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IPP 4/140 June 1976

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*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.*

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## Abstract

The influence of the density and temperature profiles on the mean fusion power density $\bar{Q}_N$ in a circular Tokamak is considered. For fixed $\beta_{\vartheta}$ and q(a) peaking of the density and temperature enlarges $\bar{Q}_N$. If for q(a) the smallest value compatible with the condition $q(r) \geq 1$ is taken, peaking of the density remains favourable but peaking of the temperature diminishes $\bar{Q}_N$ considerably.

We consider a 50 - 50 deuterium-tritium tokamak plasma with circular cross-section. The cylinder approximation is used.

The fusion power density is given by

1: $$Q_N = \frac{n_i^2}{4} \langle \sigma v \rangle E_\alpha$$ [1)]

In the following we shall use the approximate expression [1]

2: $$\langle \sigma v \rangle E_\alpha = 2{,}08 \cdot 10^{-17} \frac{e^{-\frac{19{,}94}{T_i^{1/3}}}}{T_i^{2/3}}$$

In a Tokamak there is an upper limit for the poloidal β, given by [2][3]

3: $$\beta_\theta = \frac{8\pi}{B_\theta^2(a)} V^{-1} \int d\tau \, (n_e T_e + n_i T_i)$$

$$= 4{,}02 \cdot 10^{-14} \frac{R^2 q^2(a)}{a^2 B_\varphi^2} \frac{2}{a^2} \int_0^a dr \cdot r \, (n_e T_e + n_i T_i)$$

$$\left( q(r) = \frac{r B_\varphi}{R \cdot B_\theta(r)} \right)$$

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1) All quantities are in cgs units, except T, which is in keV, and B, which is in kG, in formulae with decimal coefficients.

If the impurity concentration is not too high, one can assume that ne = ni = n. Furthermore, for not too high temperatures Te and Ti do not widely differ in a fusion plasma.

Hence we set Te = Ti = T.

We shall regard profiles of the form

$$n = n_0\left[0{,}9\left(1-\left(\frac{r}{a}\right)^{2\alpha}\right)^{\beta}+0{,}1\right];\; T = T_0\left[0{,}9\left(1-\left(\frac{r}{a}\right)^{2\gamma}\right)^{\delta}+0{,}1\right]$$

for the density and temperature respectively. With these profiles one gets from 1: and 2: for the mean nuclear energy production rate $\bar{Q}_N$

$$4{:}\quad \bar{Q}_N = V^{-1}\int dr\, Q_N = 5{,}2\cdot 10^{-18}\,\frac{n_0^2}{T_0^{2/3}}\, e^{-\frac{19{,}94}{T_0^{1/3}}}\, I(T_0, \alpha, \beta, \gamma, \delta)$$

$$I = \int_0^1 dz\, e^{-\frac{19{,}94}{T_0^{1/3}}\left\{\frac{1}{[0{,}9(1-z^{\gamma})^{\delta}+0{,}1]}-1\right\}}\,\frac{[0{,}9(1-z^{\alpha})^{\beta}+0{,}1]^2}{[0{,}9(1-z^{\gamma})^{\delta}+0{,}1]^{2/3}}$$

(obviously $I \equiv 1$ if n and T are independent of r)

and for $\beta_\vartheta$

$$5{:}\quad \beta_\vartheta = 8{,}04\cdot 10^{-14}\,\frac{A^2}{B_\varphi^2}\, q^2(a)\, n_0 T_0\, J(\alpha, \beta, \gamma, \delta)$$

(A = $^R/a$ the aspect ratio)

$$J = \int_0^1 dz\,[0{,}9(1-z^{\alpha})^{\beta}+0{,}1]\,[0{,}9(1-z^{\gamma})^{\delta}+0{,}1]$$

(obviously J = 1 if n and T are independent of r).

