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for Toroidal Field Coils  
by the Finite Element Method

M. Söll, O. Jandl, H. Gorenflo

IPP 4/142  
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Abstract

After discussing fundamental relationships of the finite element method, this report describes the calculation steps worked out for mechanical stress calculations in the case of magnetic forces and forces produced by thermal expansion or compression for toroidal field coils using the SOLID SAP IV computer program /1/.

The displacement and stress analysis are based on the 20-node isoparametric solid element. The calculation of the nodal forces produced by magnetic body forces are discussed in detail. The computer programs, which can be used generally for mesh generation and determination of the nodal forces, are published elsewhere /2, 3/.

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LIST OF SYMBOLS

- [D] Matrix relating nodal displacements to strain field
- $\vec{d}(x,y,z)$  Vector of the displacement field
- [E] Matrix of elastic constants
- $\vec{F}_i$  Two or three-dimensional force vector acting on the node with the number  $i$ .
- $\{F\}$  Vector of the nodal forces acting on all nodes
- $\{F^b\}$  Vector of the nodal forces acting on all nodes produced by the body force density  $\vec{f}_0$
- $\vec{f}_0$  Vector of body force density
- G Shear modulus
- [I] Identity matrix
- [J] Jakobian matrix
- [K] Global stiffness matrix
- [k] Element stiffness matrix
- $[N^I]$  Matrix of shape functions
- $N_i$  Shape function
- $U_E$  Strain energy
- $\frac{\partial N}{\partial x_i}$  Row vector  $\left[ \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \cdots \frac{\partial N_n}{\partial x} \right]$
- $V_E$  Potential of applied loads
- $u,v,w$  Components of the displacement field
- $U,V$  Coordinates of the planes in the coil windings
- $x,y,z$  Global Cartesian coordinates
- $x_i, y_i, z_i$  Coordinates of the node  $i$
- $\{d\}$  Vector of nodal displacements (all investigated nodes)
- $\alpha_i$  Coefficient of thermal expansion
- $\vec{d}_i$  Vector of the displacements of the nodal point  $i$
- $\delta$  Variational operator
- $\vec{\epsilon}$  General strain field vector
- $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$  Normal strains

## LIST OF SYMBOLS

$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$	Shear strains
$\xi, \eta, \psi$	Nondimensional intrinsic coordinates
$\nu$	Poisson's ratio
$\rho$	Radius of curvature
$\vec{\sigma}$	General stress field vector
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	Normal stresses
$\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$	Shear stresses
$[ \ ]^T$	Transposed matrix
$[ \ ]^{-1}$	Inverse matrix

## 1. BASIC RELATIONS OF THE FINITE ELEMENT METHOD

One of the major problems in designing toroidal field coils for large fusion experiments or for fusion reactors of the tokamak type is the problem of the mechanical stresses and strains produced by magnetic forces. The complex coil geometry, large dimensions, the support structure and the force distribution in the coil windings make it difficult to solve the problem analytically. Therefore, the powerful finite element method (FEM) is used.

The FEM is based on the principle of dividing the investigated structure into suitable elements, each of them having a distinct number of nodes on which the forces act. The element number and element geometry have to be determined by a problem analysis (force distribution, structure properties). A simple example is given in Fig. 1, showing a thin plate (plane stress) which is divided into 4 triangular elements. Each element has three nodes. A force vector with components  $F_{x1}$  and  $F_{y1}$  is applied to the node 1. If the node 1 is not fixed (supported), the force produces displacements  $\vec{\Delta}_1 = (\Delta_{x1}, \Delta_{y1})$ . In addition, the displacement  $\vec{\Delta}_1$  produce nodal forces in the nodes 2 and 3.

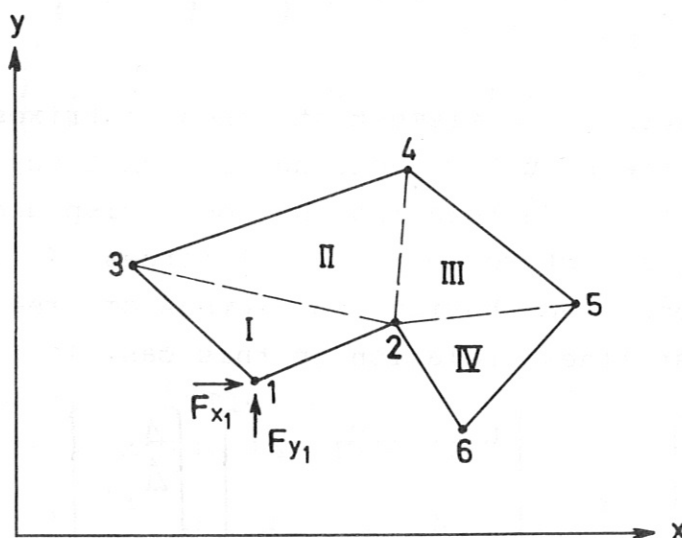


Fig. 1 Plane stress region divided into four triangular elements with six nodes under the influence of a force  $\vec{F}_1 = \{F_{x1}, F_{y1}\}$ .

The principal problem is to calculate the displacements of the three nodes of the element I from the nodal forces. A relation between the forces and displacements is given by the element stiffness equation:

$$\{F\} = [k] \{\Delta\} \quad (1)$$

The matrix  $[k]$  is the element stiffness matrix and  $\{F\}$  and  $\{\Delta\}$  are element force and displacement vectors. The  $[k]$  matrix describes the "mechanical coupling" of the nodal points and therefore determines the forces acting on all the nodal points of the element if, for example, one of the nodal points is displaced.

