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Energy Principle for Resistive Perturbations in Tokamaks

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Abstract

An energy principle has been found for all resistive perturbations of a circular plasma cylinder in the Tokamak scaling ($kr \approx \frac{B_\theta}{B_z} \approx \varepsilon$). It allows stability to be determined independently of the resistive scaling because resistivity is taken to be finite. The inclusion of F.L.R. effects and viscosity would only reduce the growth rates. Using simple test functions near the resonances always leads to instability. Estimates of growth rates can be made, these being only rough if F.L.R. effects dominate.

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In a previous paper [1] two-dimensional resistive perturbations around an equilibrium of a straight plasma cylinder of arbitrary shape were considered. The results obtained there can be used for the stability of skinned currents and of the stagnation point of magnetic islands. Helical perturbations, which are observed in all the Tokamaks, are more difficult to investigate than in the exact case above [1].

Nevertheless, if one uses the Tokamak scaling $k_z r \approx \frac{B_\theta}{B_z} \approx \varepsilon$, where ε is a small parameter, k_z is the wave number along the circular cylinder, r is the plasma radius, and B_θ and B_z are the meridional and longitudinal magnetic fields, then the two-dimensional energy principle [1] can be extended to helical perturbation, as we shall see here.

I Equilibrium

The equations governing the equilibrium are

$$\underline{j}_0 \times \underline{B}_0 = \nabla p_0 \quad , \quad \underline{v}_0 = 0 \; , \tag{1}$$

$$\nabla \times \underline{B}_0 = \underline{j}_0 \; , \tag{2}$$

$$\nabla \cdot \underline{B}_0 = 0 \; , \tag{3}$$

$$\nabla \times \eta_0 \underline{j}_0 = 0 \; , \tag{4}$$

$$\underline{B}_0 \cdot \nabla \eta_0 = 0 \; , \tag{5}$$

where $\underline{B}_0$ is the magnetic field, p_0 the pressure, and η_0 the resistivity.

In helical symmetry it is convenient [2] to introduce a coordinate $u = l\theta - hz$, where θ and z are cylindrical coordinates, and the vector

$$\underline{u} = \frac{l\, e_z + rh\, e_\theta}{l^2 + r^2 h^2} \tag{6}$$

$$\nabla \cdot \underline{u} = 0 \quad \text{and} \quad \nabla \times \underline{u} = \frac{2hl}{l^2 + r^2 h^2}\, \underline{u} \tag{7}$$

The solution of eq. (3) in helical symmetry is

$$\underline{B}_0 = f_0(r,u)\, \underline{u} + \underline{u} \times \nabla F_0(r,u) \tag{8}$$

and

$$\underline{j}_0 = \frac{2hl}{l^2 + h^2 r^2} f_0 \underline{u} + \nabla f_0 \times \underline{u} + (l^2 + h^2 r^2)\, L F_0\, \underline{u}\,, \tag{9}$$

with

$$L F_0 = \frac{1}{r} \frac{\partial}{\partial r} \frac{r}{l^2 + h^2 r^2} \frac{\partial F_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F_0}{\partial u^2} \tag{10}$$

Inserting $\underline{B}_0$ and $\underline{j}_0$ from eqs. (8) and (9) into eq. (1) yields.

$$L F_0 + \frac{2hl}{(l^2 + h^2 r^2)^2} f_0 + \frac{f_0}{l^2 + h^2 r^2} \frac{d f_0}{d F_0} + \frac{d P_0}{d F_0} = 0\,, \tag{11}$$

with $f_0(F_0)$ and $P_0(F_0)$ given arbitrary functions.

