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Comments Concerning a Numerical Model for Studying MHD Properties of Stellarators

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Abstract

The essential formulas for the asymptotic analysis of Stellarators developed at Princeton are assembled to facilitate studies of limitations on beta in Stellarators. Analogy with Tokamak studies is demonstrated. The connection between low-ß Stellarator and high-ß Scyllac work is shown.

I. Introduction

There are two possible ways to make analytical progress on Stellerator MHD studies. (1) Expansion in terms of the distance from the magnetic axis was first used by Mercier and his collaborators, and has been exploited recently by Shafranov and his colleagues and by Lortz and Nührenberg. This approach offers the only known way to analyze systems such as a Figure-Eight Stellarator where the magnetic axis has torsion. Its difficulty is in the imposition of the proper boundary conditions at the edge of the plasma. (2) The approach used by the Princeton group is to restrict consideration to systems that are periodic over a short period compared to the length of the magnetic axis and to use asymptotic expansion techniques to reduce the problem to one analogous to what is solved in Tokamak studies. It is the purpose of this note to comment on the essential equations that have been developed.[1-5] In this way I hope to show how current Tokamak studies could easily be extended to apply to Stellarators as well as to show how this low-ß work is related to the high-ß Scyllac studies. It may be useful to note that the effect of ripples (imposed on Tokamaks for "ripple trapping" when using perpendicular neutral injection) can be studied with the same formalism. In the next section I develop the equilibrium problem. In the following I consider stability. Then, finally, I make a few comments concerning work with high-ß. I have been sloppy with units so the formulae may be wrong by factors like μ_o, 4π, and possibly factors of 2 or $2\pi R$.

II. Equilibrium

We use a Stellarator expansion [1] where

$$\vec{B} = \vec{B}_o + \vec{B}_\delta + \vec{B}_\beta + \vec{B}_\kappa + \vec{B}_\sigma + \vec{B}_{\delta\delta} + \dots \qquad (1)$$

with

$$\left(\frac{B_\delta}{B_o}\right)^2 \sim \frac{B_\beta}{B_o} \sim \frac{B_\sigma}{B_o} \sim \frac{B_{\delta\delta}}{B_o} \sim \varepsilon \equiv \frac{a}{R} << 1 ,$$

i.e., the usual small inverse aspect ratio ordering employed for high-ß Tokamaks with an additional helical field. Here:

$\vec{B}_o$ is a uniform axial field.

$$\vec{B}_\delta = \nabla \sum_h \sum_\ell \frac{\varepsilon_{\ell,h}}{h} I_\ell (hr) \sin (\ell\theta - hz + \phi_{\ell,h})$$

$$\simeq \nabla \sum_h \sum_\ell \frac{\varepsilon_{\ell,h}}{2^\ell \cdot \ell !} h^{\ell-\ell} r^\ell \sin(\ell\Theta - hz + \phi_{\ell,h}) + \dots \qquad . \ (hr \sim \varepsilon \lesssim 1)$$

The different choices of h must all fit into the circumference of the torus. Since we assume that $h >> 2\pi R$ we can average over the largest of these fields and find that the components of $\vec{B}_\delta$ with different values of h do not interact. For a simple Stellarator one would only keep a single $\varepsilon_{\ell,h}$. $\vec{B}_\beta = \vec{e}_z B_\beta(\Psi)$, where Ψ is a magnetic surface label. This implies that the pressure is contained to lowest order entirely by the toroidal field, just as in high-ß Tokamaks where $\beta_p \simeq R/a$.

$\vec{B}_\kappa = \vec{\varepsilon}_z (r/R) \cos\Theta$ contains all the effects of toroidicity.