The  $[k]$  matrix is given by the geometry of the element and the mechanical properties of the material. For the element 1 in our example eq. (1) is written as

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix} = \begin{bmatrix} k_{11} & \dots & k_{16} \\ \vdots & & \vdots \\ k_{61} & \dots & k_{66} \end{bmatrix} \begin{Bmatrix} \Delta_{x1} \\ \Delta_{y1} \\ \vdots \\ \Delta_{y3} \end{Bmatrix} \quad (1')$$

After calculating the element stiffness matrixes for all elements of the structure, one has to construct the global stiffness matrix  $[K]$  relating the nodal displacements of the complete structure to the nodal forces. In our example, we have 6 nodes each having two degrees of freedom ( $\Delta_x, \Delta_y$ ). The global stiffness relation in this case is

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ \vdots \\ F_{y6} \end{Bmatrix} = \begin{bmatrix} K_{11} & \dots & K_{12} \\ \vdots & & \vdots \\ \vdots & 12 \times 12 & \vdots \\ \vdots & & \vdots \\ K_{12} & \dots & K_{12} \end{bmatrix} \begin{Bmatrix} \Delta_{x1} \\ \Delta_{y1} \\ \vdots \\ \Delta_{y6} \end{Bmatrix} \quad (2)$$



where the global [K] matrix is constructed from the element stiffness matrices [k] /4/.

To invert the system (2)

$$\{\Delta\} = [K]^{-1} \{F\}, \quad (3)$$

it is necessary that some  $\Delta_i$  are zero (this means the structure is fixed at the nodes i) because the stiffness matrix is symmetric.

In the following, we give a general description for calculating the element stiffness matrix [k]. First we introduce the concept of the shape function. The shape functions N are defined by relating the displacement-field  $\vec{d}(x,y,z)$  within the element to the node displacements  $\Delta_i$

$$\vec{d}(x,y,z) = [N] \{\Delta\}. \quad (4)$$

The shape functions are the components of the [N] matrix. For the triangular element of Fig. 1 showing only  $\Delta_x$ - and  $\Delta_y$ -displacements, eq. 4 can be written as

$$\vec{d}(x,y) = \begin{Bmatrix} u \\ v \end{Bmatrix} = [I \quad N_1 \quad I \quad N_2 \quad I \quad N_3] \begin{Bmatrix} \Delta_{x1} \\ \Delta_{y1} \\ \Delta_{x2} \\ \Delta_{y2} \\ \Delta_{x3} \\ \Delta_{y3} \end{Bmatrix} \quad (5)$$

$N_1, N_2, N_3$  are the shape functions, I is the identity matrix (2x2) and u,v are the components of the displacement field.

In this case the [N]-matrix is a (2x6) matrix and the  $\Delta_x, \Delta_y$  are the nodal point displacements.

$$[N^I] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \quad (5)$$

$N_i$  is designated as shape function because it represents the shape of the variable  $\vec{d}(xy)$  when plotted over the surface of the element for the case that only the nodal point i is

displaced and all other nodal points are fixed ( $\Delta_{xj} = 0$ ;  $\Delta_{yj} = 0$ ). In our example in Fig. 1, if only nodal point 1 is displaced then the displacement field decreases between point 1 and point 2 or 3 from the, generally normalized, value 1 to zero. This decrease is described by the shape function  $N_1$ . The shape functions play a central role in the finite element analysis, as is shown later.

In calculating the element stiffness matrix  $[k]$ , we use the concept of virtual work /4/, which can be formulated as

$$\delta V_E = \delta U_E, \quad (6)$$

where  $\delta V$  is the external work done by the nodal forces  $\{F\}$  as a consequence of the virtual nodal displacements  $\{\delta \mathbf{A}\}$ . It follows that

$$\delta V_E = \{\delta \mathbf{A}\}^T \{F\}. \quad (7)$$

$\delta U_E$  is the work of the internal forces arising from the action of the internal stresses  $\vec{\sigma}(xyz)$  through the strains  $\delta \vec{\epsilon}$  associated with the virtual displacements.

$$\delta U_E = \int_{vol} \vec{\sigma} \delta \vec{\epsilon} d(vol) \quad (8)$$

$\vec{\sigma}$ ,  $\vec{\epsilon}$  are the general stress and strain vectors including normal and shear components. For a three-dimensional state  $\vec{\epsilon}$  and  $\vec{\sigma}$  are given by

$$\vec{\epsilon} = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{Bmatrix} \quad (9); \quad \vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} \quad (10)$$

With the definition of the displacement-field  $\vec{f}(x,y,z)$  as

$$\vec{d}(x,y,z) = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix}$$

the strain vector is given by

$$\vec{\epsilon} = \left\{ \begin{array}{l} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{array} \right\} \quad (11)$$

In general, the strain displacement equation can be written as

$$\vec{\epsilon} = [D] \{\Delta\} \quad (12)$$

The [D] matrix is related to the shape function but the relation depends on the particular problem. For the example in the plane stress case (Fig. 1)  $\vec{\epsilon}$  is given by

$$\vec{\epsilon} = \left\{ \begin{array}{l} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array} \right\} = \left\{ \begin{array}{cc} \frac{\partial N}{\partial x} & 0 \\ 0 & \frac{\partial N}{\partial y} \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} \end{array} \right\} \left\{ \begin{array}{l} \Delta_{x1} \\ \Delta_{x2} \\ \Delta_{x3} \\ \Delta_{y1} \\ \Delta_{y2} \\ \Delta_{y3} \end{array} \right\} = [D] \{\Delta\} \quad (13)$$

$\gamma_{xy}$  is the shear strain and  $\frac{\partial N}{\partial x}$  is the row vector  $\left[ \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \right]$ .