We should also satisfy eq. (4) with $\eta_0 = \eta_0(F_0)$ (because of eq. (5)). This leads to

$$\eta_0 \left[\frac{2hl}{l^2 + h^2 r^2} \left((l^2 + h^2 r^2) L F_0 + \frac{2hl}{l^2 + h^2 r^2} f_0 \right) \underline{u} + \nabla \left((l^2 + h^2 r^2) L F_0 + \frac{2hl}{l^2 + h^2 r^2} f_0 \right) \times \underline{u} \right.$$
$$\left. - (l^2 + h^2 r^2) L F_0\, \underline{u} \right] + \nabla \eta_0 \times \left[\left((l^2 + h^2 r^2) L F_0 + \frac{2hl}{l^2 + h^2 r^2} f_0 \right) \underline{u} + \nabla f_0 \times \underline{u} \right] = 0 \tag{12}$$

Taking the scalar and vector products with $\underline{u}$, one obtains

$$\eta_0 \frac{2hl}{l^2+h^2r^2}\left[(l^2+h^2r^2)LF_0 + \frac{2hl}{l^2+h^2r^2} f_0\right] - \eta_0 (l^2+h^2r^2)Lf_0 - \nabla\eta_0\cdot\nabla f_0 = 0, \quad (13)$$

$$\nabla\eta_0\left[(l^2+h^2r^2)LF_0 + \frac{2hl}{l^2+h^2r^2} f_0\right] = 0 \quad (14)$$

From eq. (14) it follows that

$$\eta_0\left[(l^2+h^2r^2)LF_0 + \frac{2hl}{l^2+h^2r^2} f_0\right] = ct. \quad (15)$$

Equation (15) would be in contradiction with eq. (11) unless

$\frac{dp_0}{dF_0} = 0$ and $\eta_0 \frac{d f_0^2}{dF_0} = ct.$. The field has to be force free so that

$$\underline{j}_0 = \lambda \underline{B}_0$$

and $\int_C \eta_0 \underline{j}_0 \cdot d\underline{l} = 0$ on a $z = ct., F_0 = ct.$ curve. This leads to $\frac{df_0}{dF_0} = 0$. This means that the only nontrivial static and resistive solution is a cylinder with $\eta_0 j_{z0} = ct.$. In this case the same field can be represented as $\underline{B}_0 = f_0(r)\,\underline{u} + \underline{u}\times\nabla F_0(r)$ with arbitrary h and l, and a corresponding choice of $f_0(r)$ and $F_0(r)$.

II Perturbations and Stability

The perturbed equations for incompressible motion (see [1]) are

$$\rho_0 \ddot{\underline{\xi}} + \nabla p_1 - \underline{j}_1\times\underline{B}_0 - \underline{j}_0\times\underline{B}_1 = 0 , \quad (16)$$

$$\nabla \cdot \underline{\xi} = 0 \tag{17}$$

$$-\underline{\dot{B}}_1 + \nabla \times (\underline{\xi} \times \underline{B}_0) + \nabla \times (\eta_0 \underline{j}_1 + \eta_1 \underline{j}_0) = 0, \tag{18}$$

$$\nabla \cdot \underline{B}_1 = 0 \tag{19}$$

$$\eta_1 = - \underline{\xi} \cdot \nabla \eta_0 \tag{20}$$

Now assuming a helical perturbations around the cylindrical equilibrium and making use of eqs. (17) and (19) we can take as in eq.(8)

$$\underline{B}_1 = f(r,u,t)\,\underline{u} + \underline{u} \times \nabla F(r,u,t) \tag{21}$$

$$\xi_1 = g(r,u,t)\,\underline{u} + \underline{u} \times \nabla G(r,u,t) \tag{22}$$

We now insert $\underline{\xi}_1$ and $\underline{B}_1$ in eqs. (16) and (18) eliminate η_1 by eq. (20) and take the curl of eq. (16) and obtain after some algebra

$$\rho_0 \ddot{g} + \underline{u} \cdot \nabla f \times \nabla F_0 + \underline{u} \cdot \nabla f_0 \times \nabla F = 0, \tag{23}$$

$$\rho_0 (\ell^2 + h^2 r^2) L \ddot{G} + \nabla \rho_0 \cdot \nabla \ddot{G} + \nabla \left(\frac{L_1}{\ell^2 + h^2 r^2} \right) \times \nabla F_0 \cdot \frac{\underline{u}}{|\underline{u}|^2} +$$

