LOCAL, IDEAL AND RESISTIVE STABILITY IN TOKAMAKS WITH NONCIRCULAR CROSS SECTION

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Abstract

Recently, the conditions for stability of an arbitrarily shaped, finite pressure toroidal plasma against localized ideal and resistive modes were presented. The characteristic time scale for local instability with respect to ideal modes is small compared with that for resistive modes so that an investigation of ideal local stability is a prerequisite for an assessment of nonideal local stability. Here we consider the stability of a particular class of noncircular cross-section tokamak equilibria with respect to such modes. The equilibria are described analytically so that no ordering or expansion procedure is necessary, and both ideal and resistive stability is investigated over the whole plasma region. The effects of plasma shape (vertical elongation and triangularity), as well as aspect-ratio are investigated and particular reference is made to next generation tokamak designs (e.g. JET).

INTRODUCTION

Recently, plasma stability /1/ with respect to resistive modes was extended to finite pressure, arbitrarily shaped, toroidal configurations. In particular the modifications due to resistivity for localized (interchange) modes were presented. These modes are more sensitive to the details of the magnetic field configuration than nonlocal ones and so occupy a principal place in the theory of stability for toroidal magnetic confinement schemes. Local ideal instabilities have an associated time scale which is small compared with that of the resistive ones, so that an investigation of stability with respect to localized modes requires first an assessment of stability with respect to the ideal modes. Only when local stability with respect to these modes exists is it reasonable to investigate the local stability with respect to resistive modes. It is clear that this is also true for nonlocal modes+). Here we restrict the discussion to stability with respect to localized modes.

Ideal local linear++) stability of an equilibrium is determined by the evaluation of criteria given in various forms by Mercier /3,4/, Greene and Johnson /5/ and Solovyov /6/,

+) The linear stability of ideal nonlocal modes can probably best be treated as an initial value problem /2/. This approach divides the plasma into small but finite regions and the spatial resolution is therefore limited. This means local modes cannot be treated.

++) The significance of instability with respect to such localized modes is not yet clear, and awaits a full nonlinear treatment.

and a considerable literature concerning their application to particular configurations exists (see for example /7/ and references therein). Nevertheless, in most cases this criterion has been applied only at the magnetic axis of the equilibria considered and this has given rise to the concept of a critical rotational transform or q value (on axis) for the stability of an equilibrium. This is so because it is argued that this stability criterion is most difficult to satisfy near the magnetic axis where the stabilizing effect of magnetic shear is usually small. Because shear is not the only stabilizing factor, such investigations do not allow the ideal local stability of the whole configuration to be reliably calculated, in particular, values given for the maximum allowable ß are certainly rough estimates only.

The stability of a large aspect-ratio circular cross section tokamak equilibrium with respect to resistive interchange modes has been investigated /8,9/. The model used results from the standard low beta tokamak ordering and is equivalent to an expansion about the magnetic axis for a large aspect-ratio tokamak. The numerical evaluations of local stability for a PDX-like equilibrium /10/ and various tokamak reactor designs /11/ have been performed. The local stability of Doublet configurations appropriate to Doublet IIA /12/ has been numerically analyzed.

Here we treat the local stability of a particular class of noncircular cross-section tokamak equilibria of finite aspect-ratio. For this class of exact equilibria, analytical expressions for important quantities can be obtained and their relative importance for stability assessed. These equilibria are characterized by a "flat" toroidal current density distribution (j_T = A.R + B/R, where A and B are constants and R is the major radius variable). The equilibrium magnetic surfaces are given by G(R,z) = constant, where z is the coordinate along the major axis. The particular solution discussed here is

$$G(R,z) = \alpha\left\{\left(\frac{R^2}{R_0^2}-1\right)^2 + \frac{1}{\mu}\cdot\frac{z^2}{R_0^2}\left(\frac{R^2}{R_0^2}-\gamma\right)\right\} \qquad (1)$$

where α, γ, μ and R_o are constants characterizing each equilibrium solution (Fig. 1).

This exact solution for the flux surfaces of the magnetic field removes the necessity of an expansion about the magnetic axis (R = R_o, z = 0) or in aspect-ratio. In addition it has been shown that these equilibria are self-consistent resistive equilibria /13,14,15/ in the sense that the pressure and current distributions are determined by appropriate mass and energy sources, consistent with plasma resistivity. Using the Mercier form of the local stability criterion /3/, the ideal local stability of a particular subset of these exact equilibria has already been discussed /16/.

Here we consider the effects on local stability of the variation of plasma cross-section shape and aspect-ratio, which can be accomplished by the appropriate choice of constants in expression (1). In particular the D-shaped cross-section of JET, a next generation tokamak, will be investigated and contrasted with other cross-section shapes.

The considerations of local stability in this paper are restricted to the particular choice of equilibrium model (1), so that the specific results may be non applicable to other equilibria. Indeed, one feature of our results is that the application of local stability results at the magnetic axis can be misleading in an assessment of the local stability of the whole configuration. For example