In general, the [D] matrix is related to the shape functions by differentiation processes.

The virtual displacement field and the virtual strains are given in the same form as eqs. (4) and (12)

$$\delta \vec{d} = [N] \delta \{\Delta\} \quad (14)$$

$$\delta \vec{\epsilon} = [D] \delta \{\Delta\} \quad (15)$$

To satisfy the identity of  $\delta V$  and  $\delta U$ , a relation between the stress  $\vec{\sigma}$  and strain  $\vec{\epsilon}$  must be given. From the elasticity theory it follows that

$$\vec{\sigma} = [E] \vec{\epsilon} - [E] \vec{\epsilon}_{init} \quad (16)$$

including also initial strains  $\vec{\epsilon}_{init}$  such as are produced by, for example, temperature variations<sup>+</sup>). [E] is the elasticity matrix. Substituting (16), (12) and (15) in eq. (8), equating this value with  $\delta V$  and comparing eq. (8) with eq. (1), the [k] matrix is defined as

$$[k] = \int_{vol} [D]^T [E] [D] d(vol) \quad (17)$$

For the forces acting on the nodes as a result of the initial strains  $\vec{\epsilon}_{init}$  the corresponding force vector  $\{F_{init}\}$  is given by

$$\{F_{init}\} = \int_{vol} [D]^T [E] \vec{\epsilon}_{init} d(vol) \quad (18)$$

To account for the body forces in an element given by a force density  $\vec{f}_0$ , the integral (19) (internal work done by the body forces)

$$-\int_{vol} \delta \vec{d} \vec{f}_0 d(vol) \quad (19)$$

must be taken to supplement  $\delta V_E$ . By substitution of  $\delta \vec{d} = [N] \{\delta \Delta\}$  one obtains for this term -  $\{\Delta\}^T \{F^b\}$  with

$$\{F^b\} = \int_{vol} [N^I]^T \vec{f}_0 d(vol) \quad (20)$$

$\{F^b\}$  is the vector of the force components acting on the nodes where the forces are produced by the body forces  $\vec{f}_0$ .  $\{F^b\}$  is therefore also called element body force vector.

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<sup>+</sup>) For thermal expansion  $\epsilon_{init} = \alpha \cdot \Delta T$ , where  $\alpha$  is the coefficient of thermal expansion and  $\Delta T$  the temperature difference from the stress free state.

The equations for equilibrium are

$$\{F\} = [k] \{A\} - \{F_{int}\} - \{F^b\} \quad (21)$$

From eq. (21) the nodal displacements can be calculated and from the displacement the stresses

$$\vec{\sigma} = [E] [D] \{A\} - [E] \vec{\epsilon}_{int} \quad (22)$$

Finally, we want to discuss the so-called "isoparametric formulation" of the FEM /5/. The name derives from the use of the same interpolation functions to define the element shape as are used to define the displacement field, namely the shape function. By this method the curved shapes of the investigated structures could be mapped into one, two or three-dimensional "basic" types for which the mechanism described above can be directly used. An example is given in Fig. 2.