From 5: and 4: we elimininate $n_o$ to express $\bar{Q}_N$ for fixed $\beta_\vartheta$:

$$6: \bar{Q}_N = 8{,}04 \cdot 10^8 \frac{\beta_\vartheta^2 B_\varphi^4}{A^4 q^4(a)} \frac{e^{-\frac{19{,}94}{T_o^{1/3}}}}{T_o^{8/3}} \frac{I(T_o, \alpha, \beta, \gamma, \delta)}{J^2(\alpha, \beta, \gamma, \delta)}$$

Up to now we have not yet taken into account that there exists a lower limit for q(a) in 6:, because of the condition $q(r) \geq 1$ for $0 \leq r \leq a$, which depends on the density and temperature profiles. To determine the minimal q(a) which is compatible with this constraint, we consider Ampere's law:

$$7: \frac{1}{r} \frac{\partial}{\partial r}(r B_\vartheta) = \frac{4\pi}{c} \dot{j}_{\parallel}$$

$\dot{j}_{\parallel}$ is related to the toroidal electric field by Ohm's law:

$$8: \dot{j}_{\parallel} = E_{\parallel} \eta$$

We shall assume $E_{\parallel}$ to be independent of r and $\eta$ to obey the classical formula

$$9: \eta = \zeta \frac{n_e \bar{T}_e^{3/2}}{n_i Z_{eff}^2}$$

we can now write

$$10: \frac{1}{r} \frac{\partial}{\partial r}(r B_\vartheta) = \frac{4\pi}{c} \frac{\dot{j}_{\parallel}^0}{\eta^0} \eta(r)$$

where $\eta^o$ and $\dot{j}_{\parallel}^{o}$ are peak values of $\eta$ and $\dot{j}_{\parallel}$ respectively.
Integration of 10: yields

$$11: \quad B_\vartheta = \frac{4\pi}{c} \frac{\dot{j}_{\parallel}^{o}}{\eta^o} \frac{1}{r} \int_0^r dr' \, r' \, \eta(r') + \frac{K}{r}$$

The condition $q(r) = \frac{r}{R} \frac{B_\varphi}{B_\vartheta} \geq 1$ for $0 \leq r \leq a$ requires that K = 0.

It then follows that

$$12: \quad q(r) = \frac{B_\varphi r^2 c \, \eta^o}{4\pi R \, \dot{j}_{\parallel}^{o}} \frac{1}{\int_0^r dr' \, r' \, \eta(r')}$$

Taking into account 9: and the specified form of the profiles, one immediately sees that q(r) is an increasing function of r. The requirement that q(r) takes its minimal value q(o) = 1, when applied to 12:, gives the following condition for $\dot{j}_{\parallel}^{o}$:

$$13: \quad \dot{j}_{\parallel}^{o} = \frac{B_\varphi \cdot c}{2\pi \cdot R}$$

From 13: and 12: follows that

$$14: \quad \frac{1}{q(a)} = \frac{4\pi R}{B_\varphi c \, a^2} \frac{B_\varphi \cdot c}{2\pi R} \int_0^a dr' \, r' \left[ 0{,}9 \left( 1 - \left( \frac{r}{a} \right)^{2\gamma} \right)^{\delta} + 0{,}1 \right]^{\frac{3}{2}}$$

$$= \int_0^1 dz \left[ 0{,}9 \left( 1 - z^{\gamma} \right)^{\delta} + 0{,}1 \right]^{\frac{3}{2}}$$

where q(a) is now the minimal value such that $q(r) \geq 1$ is fulfilled. Evidently stronger peaking enlarges q(a).

In deriving 14:, Zeff was assumed to be independent of r.

If the neoclassical correction in the expression for $\eta$ had been taken into account, the result would be still more unfavourable.

We shall now regard three types of profiles for n and T:

| profile | A | B | C |
|---|---|---|---|
| $\alpha / \gamma$ | 1 | 1 | 1 |
| $\beta / \delta$ | 0 | 1 | 4 |

For a graphic representation see Fig. 1.

The constant profile A serves as a reference profile. C is the most peaked one.

To study the influence of the profiles, we compute

$$f = \frac{I}{J^2} \quad \text{and} \quad F = \frac{e^{-\frac{19,94}{T_0^{1/3}}}}{T_0^{8/3}} \frac{f}{q^4(a)} \quad \text{, with } q(a)$$

determined according to 14:, for various combinations of profiles A - C. f gives the enhancement of $\bar{Q}_N$ for fixed $\beta_\vartheta$ and q(a) relative to its value for constant density and temperature.

F is proportional to $\bar{Q}_N$ for a given system (fixed $B_\varphi$ , A) and fixed $\beta_\vartheta$

f allows the influence of profile variation on $\bar{Q}_N$ to be studied if $\beta_\ell$ and q(a) are kept fixed.

F gives the influence if $\beta_\ell$ is kept fixed but q(a) has the minimal value compatible with $q(r) \geq 1$.

Figure 2 shows f as a function of the peak temperature To. f increases with increasing To. For given To stronger peaking of T as well as of n enhances f.