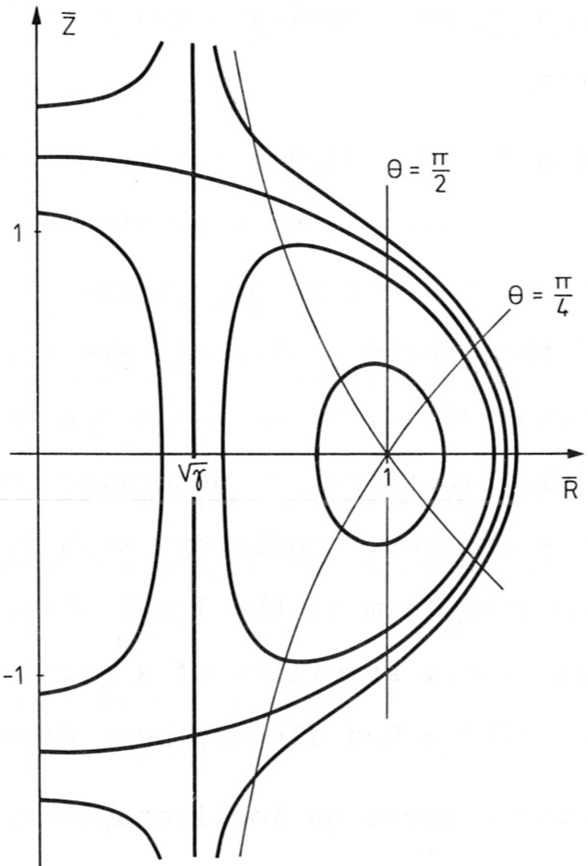

Fig. 1 Magnetic surfaces for particular values of the constants α, γ, μ and R_o ($\bar{R} = R/R_o$, $\bar{Z} = z/\sqrt{\mu}.R_o$). The lines Θ = const. result from $\bar{Z} = (\bar{R}^2-1).(\bar{R}^2-\gamma)^{-1/2}.\tan\Theta$.

some equilibria treated here exhibit instability with respect to local modes over a radial plasma region not containing the magnetic axis or the plasma boundary. Further, for a plasma stable on axis with respect to local ideal modes, the effects on local stability of finite resistivity vanish at the axis and therefore it is necessary that those cases which are stable to local ideal modes be investigated over the whole plasma region to test their stability to local resistive modes.

Already there exist many numerical codes which calculate plasma equilibria consistent with a prescribed set of external current-carrying conductors under various constraints (e.g. current density distribution and a spatial bound for the plasma) and it is clearly desirable to be able to test each solution for its macro-stability. For local stability this is conceptually straightforward as the stability criterion involves only a knowledge of some equilibrium quantities. Nevertheless, accuracy in the calculation of these quantities involving higher-order derivatives of magnetic surface functions (ie. those constant on a magnetic surface) is a problem, and calculations such as the one presented here, which remove such difficulties through exact knowledge of the equilibrium, provide an essential check on such codes. It is this, as well as the indication of qualitative features of the effect of plasma form on local stability together with specific results for JET which motivate the present work.

THE LOCAL STABILITY CRITERIA

We will use here the local stability criteria given in /1/ which are as follows

$$D_I \equiv E + F + H - \frac{1}{4} < 0 \tag{2}$$

for stability against local ideal modes, and

$$D_R \equiv D_I + (H - \tfrac{1}{2})^2 < 0 \tag{3}$$

for stability against resistive local modes, where (in MKS units)

$$E = \mu_o \frac{\langle \hat{B}^2 \rangle}{S^2} \left\{ \Omega + \mu_o p'^2 \langle \frac{1}{B^2} \rangle - S \frac{\langle j \cdot B \rangle}{\langle B^2 \rangle} \right\} \tag{4}$$

$$F = \mu_o^2 \frac{\langle \hat{B}^2 \rangle}{S^2} \left\{ \left\langle \frac{(\hat{j} \cdot \hat{B})^2}{\hat{B}^2} \right\rangle - \frac{\langle \hat{j} \cdot \hat{B} \rangle^2}{\langle \hat{B}^2 \rangle} \right\} \tag{5}$$

$$H = \mu_o \frac{\langle \hat{B}^2 \rangle}{S} \left\{ \frac{\langle j \cdot B \rangle}{\langle B^2 \rangle} - \frac{\langle \hat{j} \cdot \hat{B} \rangle}{\langle \hat{B}^2 \rangle} \right\} \tag{6}$$

The average $\langle A \rangle$ for any function $A(\underline{x})$ is defined as

$$\langle A \rangle \equiv \frac{d}{dV} \int_{V(\mathbf{x}) \leq V} A(\mathbf{x}) \, d^3x \tag{7}$$

V is the volume enclosed by a magnetic surface. Quantities with ^ are normalized by $|\nabla V|$, e.g. $\hat{\underline{B}} = B/|\nabla V|$. The magnetic field and current density are denoted (as usual) by $\underline{B}$ and $\underline{j}$ respectively.

The flux functions of the toroidal field is F(V), and the functions for both toroidal and poloidal current fluxes are I(V) and J(V) respectively. The plasma pressure p(V) is then related to the magnetic and current flux functions by the equilibrium relation

$$p' = J'F' - I'G' \tag{8}$$

The prime denotes derivatives with respect to V. Further

$$S = G'F'' - F'G'' \tag{9}$$

$$\Omega = J'F'' - I'G'' \tag{10}$$

The functions D_I and H can be written (using Hamada coordinates) as follows:

$$D_I = \frac{\mu_o p'}{S^2}\left\{(G'F')'W_2 + G''G'W_1 + F''F'W_3\right\} + \frac{(\mu_o p')^2}{S^2}(W_1 W_3 - W_2{}^2) - 1/4 \tag{11}$$

$$H = \frac{\mu_o p'(-G'\Lambda W_1 + (\mu_o IG' - F'\Lambda)W_2 + \mu_o IF'W_3)}{S(F'\Lambda + \mu_o G'I)} \tag{12}$$

where

$$\Lambda = 2\pi R_0 B_{T0} - \mu_o J = 2\pi R B_T \tag{13}$$

B_T (B_{TO} on axis) is the toroidal magnetic field.