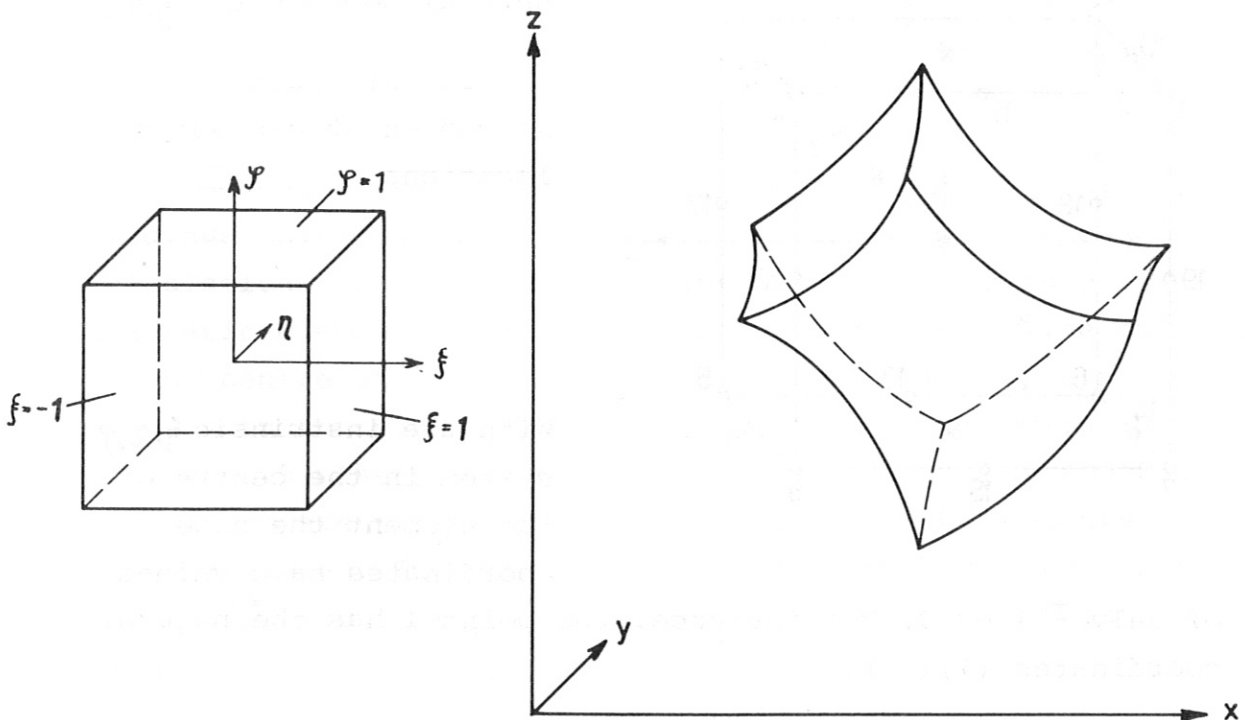


Fig. 2 Mapping of a three-dimensional curved element into a cube

The coordinates of the element are given in the form

$$x = \sum_{i=1}^m N_i x_i; \quad y = \sum_{i=1}^m N_i y_i; \quad z = \sum_{i=1}^m N_i z_i \quad (23)$$

where  $x_i$ ,  $y_i$  and  $z_i$  are the coordinates of the nodes and the  $N_i$  are the shape functions; in this case (23) they are used as transformation functions  $N_i = N_i(\xi, \eta, \zeta)$ . They transform the coordinates of the global  $x, y, z$  system to the "intrinsic" "or natural"  $\xi, \eta, \zeta$  system.

## 2. DESCRIPTION OF THE ELEMENT

For the displacement and stress analysis we use the 20-node isoparametric solid element shown in Fig. 3.

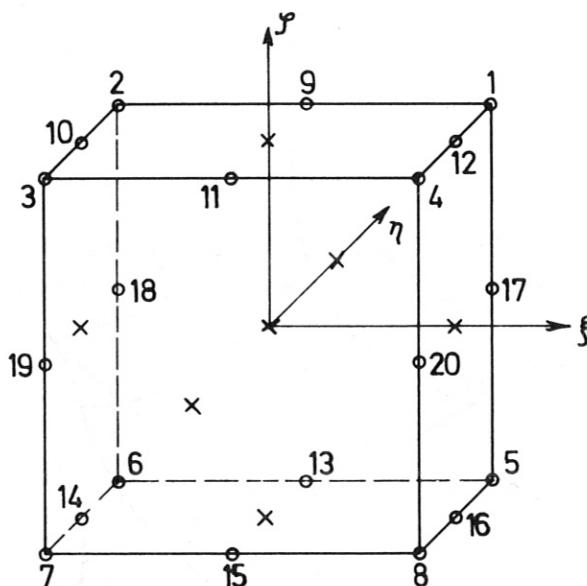


Fig. 3

Hexahedral element in natural coordinates  $\xi, \eta, \zeta$

o: element nodes

x: chosen stress output locations

With the intrinsic  $\xi, \eta, \zeta$  system in the centre of the element the node coordinates have values

of only  $\pm 1$  or 0. For instance, the point 1 has the natural coordinates  $(1, 1, 1)$ .

As shape function for calculating the force distribution produced by body forces (eq. 20) we use the following ones /4/:

$$N_i = \frac{1}{8} (1 + \xi \xi_i) (1 + \eta \eta_i) (1 + \zeta \zeta_i) (\xi \xi_i + \eta \eta_i + \zeta \zeta_i - 2) \quad (24a)$$

for nodes at the corners (nodes 1,2,3,4,5,6,7,8),

$$N_i = \frac{1}{4} (1 - \xi^2) (1 + \eta\eta_i) (1 + \zeta\zeta_i) \quad (24b)$$

for the mid-edge nodes with  $\xi_i = 0$ ,  $\eta_i = \pm 1$ ,  $\zeta_i = \pm 1$ ,

$$N_i = \frac{1}{4} (1 - \eta^2) (1 + \xi\xi_i) (1 + \zeta\zeta_i) \quad (24c)$$

for the mid-edge nodes with  $\eta_i = 0$ ,  $\xi_i = \pm 1$ ,  $\zeta_i = \pm 1$   
and

$$N_i = \frac{1}{4} (1 - \zeta^2) (1 + \xi\xi_i) (1 + \eta\eta_i) \quad (24d)$$

for the mid-edge nodes with  $\zeta = 0$ ,  $\xi_i = \pm 1$ ,  $\eta_i = \pm 1$ .

### 3. DESCRIPTION OF THE CALCULATION PROCESS

In this section we describe the procedure we worked out in order to investigate a three-dimensional displacement and stress analysis for toroidal field coils.

First we calculate magnetic force densities in the coil windings using the HEDO computer program /6/. For these calculations we divide the winding cross-section into quadratic elements (see Fig. 4) each carrying a "line current" in the centre of the element. The number of elements  $n_p$  depends on the winding cross-section; for large coils ( $a, b$  greater 0.5 m see Fig. 4)  $n_p$  is chosen larger than 40. For the circumferential segmentation about 30 - 35 segments are generally used. The program calculates the forces per unit volume  $\vec{f}_o = (f_{ox}, f_{oy}, f_{oz})$  in the centre of each volume element  $\Delta V$  /6/.

