving numerically the new fluid equations as an initial-value problem.

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Figure Captions

Fig. 1.   The normalized, circulating-particle densities $N_j(\varphi_j)/N_p$

for three values of $\hat{\delta}_0$.

Fig. 2.   The normalized, trapped-particle charge density $\rho(\varphi)/N_p$

for three values of $\hat{\delta}_0$.

Fig. 3.   The function $^{10}log\left\{n_{jo}(\varphi_j)/N_p\right\}$ for three

values of $\hat{\delta}_0$.

Fig. 4.   The normalized, effective trapped-ion collision frequency

$\hat{\nu}_i(\varphi_i)/\left[\hat{\nu}_i(0)\cdot\hat{\delta}_0^2\right]$ for three values of $\hat{\delta}_0$.

Fig. 5.   The normalized, effective trapped-electron collision

frequency $\hat{\nu}_e(\varphi_e)/\left[\hat{\nu}_e(0)\cdot\hat{\delta}_0^2\right]$ for three

values of $\hat{\delta}_0$.

Fig. 6.   The function $\overline{\hat{\delta}_j}(\varphi_j)$ for three values of $\hat{\delta}_0$.

Fig. 7.   The normalized function $f_2(\varphi)/\left[eN_p/T\right]$

for three values of $\hat{\delta}_0$.

Fig. 1

Fig. 2

$\log(n_{jo}/N_p)$

φ_j

$\delta_o = 0.5$

$\delta_o = 0.4$

$\delta_o = 0.3$

Fig. 3

$v_i(\varphi_i) / (\delta_o \, v_i(0))$

50.0

40.0

30.0

20.0

10.