W_1, W_2 and W_3 /17/ are surface averages

$$W_1 = \left\langle \frac{g_{\bar{\theta}\bar{\theta}}}{|\nabla V|^2} \right\rangle \tag{14}$$

$$W_2 = \left\langle \frac{g_{\bar{\theta}\bar{\xi}}}{|\nabla V|^2} \right\rangle \tag{15}$$

$$W_3 = \left\langle \frac{g_{\bar{\xi}\bar{\xi}}}{|\nabla V|^2} \right\rangle \tag{16}$$

and $g_{\bar{\theta}\bar{\theta}}$, $g_{\bar{\theta}\bar{\xi}}$ and $g_{\bar{\xi}\bar{\xi}}$ are elements of the metric tensor in Hamada coordinates $(V,\bar{\theta},\bar{\xi})$.

As shown in /17/, for axially symmetric equilibria $W_1 \sim W_2 = O(V^o)$ and $W_3 = O(V^{-1})$ as $V \rightarrow 0$. Thus, for reasonable distributions, $D_I = O(V^{-1})$ and $H = O(V^o)$ as $V \rightarrow 0$. A configuration which is ideally stable at the magnetic axis due to terms of the order V^{-1} is not unstable there with respect to local resistive modes. Stability with respect to local resistive modes must therefore be investigated throughout the whole plasma region.

For the particular class of equilibria given by (1), the fluxes F(V), G(V), currents I(V), J(V) and the geometric W-integrals can all be expressed in closed form and are given in the Appendix.

THE MODEL EQUILIBRIUM

The specification of a particular configuration as a preliminary to investigating its local stability proceeds as follows:

The outer boundary of the plasma is required to pass through four points in any (for reasons of axisymmetry) poloidal plane (fig.2). The points (R,z) are $(R_1,0)$, $(R_2,0)$ and $(R_M,\pm z_M)$. The latter points are chosen to be the points of maximum vertical plasma elongation. As G is an even function of z, there is up-down symmetry about the equatorial plane.

This requirement determines the constants in expression (1) as

$$R_0^2 = \frac{1}{2}(R_1^2 + R_2^2) \tag{17}$$

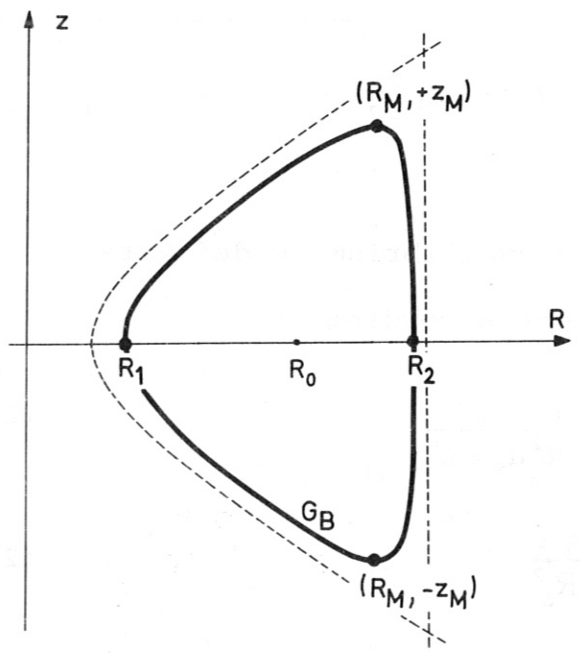

Fig.2 Geometry of the magnetic surface coinciding with the plasma boundary ($G = G_B$). (-- represents a separatrix)

$$\gamma = \frac{1}{2R_0^2} \frac{R_1^2 \cdot R_2^2 - R_M^4}{R_0^2 - R_M^2} \tag{18}$$

$$\mu = \frac{z_M^2}{2(R_0^2 - R_M^2)} \tag{19}$$

where R_o is the radius of the magnetic axis, and the poloidal magnetic flux at the boundary as

$$G_B = \alpha \left(\frac{R_2^2 - R_1^2}{R_1^2 + R_2^2} \right)^2 \tag{20}$$

If, in addition, we require the total toroidal plasma current to be I_B, then α is fixed too. Thus we have shown that a choice for R_1, R_2, R_M, z_M and I_B is equivalent to particular values for the four constants α, γ, μ, R_o and a boundary value G_B for G. The fact that G is solution of the equilibrium equation

$$\Delta^* G \equiv R^2 \mathrm{div} \frac{\nabla G}{R^2} = -\Lambda\dot{\Lambda} - 4\pi^2\mu_0 R^2 \dot{p} \tag{21}$$

$$\dot{p} \equiv \frac{dp}{dG} \qquad \dot{\Lambda} \equiv -\mu_0 \frac{dJ}{dG} \tag{22,23}$$

relates the parameters of the equilibrium to definite distributions of pressure and current according to

$$\dot{p} = -\frac{\alpha(1+4\mu)}{2\pi^2\mu_0\mu R_0^4} \tag{24}$$

$$\Lambda\dot{\Lambda} = \frac{2\gamma\alpha}{\mu R_0^2} \tag{25}$$

Because Λ itself enters the stability criteria, integration of equation (25) introduces B_{To} as further parameter

$$\Lambda^2 = \frac{4\alpha\gamma}{\mu R_0^2} G + 4\pi^2 R_0^2 \cdot B_{T0}^2 \tag{26}$$

instead of which we can also take q_o, as

$$q_0 = \left(\frac{4\mu}{1-\gamma}\right)^{1/2} \frac{\pi R_0^2 \cdot B_{T0}^2}{4\alpha} \tag{27}$$

the q-value on axis.

Summarizing, we can say that the equilibria under consideration are described by exactly six parameters and we have chosen them as R_1, R_2, R_M, z_M, I_B and B_{To} (or q_o).