This procedure can be done for:

- a) circular coils
- b) ideal "D" coils
- c) real "D" coils
- d) oval coils

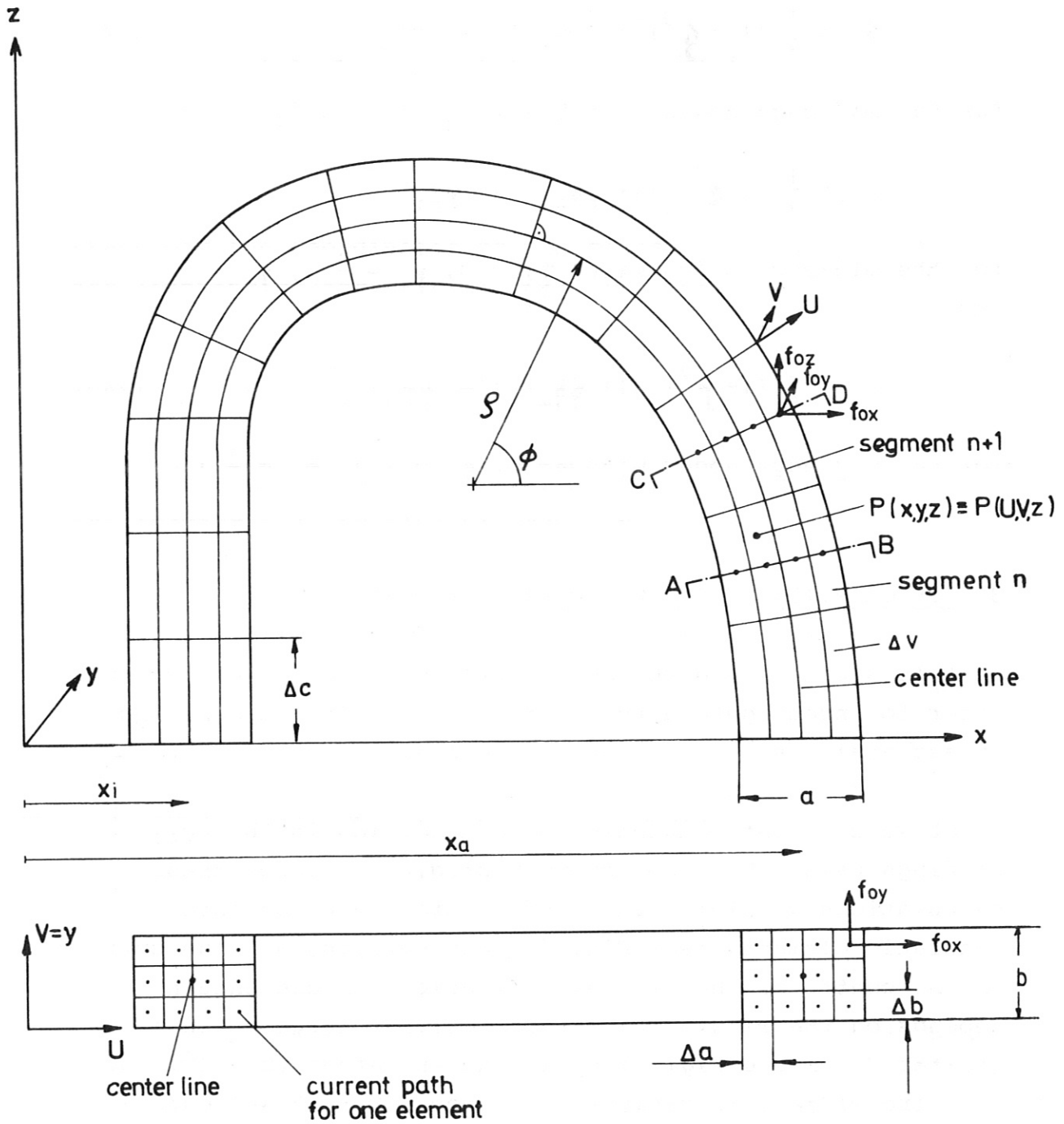


Fig. 4 Scheme of a D-coil showing the subdivision of the winding cross section and the segmentation of the coil along the coil circumference.

$f_{ox}$ ,  $f_{oy}$ ,  $f_{oz}$  are the force densities acting on one element with the volume  $\Delta V = \Delta a \cdot \Delta b \cdot \Delta c$ .

$\rho$  is the radius of curvature,  $\{U, V\}$  system for the planes across the winding section.



By ideal D-coils we mean coils for which the radius of curvature is given according to /7/ as

$$\rho = k \cdot x \quad (25)$$

where  $x$  is the distance from the main torus axis and  $k$  is a constant

$$k = \frac{1}{2} \ln (x_a/x_i) \quad (26)$$

Calculating real D-coils, we use the relation /8/

$$\rho = x \left[ \frac{\rho_a}{x_a} \left(1 + \frac{1}{n}\right) - \frac{1}{n} \ln \frac{x}{x_a} \right] / \left(1 + \frac{1}{n} \cos \varnothing\right) \quad (27)$$

where  $n$  is the number of coils.

With these definitions (25, 27) we calculate numerically the shape of the centre line for the ideal /9/ and real cases.

The next step in the calculation process was to determine the forces acting on the nodal points. In our case the nodal forces are produced by the body forces.

The simplest way to calculate the nodal forces (force distribution) would be to calculate the total force  $\vec{F}_O$  acting on the centre of the element used for the FE calculations (volume elements for the FE calculations are generally not identical with the volume elements used for magnetic field and body force determination) according to

$$\vec{F}_O = \int_{vol} \vec{f}_O d(vol) \quad (28)$$

and then distribute  $\vec{F}_O$  to the 20 nodes. The nodes at the corners (nodes 1-8; see Fig. 3) would be loaded with  $\vec{F}_O/32$  and the mid-plane points (points 9-20) with  $\vec{F}_O/16^+$ .