If allowance is made for the enhancement of the minimal q(a) compatible with $q(r) \geq 1$ that is due to peaking, the situation rapidly changes.

Since q(a) is independent of the density in our approximation, peaking of n remains favourable. On the other hand, the enhancement of f is by far compensated by the enhancement of q(a) for peaked temperature profiles.

To visualize this influence of the profile dependence of q(a), we plotted F(To) for some typical profiles in Fig. 3. It can be seen that peaking of T strongly diminishes F.

Usually $\bar{Q}_N$ is computed with constant n and T, but taking some reasonable q(a) (say q(a) = 2.5) instead of q(a) as determined according to 14:. The corresponding F is given by the dotted curve in Fig. 3.

It is seen that F for the most peaked profile is almost one order of magnitude smaller, even in the case of the most favourable density profile.

Profiles of type C for n and T are not unrealistic. Profiles of similar shape have been found by Conn et al. [3] by 1-dimensional simulation of a reactor-like plasma.

## References


[1] S. Glasstone and R. Lovberg: Controlled thermonuclear reactions, New York 1960

[2] V.S. Mukhovatov and V.D. Shafranov: Nucl. Fus. 11 (1971)

[3] H.P. Furth: Nucl. Fus. 15 (1975)

[4] J. Kesner and R.W. Conn: To be published in Nucl. Fus.