LOCAL STABILITY ABOUT THE MAGNETIC AXIS

Although results later will show that the stability behaviour of the equilibrium in the neighbourhood of the magnetic axis (V = O) is, in general, not an index for the stability of the whole system, it is useful to investigate this behaviour to be able to retrieve the generally accepted results concerning the effects of magnetic surface shape on the local stability behaviour.

An expansion in V about V = O gives for D_I and H

$$D_I(V) = D_I^{(-1)} V^{-1} + D_I^{(0)} + D_I^{(1)} V + \cdots \tag{28}$$

$$H(V) = H^{(0)} + H^{(1)} V + \cdots \tag{29}$$

From the latter it is clear that for $D_I^{(-1)} \neq 0$, resistive effects on local stability in the neighbourhood of the axis are negligible. An expression for $D_I^{(-1)}$ will be given in the Appendix.

The zero of $D_I^{(-1)}$ defines a critical q on axis and is given by

$$q_c^2 = \frac{1}{4} \cdot \frac{m\beta^* - \gamma(1-\gamma)^{-1}}{1 + \frac{1}{2}(1-\gamma)^{-1} - m^{1/2}\beta^*(1+m^{1/2})^{-1}} \tag{30}$$

where

$$m = \frac{4\mu}{1-\gamma} \quad , \quad \beta^* = 1 + \frac{1}{4\mu} \tag{31,32}$$

When $q_o = q_c$ both ideal and resistive terms come in at the same order in V so that more detailed considerations are needed. For ideal local stability of the complete configuration it is necessary that $q_c^2 > 0$ and (at least) $q_o \geqslant q_c$.

RESULTS

For a particular aspect-ratio and major radius we examined the effects of varying the plasma ellipticity (or elongation) and triangularity on the local stability of the equilibria considered. The aspect-ratio A and the radius R_o of the magnetic axis (cf. (17)) are given as

$$A = \frac{R_2 + R_1}{R_2 - R_1} \quad , \quad R_0 = \left\{\frac{1}{2}(R_1^2 + R_2^2)\right\}^{1/2} \tag{33,34}$$

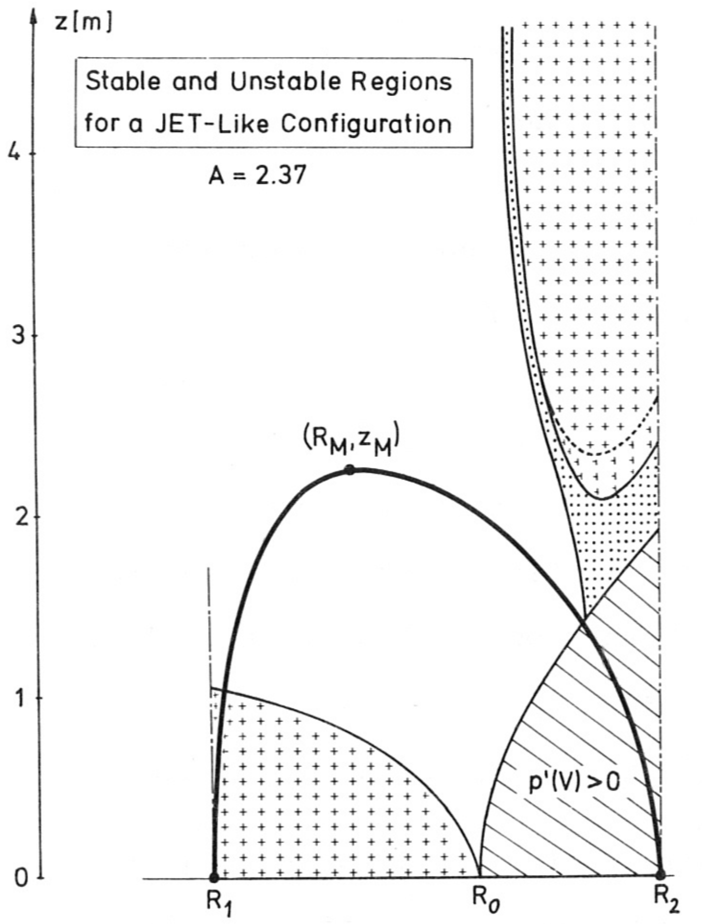

Fig. 3 Local stability diagram of equilibria of type (1) with plasma cross-sections determined by R_1, R_2, R_M and z_M.

+++ indicates regions where the equilibria are unstable to ideal modes (near the axis);

+++ indicates regions where the equilibria are unstable to local ideal modes at some radial point, for $q(0) = q_c$;

... indicates regions where the equilibria are unstable to local resistive modes for $q(0) = q_c$.

so that fixing A and R_o is equivalent to fixing R_1 and R_2, i.e. the radial extremities of the plasma configuration. The "ellipticity" E_B of the outer magnetic surface where $E_B = 2.z_M/(R_2-R_1)$, is varied by altering z_M, and the triangularity of the configuration depends essentially on R_M ($R_1 \leq R_M \leq R_2$). D-shaped configurations where the curved part of the D faces outward away from the major axis are characterized by $R_M/R_o < 1$ ($\mu > 0$, $\gamma < 1$).

The stability of this subset of configurations with respect to ideal local modes was investigated in /16/. D-shaped configurations with the curved part of the D facing inwards towards the major axis will be called anti-D shaped and are characterized by $R_M/R_o > 1$ ($\mu < 0$, $\gamma > 1$).

For fixed A and R_o, R_M and z_M were varied over the (R,z)-plane, and the local stability of each configuration obtained in this manner was examined. Results of such a procedure are shown, for example, in Fig. 3 where parameters corresponding to JET have been chosen, i.e. $R_1 = 1.71$ m, $R_2 = 4.21$ m or $A = 2.37$ and $R_o = 3.21$ m.

