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On the Stability of Force-free Fields

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Abstract

Stability of force-free fields is considered with allowance for dissipation. A general sufficient condition for stability is derived. Its relation to Taylor's constraint is emphasized.

Force-free fields have been considered a long time ago for astrophysical problems relating to solar flares and tails of galaxies. In many toroidal discharges, their presence seems to be beneficial to the macroscopic stability. We consider here their stability taking dissipation into account. Dissipation makes these fields time dependent and therefore investigation of their stability is similar to the study of the stability of motion. Let us first look for time-dependent force-free field solutions.

I Time-dependent Force-free Fields

We have to find solutions satisfying

$$\nabla \times \underline{B} = \underline{j} = \lambda(\underline{r}, t)\, \underline{B} \qquad (1)$$

and

$$-\dot{\underline{B}} = \nabla \times \eta \lambda \underline{B} \quad , \qquad (2)$$

with $\underline{B} \cdot \nabla\eta = \underline{B} \cdot \nabla\lambda = 0,$

where $\underline{B}$ is the magnetic field, $\underline{j}$ is the current density and the resistivity η and λ are functions constant along the magnetic field lines. Equation (2) is Ohm's law with zero mass velocity.

Assuming constant resistivity, Jette [1] showed that the only force-free field solution is such that λ must be a constant. Generally, it is necessary that

$$(\underline{j} \times \underline{B})^{\bullet} = 0 \ . \tag{3}$$

$\underline{j}$ can be calculated by taking the curl of equation (2), which leads to

$$- \nabla \times \dot{\underline{B}} = \eta\lambda^3 \underline{B} + \nabla(\eta\lambda^2) \times \underline{B} + \nabla \times \left[\nabla(\eta\lambda) \times \underline{B}\right] \ . \tag{4}$$

Inserting in equation (3) and combining with equations (1) and (2), one obtains

$$- B^2 \eta\lambda\nabla\lambda + \left[\nabla \times (\nabla(\eta\lambda) \times \underline{B})\right] \times \underline{B} = 0 \ . \tag{5}$$

The second term of equation (5) can be written as

$$\lambda B^2 \nabla(\eta\lambda) - 2(\nabla(\eta\lambda) \cdot \nabla \underline{B}) \times \underline{B}$$

so that equation (5) becomes

$$\lambda^2 B^2 \nabla\eta - 2(\nabla(\eta\lambda) \cdot \nabla \underline{B}) \times \underline{B} = 0 \ . \tag{6}$$

Equation (6) reduces to the starting equation of Jette [1] if $\nabla\eta = 0$.

In cylindrical geometry equations (1) and (6) lead to a non-linear coupled problem, but if η and λ are functions of the radial coordinate r there is enough freedom to have solutions at least in a finite region in r. In general, condition (6) will probably not be verified. Let us now consider the stability of time-dependent force-free field solutions.

II Stability

The linearized equations of motion around such solutions are:

$$\rho_o \ddot{\underline{\xi}} = \underline{j}_1 \times \underline{B}_o - \lambda \underline{B}_1 \times \underline{B}_o \tag{7}$$

$$- \dot{\underline{A}}_1 + \dot{\underline{\xi}} \times \underline{B}_o = \eta \underline{j}_1 + \eta_1 \underline{j}_o \ , \tag{8}$$

$$\underline{B}_1 = \nabla \times \underline{A}_1 \ , \quad \underline{j}_1 = \nabla \times \nabla \times \underline{A}_1 \ , \tag{9}$$

The investigation is restricted to the case λ = ct, η = ct consistent with $\eta_1 = 0$ and the gauge is chosen such that $\underline{E} = - \dot{A}$. The scalar product of equation (7) with $\dot{\underline{\xi}}$ yields

$$\rho_o \dot{\underline{\xi}} \bullet \ddot{\underline{\xi}} = - (j_1 - \lambda \underline{B}_1) \cdot (\dot{\underline{A}}_1 + \eta \underline{j}_1)$$

Integrating over the plasma volume limited by a perfectly conducting wall, we obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \Big[(\rho_o \dot{\xi}, \dot{\xi}) + (\nabla \times \underline{A}_1 , \nabla \times \underline{A}_1) - (\lambda A_1 , \nabla \times \underline{A}_1) \Big] =$$

$$- \eta \Big[(j_1 , j_1) - (\lambda j_1 , B_1) \Big] \ . \tag{10}$$

where $(\alpha, \zeta) = \int_V d\tau \, \underline{\alpha} \cdot \underline{\zeta}$.

Let δW_R be defined as

$$\delta W_R = (\nabla \times \underline{A}_1 , \nabla \times \underline{A}_1) - \lambda (A_1 , \nabla \times \underline{A}_1) \tag{11}$$

The variation of δW_R leads to the following Euler eigenvalue equation:

$$\nabla \times \nabla \times \underline{A}_1 - \lambda \nabla \times \underline{A}_1 = \alpha \underline{A}_1 \tag{12}$$

The variation of the right-hand side of equation (10) leads to

$$\nabla \times \nabla \times \underline{B}_1 - \lambda \nabla \times \underline{B}_1 = \beta \underline{B}_1 \tag{13}$$

The curl of equation (12) is identical with equation (13). This means that any solution of equation (13) verifying $\underline{n} \cdot \underline{B}_1 = 0$ at the boundary is also a solution of equation (12) with $\underline{n} \times \underline{A}_1 = 0$ at the boundary.

It follows that $\delta W_R > 0$ implies the negativeness of the right-hand side of equation (10). This means that $\delta W_R > 0$ is sufficient for stability with respect to MHD + resistive modes. This condition is found necessary and sufficient if one ignores resistivity and uses instead Taylor's hypothesis of a global invariant [2, 3]. The result obtained here supports Taylor's assumption for the case of λ = ct force-free fields. Finally, it is easy to prove that Hall term, gyroviscosity and collisional viscosity do not alter the sufficient condition derived here.

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References

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