Fig. 1

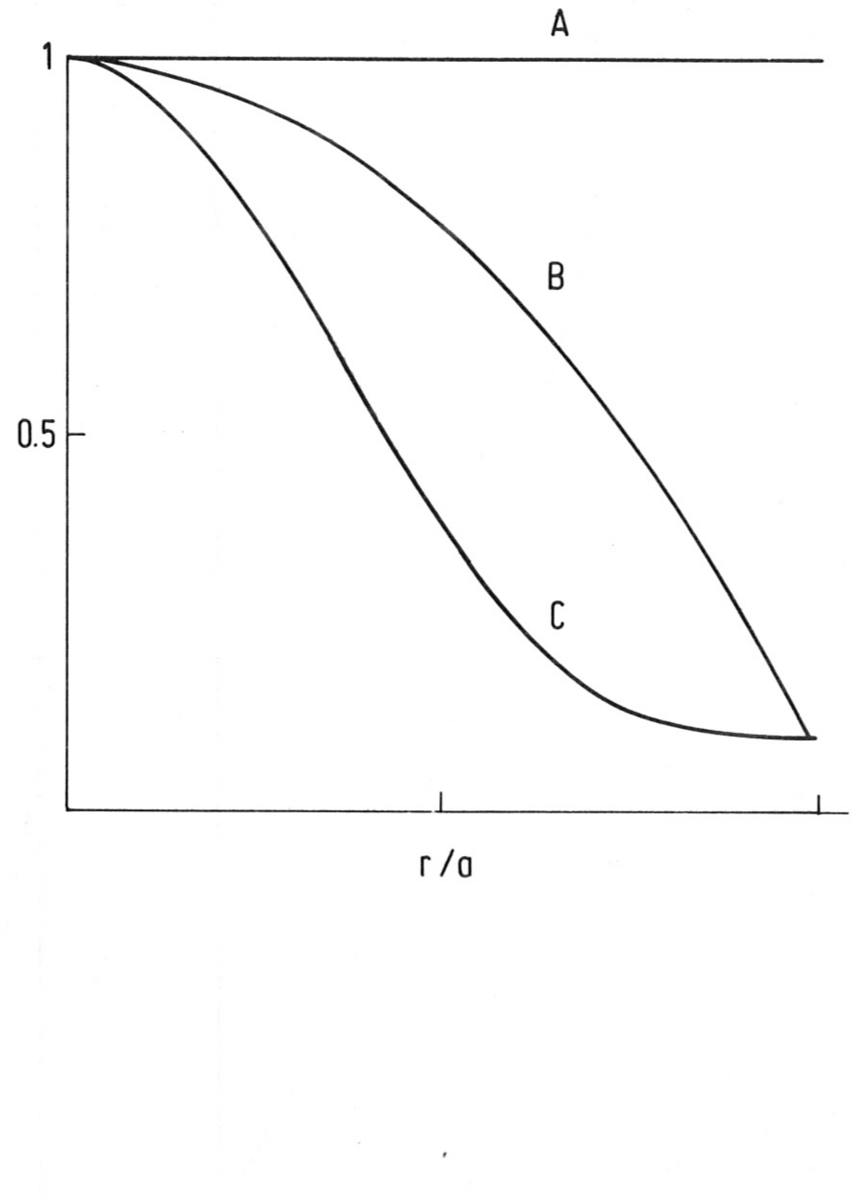
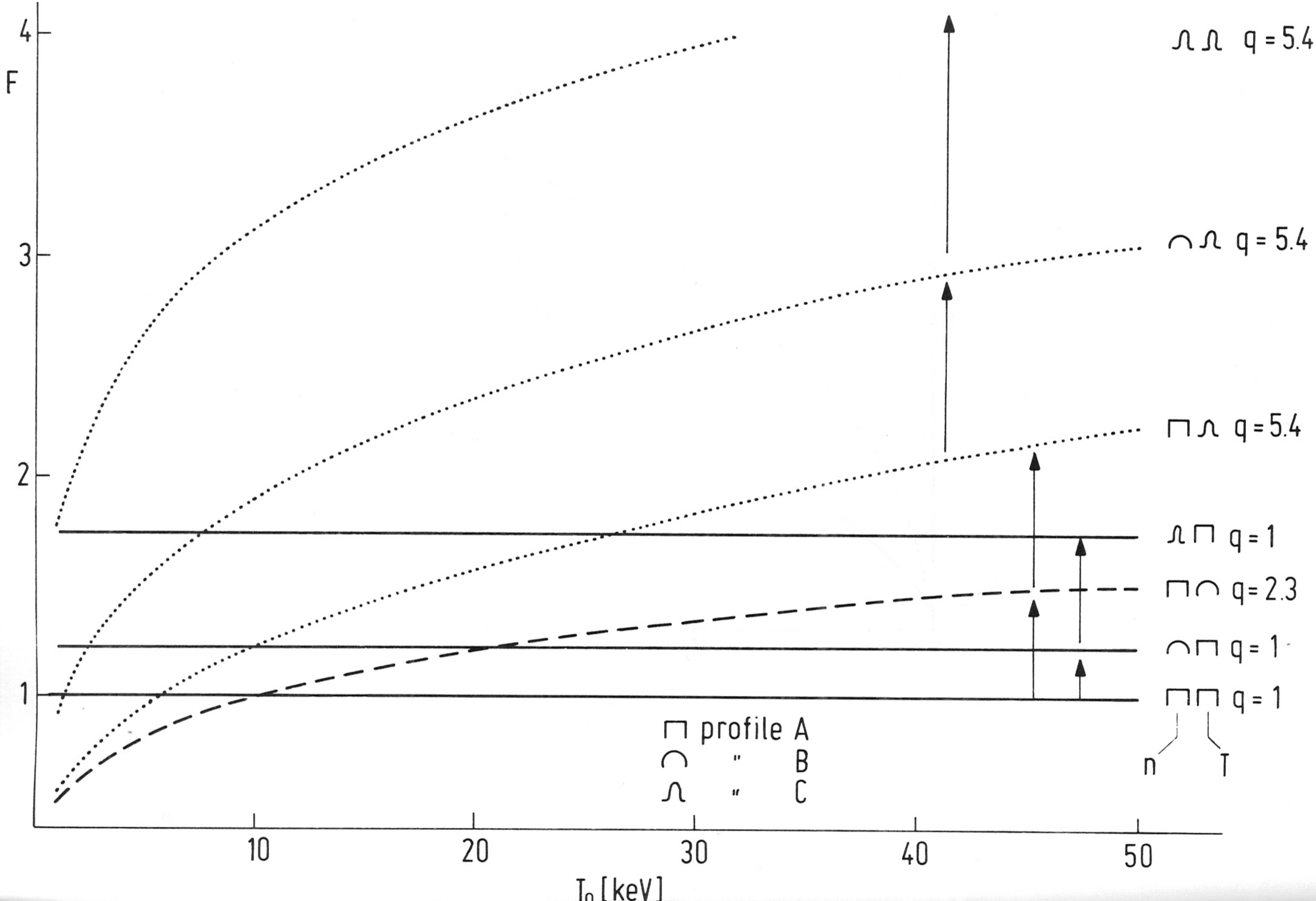


Fig. 2



F
1
2
3
4
10
20
30
40
50
$T_0$ [keV]
q = 5.4
q = 5.4
q = 5.4
q = 1
q = 2.3
q = 1
q = 1
n
T
profile A
" B
" C

Fig. 3