The (R,z)-region is divided up into several subregions:

(1) a region (hatched) where $p'(V) > 0$, i.e. such equilibria involve positive pressure gradients and will not be considered;

(2) regions (+++) where $D_I^{(-1)}$ is always positive, i.e. no real q_c value exists so that such configurations are unstable to ideal local modes in the neighborhood of the axis;

(3) a region (┼┼┼) where the configuration are ideally stable on axis for some value of q_o, but for $q_o = q_c$ are nevertheless unstable to local ideal modes in some radial region of the plasma (see Fig. 4);

(4) a region (...) in which the equilibria are completely stable to local ideal modes but are unstable to local resistive modes (see Fig. 5);

(5) a region (blank) in which the equilibria are completely stable to both local ideal and resistive modes.

Note that in Fig. 4(a) the radial instability region is not about the axis or at the edge (as in Fig. 4(b)) but in an internal region.

The effect on stability of increasing the toroidal magnetic field is shown. Reasonable increases in the toroidal field do not lead to complete stabilization and essentially influence only some neighborhood of the magnetic axis.

In Fig. 5 several different cases of instability with respect to local resistive modes are shown. All these cases are completely stable to local ideal modes. The effect on stability of increasing the toroidal magnetic field is shown. As in the case of local ideal modes, the stabilizing influence of increased toroidal field extends not far enough from the magnetic axis to achieve stability of the whole plasma region.

The effect of aspect-ratio on the stability properties of the model equilibria is shown in Figs. 6(a) and 6(b). It can be seen that D-shaped configurations above a certain elongation are always stable to local modes. Anti-D shaped configurations have a smaller unstable region in elongation -triangularity space as the aspect-ratio is reduced.

CONCLUSIONS

For the particular class of exact, resistively self-consistent, toroidal equilibria considered here we have shown:

(1) Stability with respect to both local ideal and resistive modes should be investigated not just at the magnetic axis but throughout the plasma configuration, as the former procedure can be misleading.

(2) In general a reasonable increase of toroidal field is of limited effect on the overall linear stability of the configuration.

(3) D-shaped configurations are more stable to such modes than anti-D shaped configurations.

(4) Anti-D shaped configurations can be made more stable to local modes by increasing the effect of toroidicity.

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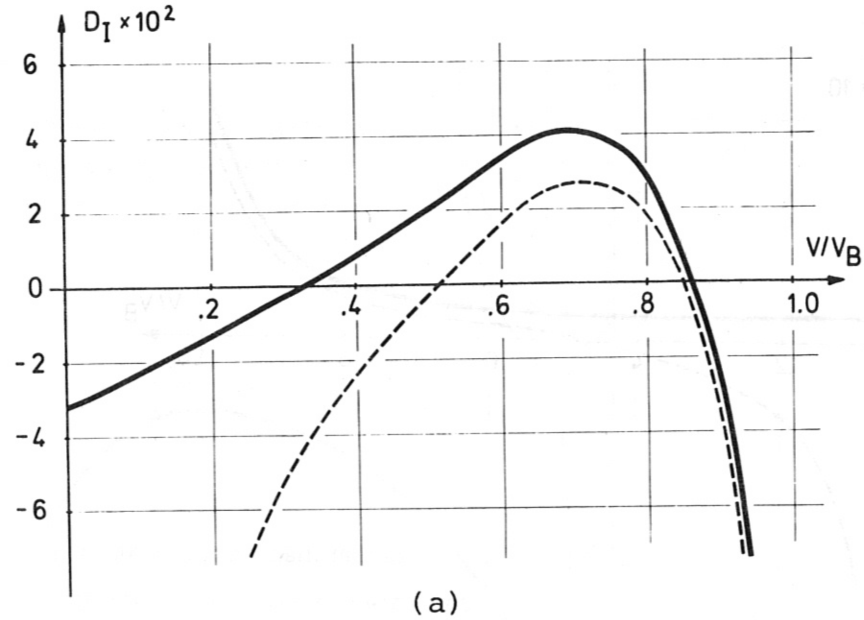
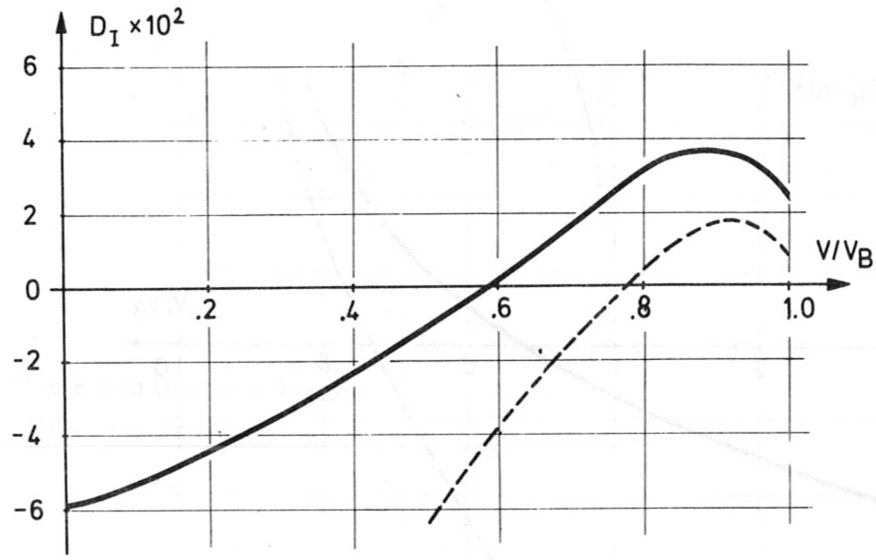