$$+ \nabla \left(\frac{L_0}{\rho^2 + h^2 r^2} \right) \times \nabla F \cdot \frac{\underline{u}}{|\underline{u}|^2} + \nabla \left(\frac{f_0}{\rho^2 + h^2 r^2} \right) \times \nabla f \cdot \frac{\underline{u}}{|\underline{u}|^2} + \tag{24}$$

$$+ \nabla \left(\frac{f}{\ell^2 + h^2 r^2} \right) \times \nabla f_0 \cdot \frac{\underline{u}}{|\underline{u}|^2} = 0$$

$$\dot{F} + u \cdot \nabla \dot{G} \times \nabla F_0 + (\underline{u} \cdot \nabla G \times \nabla \eta_0) L_0 - \eta_0 L_1 = 0 \tag{25}$$

$$- \dot{f} - \nabla\left(\frac{\dot{g}}{\ell^2+h^2r^2}\right) \times \nabla F_o \cdot \frac{\underline{u}}{|\underline{u}|^2} + \nabla\left(\frac{f_o}{\ell^2+h^2r^2}\right) \times \nabla\dot{G} \cdot \frac{\underline{u}}{|\underline{u}|^2} +$$
$$+ \left(\underline{u}\cdot\nabla\dot{G}\times\nabla F_o\right)\frac{2h\ell}{\ell^2+h^2r^2} + \left(\underline{u}\cdot\nabla G\times\nabla\eta_o\right) L_o \frac{2h\ell}{\ell^2+h^2r^2}$$
$$- \left(\underline{u}\cdot\nabla G\times\nabla\eta_o\right)(\ell^2+h^2r^2) L f_o - \nabla\left(\underline{u}\cdot\nabla G\times\nabla\eta_o\right)\cdot\nabla f_o$$
$$- \eta_o L_1 \frac{2h\ell}{\ell^2+h^2r^2} + \eta_o\,(\ell^2+h^2r^2) L f + \nabla\eta_o\cdot\nabla f = 0 , \qquad (26)$$

with

$$L_o = (\ell^2+h^2r^2) L F_o + \frac{2h\ell}{\ell^2+h^2r^2} f_o , \qquad (27)$$

$$L_1 = (\ell^2+h^2r^2) L F + \frac{2h\ell}{\ell^2+h^2r^2} f . \qquad (28)$$

Instead of two equations on two scalars as in ref [1], we have four rather complicated equations apparently without single symmetry properties as in ref. [1]

III Tokamak Scaling and Stability Equations

We consider a rather long but finite cylinder and consider priodic perturbations, which means that $h = 2\pi\frac{n}{L}$, and replace ℓ by the natural m usually used in the literature. θ varies between o and 2π , and z between o and L.

Let us then assume that $2\pi n\frac{r}{L} \approx \frac{|\nabla F_o|}{f_o} \approx \varepsilon << 1$, which is known in the literature as the tokamak scaling.

This keeps the rotational transform finite but will help us to decouple the previous system of four equations.

The poloidal currents being absent in the equilibrium $\frac{r}{f_o}\frac{d f_o}{dr} \approx \varepsilon^2$ because the main part of f_o is a vacuum field.

Equations (24) and (25) decouple in lowest order and become

$$\rho_o m^2 L\ddot{G} + \nabla\rho_o \cdot \nabla\ddot{G} + \nabla LF \cdot \nabla F_o \times \underline{u} + \nabla LF_o \times \nabla F \cdot \underline{u} = 0 \qquad (29)$$

$$\dot{F} - \underline{u} \times \nabla F_o \cdot \nabla\dot{G} - m^2\left(LF_o + \frac{2 \times 2\pi n}{Lm} f_o\right) \underline{u} \times \nabla\eta_o \cdot \nabla G - \eta_o m^2 LF = 0, \qquad (30)$$

with $\underline{u}$ and L redefined as

$$\underline{u} = \frac{e_z}{m} \quad , \quad L = L_{old}\,(h = 0)$$

We now insert LF from eq. (30) in eq. (29) and obtain

$$\rho_o m^2 L\ddot{G} + \nabla\rho_o \cdot \nabla\ddot{G} + \frac{1}{\eta_o m^2} \nabla\left[\dot{F} - \underline{u} \times \nabla F_o \cdot \nabla\dot{G} - (m^2 \underline{u} \times \nabla\eta_o \cdot \nabla G) \cdot \right.$$

$$\left.\left(LF_o + \frac{2\pi n}{Lm} \cdot 2 f_o\right)\right] \cdot \nabla F_o \times \underline{u} + \nabla LF_o \times \nabla F \cdot \underline{u} = 0 . \qquad (31)$$