$\delta_o = 0.3$

$\delta_o = 0.4$

$\delta_o = 0.5$
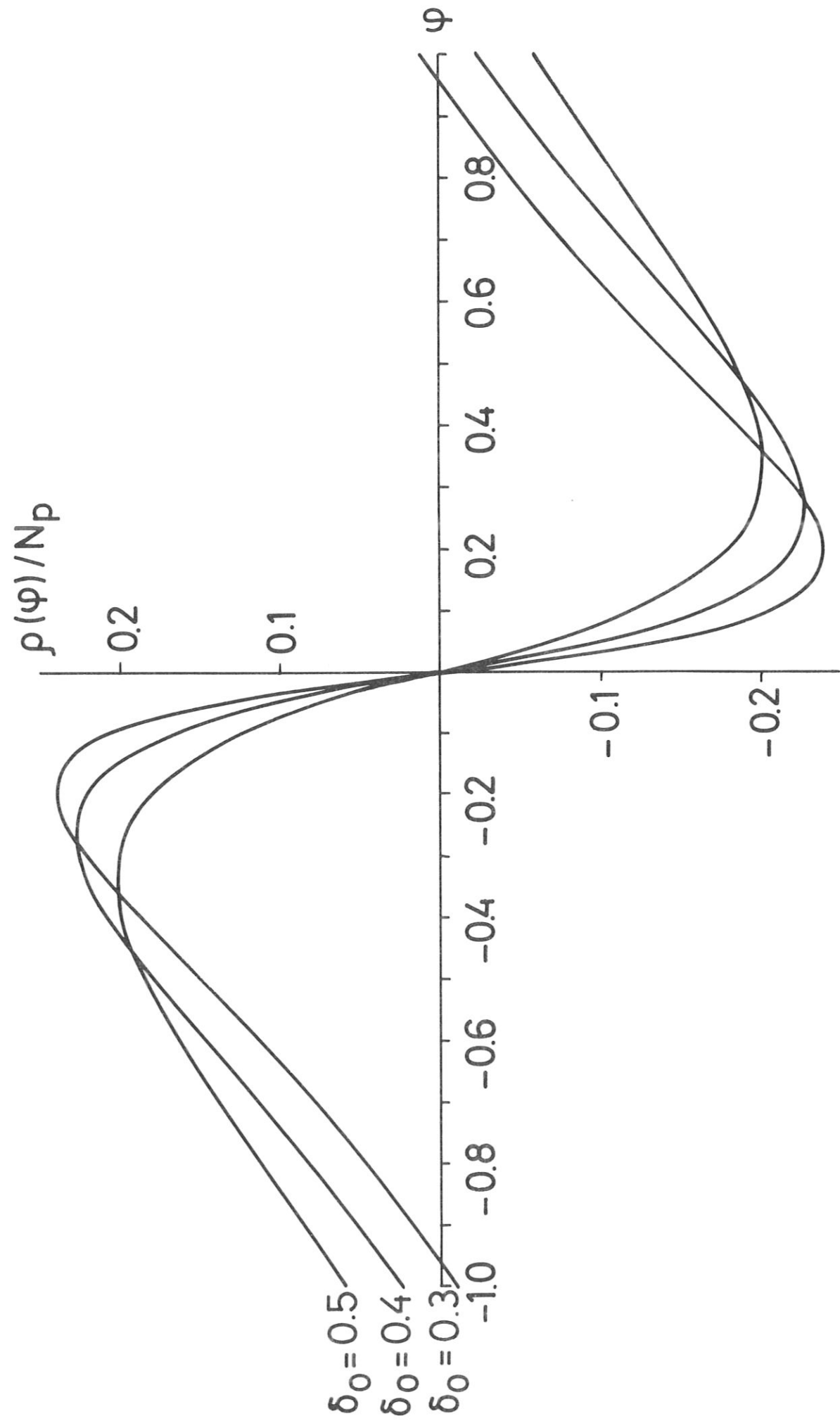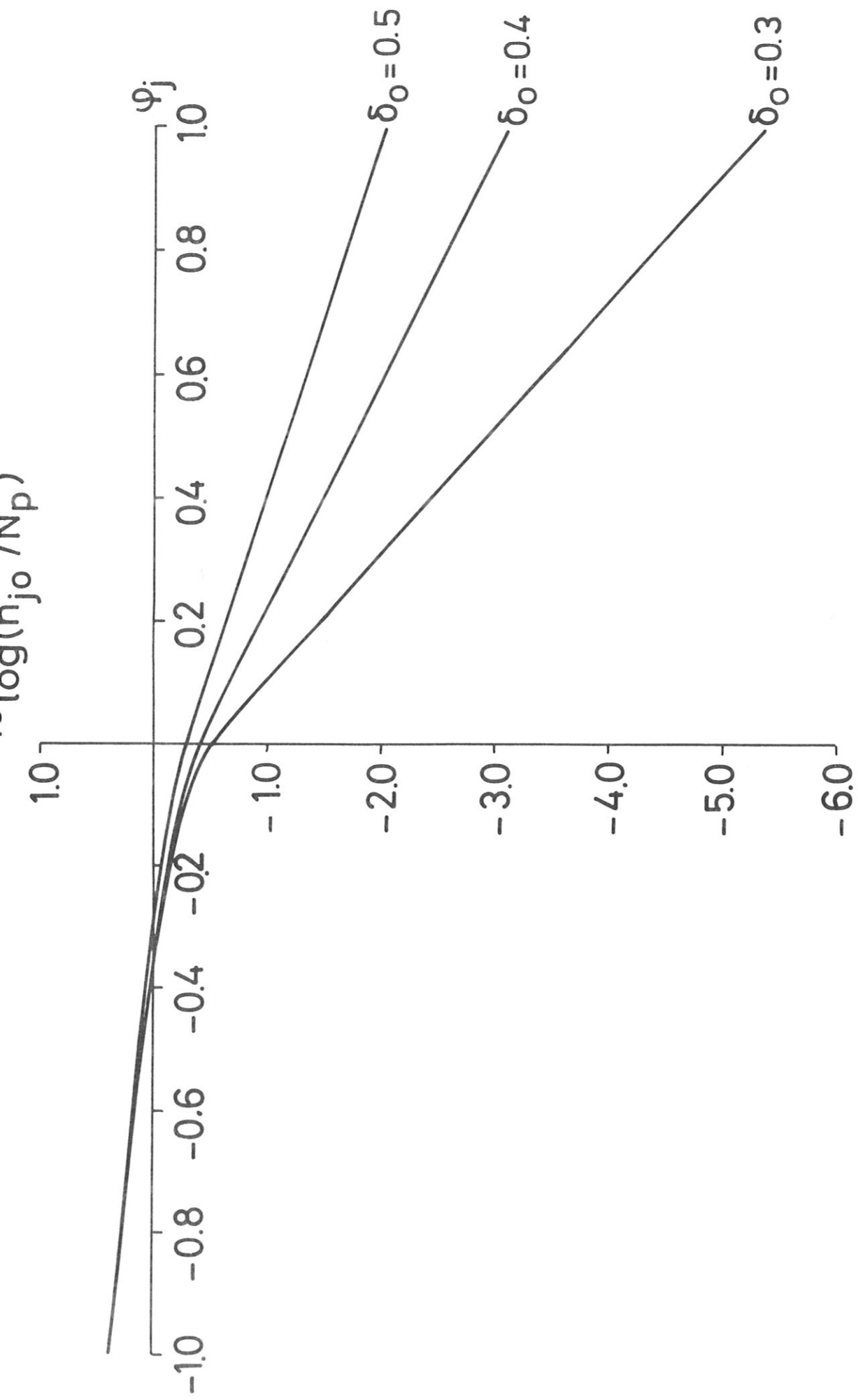