(a)



(b)

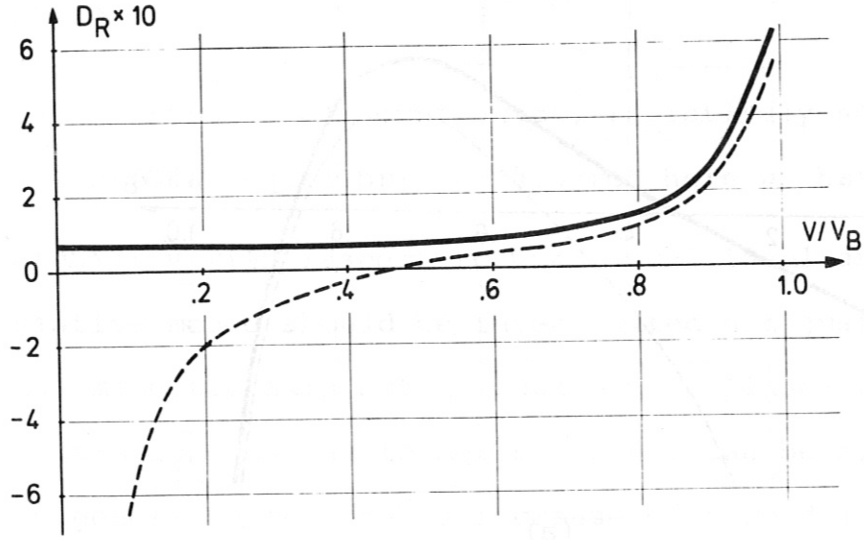
Fig. 4 D_I as function of V/V_B where V_B is the volume enclosed by the plasma boundary. $D_I < 0$ for stability with respect to local ideal modes.

— corresponds to a toroidal field such that $q_o = q_c$

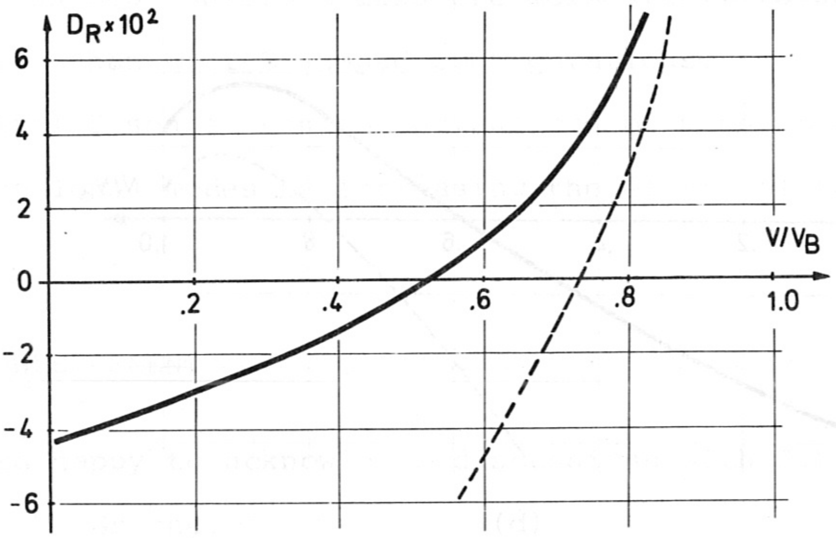
-- corresponds to a toroidal field such that $q_o = 1.5\ q_c$

(a) $R_o = 3.21$ m, $A = 2.37$ (JET); $R_M = 4.21$ m, $z_M = 2.58$ m, $q_c = 5.5$

(b) as (a), but $R_M = 3.96$ m, $z_M = 2.25$ m, $q_c = 4.2$

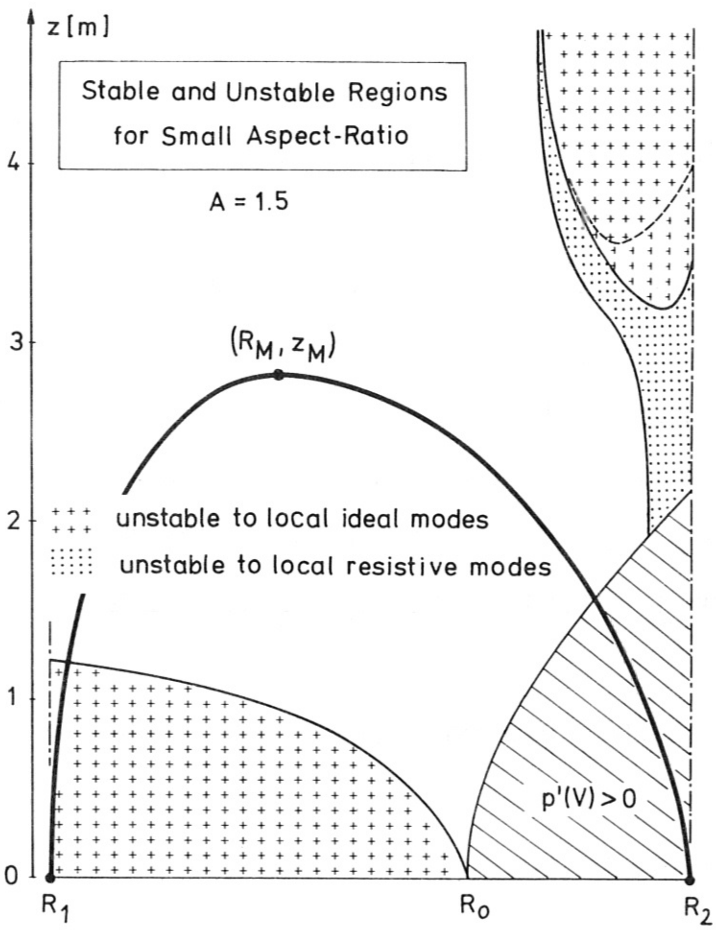


(a)