$\vec{B}_\sigma = \vec{B}_{\sigma\perp}$ (r,Θ) is the field perpendicular to z and independent of z. It is associated with axial currents in the plasma, both associated with Ohmic heating and with the Pfirsch-Schlüter currents, as well as with currents parallel to z outside the plasma, for example in Ioffe bars. Since it is perpendicular to z and independent of z, it can be represented by a stream function $\vec{A}_\sigma$,

$$\vec{B}_\sigma = - B_o \vec{e}_z \times \vec{\nabla} A_\sigma .$$

Then, the axial current in the plasma is, to lowest order,

$$J_\sigma = - B_o \nabla^2 A_\sigma . \quad (2)$$

$\vec{B}_{\delta\delta}$ contains that part of the second order field that varies with z over the short period.

We introduce an expansion in $\varepsilon^{1/2}$ for the surface label

$$\Psi = \Psi^{(o)} + \Psi^{(1)} \quad \Psi^{(2)} + \ldots$$

and solve the equation

$$\vec{B} \cdot \vec{\nabla} \Psi = 0 \quad (3)$$

order by order. Then, in zero order,

$$B_o \frac{\partial \Psi^{(o)}}{\partial z} = 0$$

or

$$\Psi^{(o)} = \Psi^{(o)}(r,\Theta).$$

In first order,

$$B_o \frac{\partial \Psi^{(1)}}{\partial z} + \vec{B}_\delta \cdot \vec{\nabla} \Psi^{(o)} = 0$$

or

$$\Psi^{(1)} = - \left[\int B_\delta dz \right] \cdot \nabla \Psi^{(o)} + \hat{\Psi}^{(1)}(r,\Theta) .$$

The last term is a constant of integration. The presence of this and similar terms in the higher orders makes it possible to have asymptotic convergence. In second order,

$$B_o \frac{\partial \Psi^{(2)}}{\partial z} + \vec{B}_\delta \cdot \vec{\nabla} \Psi^{1} + (\vec{B}_\beta + \vec{B}_\kappa + \vec{B}_\sigma + \vec{B}_{\delta\delta}) \cdot \vec{\nabla} \Psi^{(o)} = 0.$$

Since $\Psi^{(o)}$ is independent of z, the terms in $\vec{B}_\beta$ and $\vec{B}_\kappa$ do not contribute. The terms that vary with z determine $\Psi^{(2)}$ up to an arbitrary function of r and Θ. The solvability condition, that all the terms independent of z vanish, is a first order equation of the form $\vec{F} \cdot \vec{\nabla} \Psi^{(o)} = 0$, with the solution

$$\Psi^{(o)} = R B_o \left[\sum_h \sum_{\ell,m} \frac{m\varepsilon_{\ell,h} \cdot \varepsilon_{m,h}}{} I'_\ell(hr) I_m(hr) \cos\left[(\ell-m)\Theta + \phi_{\ell,h} - \phi_{m,h} \right] - A_\sigma \right] . \qquad (4)$$

Similarly, from the condition that $\vec{\nabla} \cdot \vec{J} = 0$, with $\vec{J}_\perp = \vec{B} \times \vec{\nabla} p / B^2$, we have

$$\vec{B} \cdot \vec{\nabla} \times (\vec{J} \cdot \vec{B}/B^2) = (\vec{B} \times \vec{\nabla} p \cdot \vec{\nabla} B^2)/B^4 . \qquad (5)$$

Going through the same steps that we did with Ψ, we find

$$J_\sigma = - 2\pi R \frac{dp}{d\Psi} \Omega(r,\Theta) + \hat{J}_\sigma(\Psi) , \qquad (6)$$

$$\Omega(r,\Theta) \equiv \sum_h \sum_{\ell,m} \frac{\varepsilon_{\ell,h} \varepsilon_{m,h}}{2} \left[I'_\ell (hr) I'_m (hr) + \frac{(\ell m + h^2 r^2}{h^2 r^2} I_\ell(hr) I_m(hr) \right]$$

$$\times \cos \left[(\ell-m)\Theta + \phi_{\ell,h} - \phi_{m,h} \right] - \frac{2r}{R} \cos \Theta . \qquad (7)$$

Note that the first term in Ω is $< B_\delta^2 >$, the magnitude of the helical field averaged over the periodicity length, and represents an average curvature associated with this field. The second term is the curvature arising from toroidicity. It was shown in reference 1 that Ω is essentially a measure of the length $d\ell/B$ of a line of force over this periodicity length, so that the criterion for a closed-line device to have equilibrium is that $\Omega = \Omega(\Psi)$.