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+) The distribution regards the fact that a corner point belongs to two or four elements according to whether the element is located at the surface or not, whereas a midplane point belongs to 1 or 2 elements.

We use a more realistic distribution (an example for this assumption is given later) using eq. (20) which distributes the body forces according to the shape functions.

In order to carry out this integration, we had to calculate the volume forces for any arbitrary point  $P(x,y,z)$  in the coil winding.

From the output of HEDO we have the volume forces  $\vec{f}_O$  at the current paths in the centre of each element  $\Delta V = \Delta a \cdot \Delta b \cdot \Delta c$ . In Fig. 4 for two segments (in the circumferential direction)  $n$  and  $n+1$ , the centre points of the body forces, are shown as dots in the planes  $\overline{AB}$  and  $\overline{CD}$ .

To calculate in the plane  $\overline{AB}$ , the force density  $\vec{f}_O$  for every point in the plane, we use the spline technique /10/. For this we divide the plane  $\overline{AB}$  in elements as was done for the force density calculations with HEDO (see Fig. 4). One component  $f_{Ox}$ ,  $f_{Oy}$  or  $f_{Oz}$  of the force density  $\vec{f}_O$  (for  $f_{Ox} \dots$  we use the symbol  $S$ ) is given by

$$S(U,V) = \sum_{i=1}^4 \sum_{j=1}^4 e_{ij} (U-X_{LX})^{i-1} (V-Y_{LY})^{j-1} \quad (29)$$

where  $U,V$  are the coordinates of an arbitrary point in the plane,  $X_{LX}$  and  $Y_{LY}$  are the coordinates of the element centre (centre of  $\vec{f}_O$ ) and  $e_{ij}$  are the spline coefficients of the "centre points"  $(X_{LX}, Y_{LY})$ .

In carrying out the summation process of eq. (29), one has to calculate the 16 spline coefficients for every "centre point", then the element in which the point with the coordinates  $(U,V)$  has to be found. A detailed description for this case is given in Ref. /3/.

With the  $\vec{f}_O(x,y,z)$ -values the nodal forces acting on the nodes could be calculated with eq. (20)

$$\{F^b\} = \int_{vol} [N^I]^T \vec{f}_O(x,y,z) dx dy dz. \quad (20)$$

We carried out this integration in the  $(\xi, \eta, \psi)$  space. For the integration in the  $(\xi, \eta, \psi)$  space we have to construct the Jakobi matrix  $[J]$  because the volume element  $d(\text{vol}) = dx dy dz$  transforms to the volume element  $d\xi d\eta d\psi$  as eq. (30)

$$dx dy dz = \det [J] d\xi d\eta d\psi \quad (30)$$

The integral (20) is now written as

$$\{F^b\} = \iiint [N^I]^T \vec{f}_0(\xi, \eta, \psi) \det [J] d\xi d\eta d\psi \quad (31)$$

For the 20-node element with  $\Delta_{xi}$ ,  $\Delta_{yi}$ , and  $\Delta_{zi}$  displacements the components of eq. (31) should be described in more detail.

The  $[N^I]$ -matrix

$$[N^I] = [IN_1 \quad IN_2 \quad \dots \quad IN_{20}] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \dots & N_{20} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_{20} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \dots & 0 & 0 & N_{20} \end{bmatrix} \quad (32)$$

is a matrix with 3 rows and 60 columns

$$[N^I] = [3 \times 60] \quad (33)$$

$$\text{and } [N^I]^T = [60 \times 3]. \quad (33')$$

The shape functions  $N_i$  are given by eq. (24a) - (24d). The multiplication of  $[N^I]^T$  by the force density vector  $\vec{f}_0 = [3 \times 1]$  leads to the  $\{F^e\}$ -vector with

$$\{F^b\} = [60 \times 1] = \begin{Bmatrix} F_{x1}^b \\ F_{y1}^b \\ F_{z1}^b \\ \vdots \\ F_{z20}^b \end{Bmatrix}. \quad (34)$$

The advantage of calculating the nodal forces  $\{F^b\}$  according to the shape function distribution (eq. 20) in contradiction to the simple nodal force distribution (eq. 28) is demonstrated in Fig. 5a, 5b. Two cubes, fixed in the  $z = 0$  plane, are loaded with a body force of 1 N per unit volume which is directed parallel to the  $z$ -axis. The produced displacement field of the points 1, 2, 3, 4 and 5 is shown in Fig. 5b for three cases: the realistic case (a), the calculation with the "shape function distribution" (eq. 20) case b, and case c with the simple nodal force distribution ( $\vec{F}_O/16$  resp.  $\vec{F}_O/32$ ).

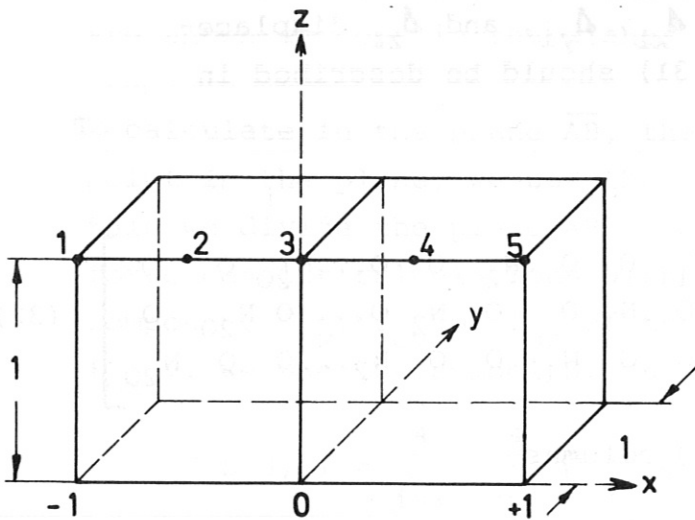


Fig. 5a

Arrangement of the two cubes for demonstrating the displacement fields for different nodal force distribution