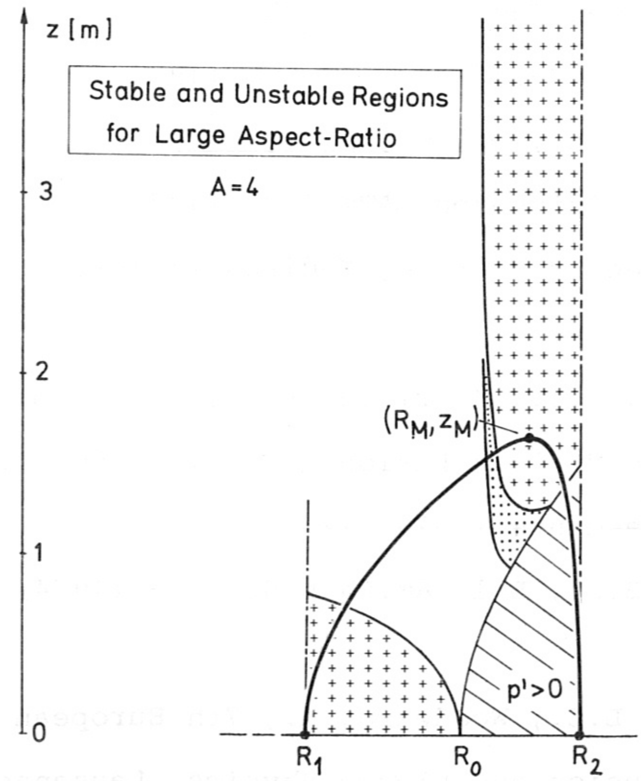


(b)

Fig. 5 D_R as function of V/V_B (R_o = 3.21 m, A = 1.5).
$D_R < 0$ for stability with respect to local resistive modes.
— corresponds to a toroidal field such that $q_o = q_c$
-- corresponds to a toroidal field such that q_o = 1.5 q_c
(a) R_M = 4.34 m, z_M = 3.21 m, q_c = 3
(b) R_M = 4.34 m, z_M = 2.85 m, q_c = 2.1



(a)



(b)

Fig. 6 The effect of aspect-ratio A on local stability
(a) A = 1.5, R_o = 3.21 m
(b) A = 4 , R_o = 3.21 m

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APPENDIX

Using expression (1) the volume V(G) enclosed by the magnetic surface $G(R,z) = G$ can be found to be

$$V = \frac{16\pi R_0^3 |\mu|^{1/2} \bar{G}^{3/4}}{3\sqrt{2}\, k^3} \left\{ (2-k^2) E(k) - 2(1-k^2) K(k) \right\} \tag{A1}$$

where

$$\bar{G} = G/\alpha \quad , \qquad k^2 = \frac{2\bar{G}^{1/2}}{|1-\gamma| + \bar{G}^{1/2}} \tag{A2,A3}$$

and K(k) and E(k) are the complete elliptic integrals of the first and second kind /18/. For $G'(V) = dG/dV$ we obtain

$$G' = \frac{\alpha \bar{G}^{1/4}}{\sqrt{2}\pi R_0^3 K(k) |\mu|^{1/2} k} \tag{A4}$$

From

$$B = \frac{1}{2\pi} (\nabla\zeta \times \nabla G + \Lambda \nabla\zeta) \tag{A5}$$

where ζ is the angle about the axis of symmetry, we find

$$\mu_0 j = \frac{1}{2\pi} (\Lambda' \nabla V \times \nabla\zeta + \Delta^* G \nabla\zeta) \tag{A6}$$

where Δ^* is defined in (21). Thus

$$\mu_0 I \equiv \frac{\mu_0}{2\pi} \int_{V(x) \le V} j \cdot \nabla\zeta \, d^3x = \frac{4\sqrt{2}}{3\pi} \frac{\alpha |\mu|^{1/2} \bar{G}^{3/4}}{R_0 k^3} \left\{ \frac{E_1}{\mu} + E_2 \right\} \tag{A7}$$

with

$$E_1 \equiv K(k) - E(k) - k^2(K(k) + E(k)) + \frac{3k^2}{\xi} E(k) - \frac{3k^4}{\xi^2}(1-\xi) K(k) + \frac{3k^2}{\xi^2}(k^2 - \xi)(1-\xi)\Pi(\xi,k) \tag{A8}$$

$$E_2 \equiv 4k^2 (2K(k) - E(k)) - 8(K(k) - E(k)) \tag{A9}$$

$\Pi(\xi,k)$ is the complete elliptic integral of the third kind as defined in /18/. ξ is constant on a magnetic surface and given by

$$\xi = \frac{2\bar{G}^{1/2}}{\mathrm{sign}(\mu) + \bar{G}^{1/2}} \tag{A10}$$

(A7) shows the relation the total plasma current I_B (i.e. I within $V \leq V_B$) and α (cf. the remark after equation (20)).