After using eq. (14) we can write down the final system of two equations in a matrix operational form:

$$\begin{pmatrix} \rho_o m^2 L + \nabla\rho_o \cdot \nabla & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{G} \\ \ddot{F} \end{pmatrix} + \begin{pmatrix} -\frac{\underline{u} \times \nabla F_o}{\eta_o m^2} \cdot \nabla\, \underline{u} \times \nabla F_o \cdot \nabla & \frac{\underline{u} \times \nabla F_o}{\eta_o m^2} \cdot \nabla \\ -\frac{\underline{u} \times \nabla F_o}{\eta_o m^2} \cdot \nabla & \frac{1}{\eta_o m^2} \end{pmatrix} \begin{pmatrix} \dot{G} \\ \dot{F} \end{pmatrix}$$

$$+\begin{pmatrix} -\dfrac{\underline{u}\times\nabla F_o}{\eta_o}\cdot\nabla(LF_o)\,\underline{u}\times\nabla\eta_o\cdot\nabla & \underline{u}\times\nabla(LF_o)\cdot\nabla \\ & \\ -\underline{u}\times\nabla(LF_o)\cdot\nabla & -L \end{pmatrix}\begin{pmatrix} G \\ \\ F \end{pmatrix} = 0 \tag{32}$$

It can be shown as in ref. [1] that the three matrix operators of eq.(32) are symmetric and that on $\begin{pmatrix}\dot{G}\\ \dot{F}\end{pmatrix}$ is also positive definite. This allows previous techniques [3] to be used. The necessary and sufficient condition for stability in the tokamak scaling and circular cross-sections is

$$\begin{aligned} \delta W = & \int\left(-\frac{dj_o}{dr}\right)(\underline{u}\times\underline{e}_r\cdot\nabla G)(\underline{u}\times\nabla F_o\cdot\nabla G)\,d\tau \\ & +2\int\left(-\frac{dj_o}{dr}\right)(\underline{u}\times\underline{e}_r\cdot\nabla G)\,F\,d\tau \\ & -\int FLF\,d\tau \end{aligned} \tag{33}$$

where $\underline{e}_r$ is a radial unit vector

It holds that $\nabla F_o = 0$ at the position r where the pitch of the mode is the same as that of the magnetic lines. (This singularity is well known in plasma Physics.) So $\underline{u}\times\nabla F_o$ has both signs around this singluarity, and δW can always be made negative by concentrating the test function G at the negative side of the first integrand and choosing $F \simeq 0$. Instability is always possible. This is well verified by the numerical calculations done in [7] .

Discussion and Conclusions

The criterion found in this paper would persist even if finite Larmor radius effects and viscosity were taken into account. These effects and resistivity would lead to an equation of the type:[4]

$$N \ddot{\xi} + (F + M) \dot{\xi} + Q \xi = 0 ,$$

with N and M symmetric and positive, Q symmetric, and F antisymmetric [5]. Q would be the last operator of eq. (32), F would be due to Larmor radius effects and M would contain dissipation due to viscosity and resistivity. This equation is treated in ref. [4], where it is proved that $(\xi, Q\xi) > 0$ is necessary and sufficient for stability, this being criterion (33). Of course, the growth rates would be altered by the new physical effects.

The validity of eq. (20) is also discussed in refs. [4,6]. Inclusion of heat conductivity along the magnetic lines could perturb [4] the symmetry of the operators in eq. (32).

The importance of criterion (33) is that it allows a scaling-free descision on all resistive instabilities in the circular cylinder including resistive kinks, tearings etc......

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