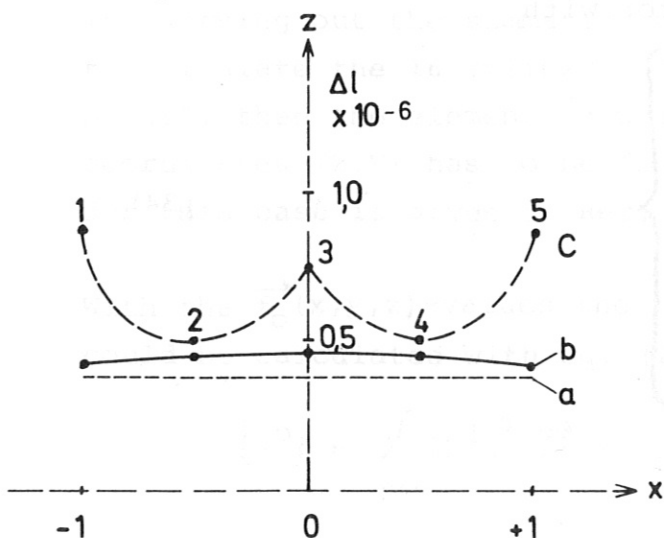


Fig. 5b

Displacement field of the points 1, 2, 3, 4, 5 for

- a) realistic case (elasticity theory)
- b) nodal forces calculated with "shape function distribution" (eq. 20)
- c) nodal forces calculated by a simple distribution ( $\vec{F}_O/32$ ;  $\vec{F}_O/16$ ; eq. 28)

For our 20-node element the Jakobi matrix is given as

$$[J] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_{20}}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_{20}}{\partial \eta} \\ \frac{\partial N_1}{\partial \psi} & \frac{\partial N_2}{\partial \psi} & \cdots & \frac{\partial N_{20}}{\partial \psi} \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_{20} & y_{20} & z_{20} \end{bmatrix} \quad (35)$$

$x_1, y_1, z_1, \dots, z_{20}$  are the coordinates of the element nodes in xyz space and the  $N_i$  are the shape functions listed in eq. (22a- 22d).

For the numerical integration of eq. (28) we use the Gauss method.

$$\int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} [N^I]^T \vec{f}_O(\xi, \eta, \psi) \det[J] d\xi d\eta d\psi = \sum_I \sum_J \sum_K [N^I]^T \vec{f}_O(\xi_i, \eta_j, \psi_h) W_i W_j W_k \det[J] \quad (36)$$

where  $(\xi_i, \eta_j, \psi_h)$  are the coordinates of the sampling points and  $W_i, W_j, W_k$  are "weights". In Fig. 6 the 9 sampling points for the midplane ( $\eta = 0$ ) of the used element are shown.

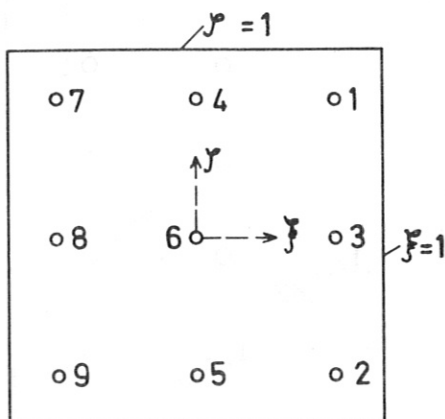


Fig. 6

Sampling points in the midplane of the used hexahedral element for Gauss integration.

The location and "weights" for the points 4,5 and 6 ( $\eta = 0; \xi = 0$ ) are listed in Table 1.

Table 1: Gauss quadrature coefficients (for n = 3)

<u>Point</u>	<u>Location</u>	<u>Weight</u>
4	$\eta = 0.7745966..$	5/9
5	$\eta = 0.7745966..$	5/9
6	$\eta = 0$	8/9

The extension of the Gauss quadrature coefficients in the mid-plane and to the points for  $\xi \neq 0$  is straightforward. The number of sampling points in an element is 27. Finally, a transformation of the coordinates of the sampling points is done with the aid of the shape functions (eq. 23) to get the corresponding body force values  $\vec{f}_0(\xi_i, \eta_i, \gamma_i)$  from the  $\vec{f}_0(x, y, z)$  values.

To get the displacements  $\vec{d}_1$  and stresses  $\vec{\sigma}$  the global stiffness matrix [K] must be known. The SOLID SAP program calculates the element stiffness matrix [k] and the [K] matrix if the element type, the number of elements, the geometry of the structure, the node numbering and the material constants are collected as input data.

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} 1/E_{xx} & -\nu_{xy}/E_{yy} & -\nu_{xz}/E_{zz} & 0 & 0 & 0 \\ -\nu_{yx}/E_{xx} & 1/E_{yy} & -\nu_{yz}/E_{zz} & 0 & 0 & 0 \\ -\nu_{zx}/E_{xx} & -\nu_{zy}/E_{yy} & 1/E_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{yz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{zx} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} - \begin{bmatrix} \Delta T \alpha_x & \Delta T \alpha_y & \Delta T \alpha_z & 0 & 0 & 0 \end{bmatrix}^T .$$

In our notation eq. (37) is written as

$$\vec{\epsilon} = [E]^{-1} \vec{\sigma} \quad (38)$$

$\epsilon_{ii}$  and  $\sigma_{ii}$  are normal strains and stresses,  $\epsilon_{ij}$  and  $\sigma_{ij}$  are shear strains and stresses. The elastic and shear moduli are denoted as  $E_{ii}$  and  $G_{ij}$ ;  $\nu_{ij}$  are the Poisson's ratios and  $\alpha_i$  are the coefficients of thermal expansion.