The equilibrium problem then consists of solving Eqs. (2),(6), and (7). Note that if $B_\delta \to 0$, $\Omega \propto 2(r/R) \cos \Theta$, so that this system reduces to the well known Grad-Shafranov equation in the large aspect ratio limit. It would not be difficult to modify any numerical equilibrium code (such as Lackner's) to include the extra terms.

In reference 1 we showed how to write this set as an integral equation so that the boundary conditions are made manifest. Obviously, the boundary conditions are the same as those treated by standard Tokamak equilibrium codes.

III. Stability

The same expansion that was employed in the equilibrium analysis can be used to reduce the stability problem to one similar to that treated in Tokamak studies. This is done in reference 2, with the stability criterion for localized modes derived in reference 3. Some possibly useful expressions were derived earlier [4].

In lowest order we must set $\vec{\nabla} \cdot \vec{\xi}_\perp^{(o)} = 0$ to prevent stabilization associated with fast waves and $\partial \vec{\xi}_\perp^{(o)}/\partial z \sim \varepsilon$ to eliminate the stable short wavelength shear Alfvén waves. This is exactly the same as in Tokamak work. We can introduce a stream function for $\vec{\xi}_\perp^{(o)}$,

$$\vec{\varepsilon}_\perp^{(o)} = \vec{e}_z \times \nabla \zeta .$$

In the next significant order we find that $\vec{\nabla} \cdot \vec{\xi}_\perp^{(1)} = 0$ and that $B_o \, d \vec{\xi}_\perp^{(1)}/dz = - \vec{B}_\delta \cdot \nabla \vec{\xi}_\perp^{(o)}$. In the next order we pick $\xi_z^{(o)} = 0$ to remove the stabilization associated with the slow wave, adjust $\nabla \cdot \vec{\xi}_\perp^{(2)}$ to eliminate the fast wave, and reduce δW to

$$\delta W = \frac{1}{2} \int d\tau \left[\vec{Q}_\perp^{(2)\,2} + J_\sigma^{(2)} \times \vec{\varepsilon}_\perp^{(o)} \cdot Q_\perp^{(2)} - \vec{\xi}_\perp^{(o)} \cdot \nabla p^{(2)} \vec{\xi}_\perp^{(o)} \cdot \nabla \Omega^{\,2} \right] + \delta W_v, \quad (9)$$

with

$$\vec{Q}_\perp^{(2)} = \vec{e}_z \times \vec{\nabla} \left[(\vec{e}_z \times \vec{\nabla} \Psi^{(o)}) \cdot \vec{\nabla}\zeta - \vec{B}_o \partial\zeta/\partial z \right] \quad (10)$$

These equation are very similar to those treated with Tokamaks (see reference b, for example). The two differences are in the extra curvature term in Ω and the more complicated form of Ψ which determines the effective poloidal field in Eq. (10). It would be easy to modify the Princeton stability code PEST to include this since it has $\vec{\xi}$ projected in such a way that representation of $\vec{\xi}^{(o)}$ by a stream function is trivial. Modification of the Lausanne code ERATO should be possible but may be more difficult.

IV High-ß

One can get an understanding of many of the features of Scyllac the Los Alamos toroidal theta pinch, by looking at Eqs. (4), (6), and (7). If J_σ is to be small enough that B_σ is $O(\alpha/R)$, we must have either $p/B_o^2 \sim \alpha/R$ or $\Omega = \Omega\ (\Psi)$. This latter possibilty is the one utilized in the Meyer - Schmidt model and in standard Scyllac orderings. In particular, the standard Scyllac approach is to use a mixture of $\ell = o$ and $\ell = 1$ fields so that the $\cos\Theta$ part of the $\varepsilon_{\ell,h}\ \varepsilon_{m,h}$ term in Eq. (7) combines with the $(r/R)\cos\Theta$ term so that the formula in Eq. (4) is obtained. In a sharp surface model where $p'(\Psi) = 0$ except on one surface this can be done in a straightforward manner. It is actually accomplished by making a subsidiary expansion where r/R and $\varepsilon_o/\varepsilon_1$ are made small or by initially assuming them smaller).