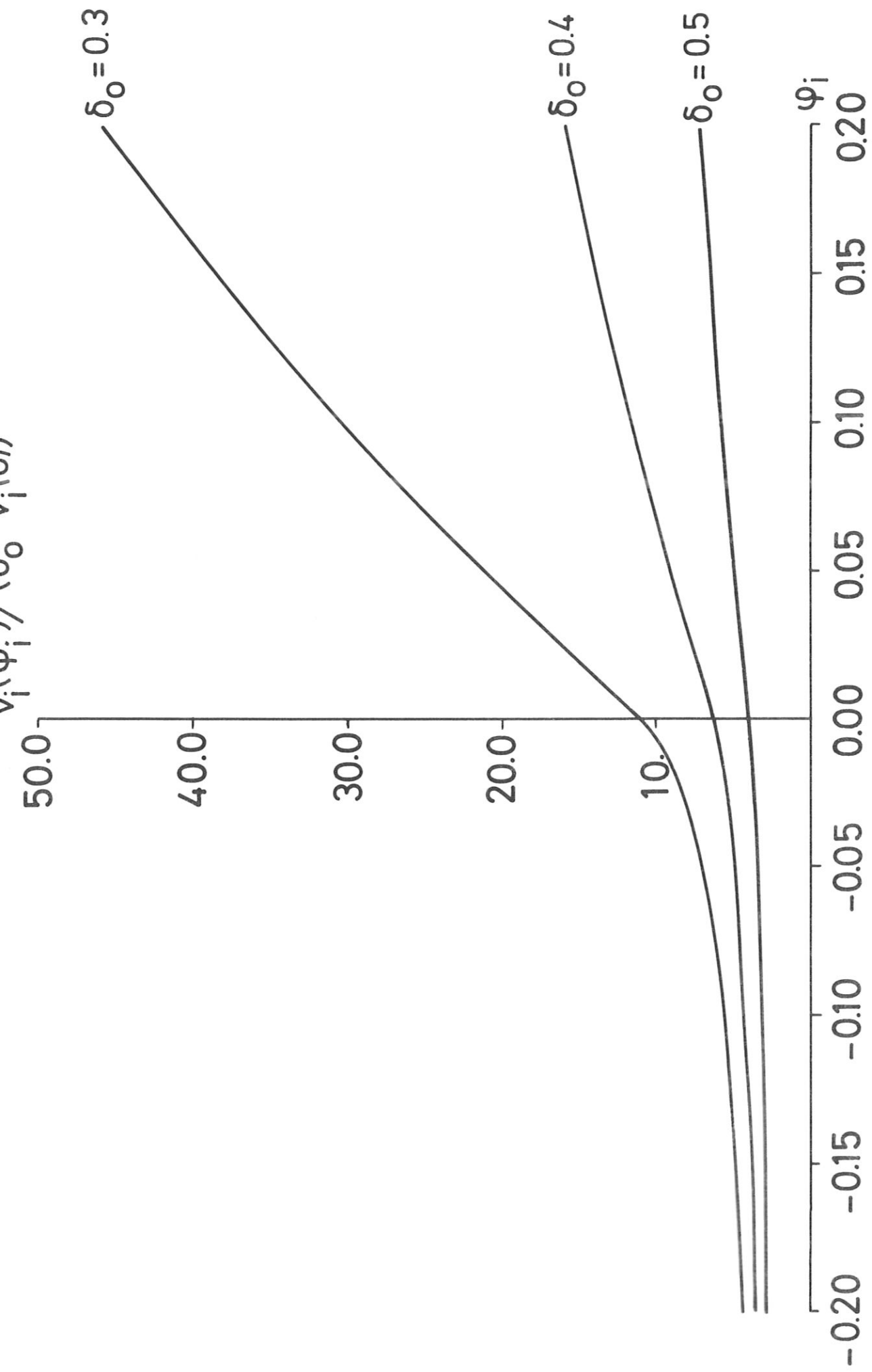
φ_i

-0.20   -0.15   -0.10   -0.05   0.00   0.05   0.10   0.15   0.20

Fig. 4

Fig. 5

Fig. 6

Fig. 7