With the SOLID SAP computer program the calculation of the stresses produced by thermal expansion or compression reduces, in principle, to determination of the mesh (nodal points) and the input of the  $\alpha$ -, and  $\Delta T$  values in addition to the data necessary for calculating the  $[k]$  matrix. The nodal forces  $\{F_{int}\}$  produced by temperature variations which are given by eq. (18) are calculated with SOLID SAP IV.

The stresses and strains produced by the cooling down process of superconducting coils and the stress distribution in parts of a coil after a local normal transition can be calculated with the Mesh Generation Program /2/ and SOLID DAP IV (see Fig.7).

The  $[D]$  matrix for solid elements used in this analysis is given by:

$$[D] = \begin{bmatrix} \frac{\partial N}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N}{\partial z} \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} & 0 \\ 0 & \frac{\partial N}{\partial z} & \frac{\partial N}{\partial y} \\ \frac{\partial N}{\partial z} & 0 & \frac{\partial N}{\partial x} \end{bmatrix} \quad (39)$$

#### 4. PROGRAMMING

A scheme of the programming process runs as shown in Fig. 7.

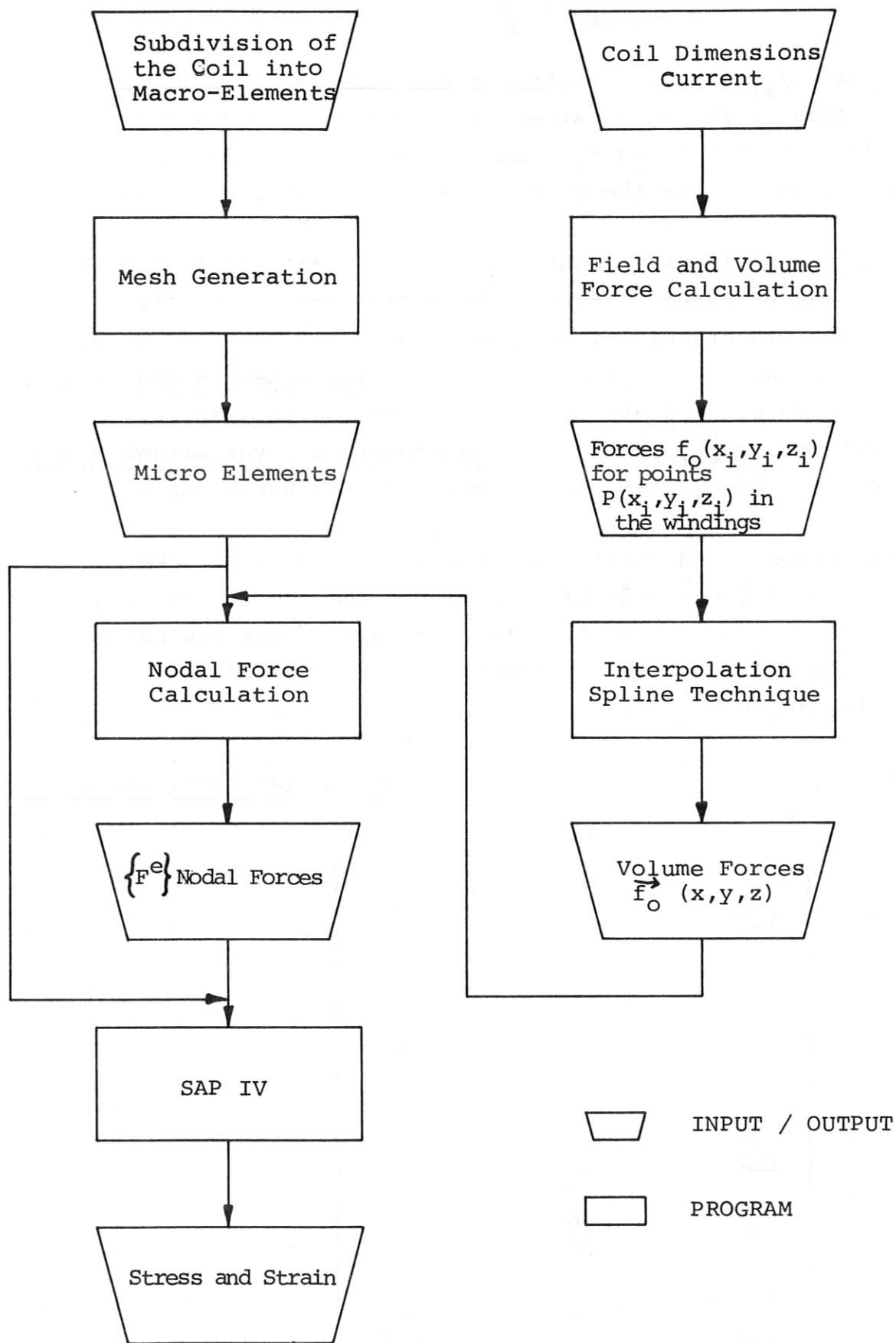


Fig. 7 Scheme of the calculation process



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