Then

$$\Psi^{(o)} = \Psi^{(o)}_{(o)}\ (r) + \Psi^{(o)}_{(1)}\ (r)\ \cos\Theta + \ldots$$

with Eq. (2) reducing to an ordinary equation for $\Psi^{(o)}_{(1)}$ (see Ref. 5)

$$\frac{1}{r}\frac{d}{dr}\, r\, \frac{d\Psi_{(1)}}{dr} - \left[\frac{1}{r^2} - \frac{dp/dr}{d\Psi_{(o)}/dr}\ \frac{d\Omega_o/dr}{d\Psi_o/dr} + \frac{d\hat{J}_{\sigma(o)}/dr}{d\Psi_{(o)}/dr} \right] \Psi_{(1)}$$

$$= - \frac{2r dp/dr}{d\Psi_o/dr} \quad . \tag{11}$$

Numerical solutions of this equation indicate that as ß is increased the shift of the surfaces from being concentric grows as shown in Fig. 1. This is the familiar Shafranov shift associated with Pfirsch-Schlüter currents. At some critical ß, $ß_c$, the shift becomes infinite.

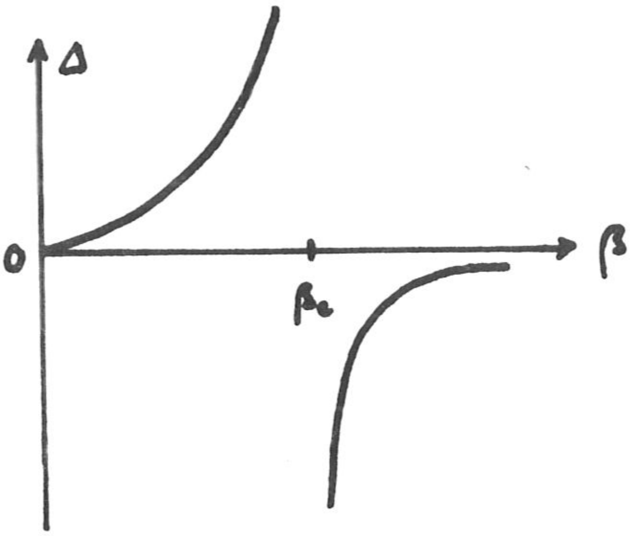

Fig. 1

For $ß > ß_c$ equilibria can be found, but with the magnetic axis shifted inward. It is interesting that the homogeneous part of Eq. (11) is just the Euler-Lagrange equation obtained from Eq. (9) for a straight cylincrical system [4] for a mode with $\vec{\xi}^{(o)}$ independent of z and varying as cosΘ. For $ß < ß_c$ this equation indicates stability with respect to such a mode; for $ß > ß_c$ it indicates instability. Thus one can visualize the Scyllac difficulties as being due to the choice of an unstable equilibrium with the inhomogeneous term in Eq. (11) serving to drive the instability.

It has been, and still is, my opinion that there are enough parameters available that one could get out this operating regime at the upper end of ß in Fig. 1 where the shift Δ makes the average curvature along a field line unfavorable and where the $m = 1$, $n = 0$ mode is so apparent. So far I have not succeeded in doing so. Indeed, no analytic progress and only partially successful numerical work on diffuse three dimensional equilibria of this type has been done, due to the need for keeping $\Omega = \Omega\,(\Psi)$. I think changing the externally imposed contribution to A_σ could lead to improvement. I think Lortz and Nührenberg would argue that this approach is limited by external separatices. Since the solution found by Garabedian and Betancourt utilizes a superposition of fields varying as $\sin \ell(\Theta - h z)$, I find it hard to believe these fields should interact to provide a positive effect.

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