From

$$F' = \frac{1}{2\pi} \frac{d}{dV} \int_{V(x) \leq V} B \cdot \nabla \zeta \, d^3x \tag{A11}$$

we find through (A5) for the safety factor $q = F'/G'$

$$q = \frac{\Lambda}{4\pi^2 G'} \left\langle \frac{1}{R^2} \right\rangle = \frac{|\mu|^{1/2} R_0}{4\sqrt{2}\,\alpha\,\pi} \; \frac{\Lambda(2-\xi) k \Pi(\xi,k)}{\bar{G}^{1/4}} \tag{A12}$$

Λ is to be taken according to (26). It is related to the distribution of the poloidal current J(V) through (13):

$$\mu_0 J' = - \frac{\gamma\sqrt{2}\,\alpha^2}{\mu|\mu|^{1/2}\pi R_0^5} \cdot \frac{\bar{G}^{1/4}}{\Lambda k K(k)} \tag{A13}$$

For I'(V) we obtain

$$I' = \frac{2\alpha}{\mu_0 \pi^2 R_0^4} \left\{ 1 + \frac{1}{4\mu} - \frac{\gamma(2-\xi)\Pi(\xi,k)}{8\mu K(k)} \right\} \tag{A14}$$

The second derivatives of the magnetic fluxes F and G are:

$$F'' = \frac{F'G'}{2G} \left\{ \frac{4\alpha\gamma G}{\mu R_0^2 \Lambda^2} - \frac{1}{4}\Xi - \frac{1}{2}\xi \right\} \tag{A15}$$

$$\Xi(\xi,k) \equiv \frac{2-k^2}{1-k^2} \frac{E(k)}{K(k)} + \frac{\xi^2-2}{1-\xi} + \frac{2-\xi}{1-\xi} \frac{K(k)}{\Pi(\xi,k)} + \left(\frac{2-\xi}{1-\xi} - \frac{k^2}{\xi} \cdot \frac{2-k^2}{1-k^2} \right) \frac{\xi}{(k^2-\xi)} \frac{E(k)}{\Pi(\xi,k)} \tag{A16}$$

$$G'' = \frac{G'^2}{4G} \left\{ 1 - (1 - \frac{1}{2}k^2) \frac{E(k)}{(1-k^2)K(k)} \right\} \tag{A17}$$

To determine the metric elements of magnetic surfaces in Hamada co-ordinates $(V,\bar{\theta},\bar{\zeta})$ needed for the calculation of the W-integrals (14,15,16) we make the co-ordinate transformations $(R,z,\zeta) \rightarrow (V,\ \theta = \cos^{-1}\{(R^2/R_o^2-1)/\sqrt{\bar{G}}\},\zeta) \rightarrow$ $(V,\bar{\theta},\bar{\zeta})$ where for the latter transformation the following relations hold

$$\begin{pmatrix} \frac{\partial\bar{\theta}}{\partial\theta} & \frac{\partial\bar{\theta}}{\partial\zeta} \\ \frac{\partial\bar{\zeta}}{\partial\theta} & \frac{\partial\bar{\zeta}}{\partial\zeta} \end{pmatrix} = \begin{pmatrix} \frac{\pi|\mu|^{1/2}R_0^3 G'}{2\alpha|N|^{1/2}} & 0 \\ \frac{\pi|\mu|^{1/2}R_0^3}{2\alpha|N|^{1/2}}\left(F' - \frac{\Lambda}{4\pi^2R^2}\right) & \frac{1}{2\pi} \end{pmatrix} \tag{A18}$$

Now the W-integrals read

$$W_1 = \frac{\bar{G}^{1/4}}{4\sqrt{2}\pi^2R_0^2\,k\,K(k)}\int_0^{\pi}\frac{d\theta}{|N|^{1/2}(N+\gamma)} \cdots$$

$$\cdots\left\{1 + \frac{\mu R_0^2\Lambda^2}{4\alpha^2\bar{G}N(N+\gamma)\Gamma_{\theta\theta}}\left(1 - \frac{(2-\xi)(N+\gamma)\Pi(\xi,k)}{2K(k)}\right)^2\right\} \tag{A19}$$

$$W_2 = \frac{|\mu|^{1/2}\Lambda\,\bar{G}^{-1/2}}{4\alpha\pi R_0\,k^2K^2(k)}\int_0^{\pi}\frac{d\theta}{2|N|^{3/2}(N+\gamma)\Gamma_{\theta\theta}}\left(1 - \frac{(2-\xi)(N+\gamma)\Pi(\xi,k)}{2K(k)}\right) \tag{A20}$$

$$W_3 = \frac{\sqrt{2}\bar{G}^{-1/4}}{4R_0^2k^3K^3(k)}\int_0^{\pi}\frac{d\theta}{|N|^{3/2}\Gamma_{\theta\theta}} \tag{A21}$$

In (A19-A21) use has been made of the following abbreviations:

$$N = \text{sign}(\mu)\left\{|1-\gamma| + \bar{G}^{1/2}\cos\theta\right\} \tag{A22}$$

$$\Gamma_{\theta\theta} = \frac{\sin^2\theta}{N+\gamma} + \frac{\mu}{N}\left\{2\cos\theta + \frac{\sin^2\theta\,\bar{G}^{1/2}}{|1-\gamma| + \bar{G}^{1/2}\cos\theta}\right\}^2 \tag{A23}$$

Expanding (2) in V about V=0 gives for $D_I^{(-1)}$ in expression (28)

$$D_I^{(-1)} = -\frac{8\beta^* m\pi^2 q_0^2(1-\gamma)^2}{\Delta^2}\left(1+\frac{1}{2(1-\gamma)}-\frac{\sqrt{m}\,\beta^*}{1+\sqrt{m}}\right)(q_0^2-q_c^2)\cdot R_0^3 \qquad \text{(A24)}$$

where

$$\Delta \equiv \gamma + q_0^2\left(2+4(1-\gamma)+\frac{3}{2(1-\gamma)}\right) \qquad \text{(A25)}$$

and m, q_c, q_o, β^* are defined in equations (27,30-32).