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Stability of Tokamaks with Respect to Slip Motions

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Abstract

Using the energy principle in tokamaks we investigate a class of perturbations which, if unstable, cannot be stabilized by the toroidal main field. On the assumptions of usual tokamak ordering and in the limit of infinite aspect ratio, these perturbations are shown to be minimizing among all axisymmetric perturbations. In the case of finite aspect ratio, a detailed stability analysis is carried out using a constant pressure surface current model with elliptic, triangular or rectangular plasma cross - section. Definite stabilization by toroidal effects and by β_p is demonstrated.

1. Introduction

In tokamak plasmas surrounded by vacuum instabilities may occur which cannot be stabilized by the toroidal magnetic field, however strong it may be. If they occur, the plasma slips, so to speak, through the main field without increasing the energy of the field.

In Section 2, a general investigation of these slip motions for toroidal plasmas with arbitrary, smooth pressure profile is carried out on the basis of the energy principle. In the limit of infinite aspect ratio and on the assumptions of usual tokamak ordering, they are compared with the most general minimizing perturbations previously found under such conditions [1], [2], [3].

For a toroidal constant-pressure tokamak model with surface currents, a detailed evaluation of stability is carried out, Section 3 being concerned with equilibrium properties and Section 4 with the reduction of the stability problem to a one-dimensional problem. Section 5 provides some comments about the numerical procedure which was used for solving this one-dimensional problem. Finally, Section 6 is devoted to a discussion of the results obtained.

Wall stabilization is not considered for reasons to be discussed in Section 6. Correspondingly, the external conductors producing the confining equilibrium field are assumed to be completely permeable to the perturbational field.

2. General theory of slip motions.

In ideal MHD theory, inside the plasma the magnetic field $\underline{B}$ and current $\underline{j}$ are given by

$$\underline{B} = \nabla\theta \times \nabla\psi + \Lambda \nabla\theta, \quad (2.1)$$

$$\underline{j} = -(\Lambda\Lambda' + R^2 p') \nabla\theta + \Lambda' \nabla\psi \times \nabla\theta, \quad (2.2)$$

where the pressure p and the quantity Λ are functions of the stream function ψ , and R, θ, z are cylindrical coordinates (see Fig. 1). The geometry of equilibrium is determined by prescribing the functions p , $g(\psi) = \Lambda\Lambda'$ and by prescribing the plasma boundary S by $\psi = \psi_s$. Since the equilibrium equation

$$\frac{\partial^2 \psi}{\partial R^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} = -R^2 p' - g' \quad (2.3)$$

contains only the derivative of g , in Λ an integration constant Λ_v is free which may be chosen to represent the external toroidal vacuum field surrounding the plasma: $\Lambda = (\Lambda_v^2 + 2g)^{1/2}$ and $\underline{B}_v^{\text{tor}} = \Lambda_v \nabla\theta$ if g is normalized by $g(\psi_s) = 0$. In other words, the strength of the external vacuum field may be chosen independently of $p(\psi)$, $g'(\psi)$ and of the geometry of flux surfaces.

We shall now look for the implications which this well known arbitrariness has on stability. For smooth pressure profiles with $p(\psi_s) = 0$ according to [4] the second variation of the energy is given by

$$\begin{aligned} \delta^2 W &= \delta W_{Pl} + \delta W_v, \\ \delta W_{Pl} &= \frac{1}{2} \int_{Pl} \left[\text{curl}^2(\underline{\xi} \times \underline{B}) + (\underline{j} \times \underline{\xi}) \cdot \text{curl}(\underline{\xi} \times \underline{B}) + (\underline{\xi} \cdot \nabla p) \nabla \cdot \underline{\xi} + \frac{5}{3} p (\nabla \cdot \underline{\xi})^2 \right] d\tau, \\ \delta W_v &= \frac{1}{2} \int_V (\delta \underline{B}_v)^2 d\tau = \frac{1}{2} \int_S (\underline{B}_v \cdot \delta \underline{B}_v) \underline{\xi} \cdot d\underline{\sigma}. \end{aligned} \quad (2.4)$$

The surface integral representation of δW_v is related to the first form in the usual way via Gauss's theorem using the assumption of completely permeable external wires made in the introduction. In δW_{Pl} only the first two terms depend on Λ_v . Hence, for large Λ_v we have the expansion

$$\begin{aligned} \delta W_{Pl} &= \frac{1}{2} \Lambda_v^2 \int_{Pl} d\tau \text{curl}^2(\underline{\xi} \times \nabla \theta) + \frac{1}{2} \Lambda_v \int_{Pl} d\tau [\text{curl}(\underline{\xi} \times \nabla \theta)] \cdot \\ &\quad \cdot \{ 2 \text{curl}[\underline{\xi} \times (\nabla \theta \times \nabla \psi)] + (g' + R^2 p')(\underline{\xi} \times \nabla \theta) \} + \dots \end{aligned}$$

It is seen that any fixed perturbation $\underline{\xi}$ which is unstable for certain values of Λ_v may be stabilized by letting Λ_v become large enough, provided that $\int d\tau \text{curl}^2(\underline{\xi} \times \nabla \theta) \neq 0$. This stabilization is not possible for all perturbations with

$$\text{curl}(\underline{\xi} \times \nabla \theta) = 0 \quad (2.5)$$

since, in addition, δW_v becomes independent of Λ_v : according to ref. [5], eq. (43), δW_v depends on Λ_v only through the values of

$\nabla \cdot \underline{B}_v(\underline{\xi} \cdot \nabla \psi)$ on S , which with $\underline{B}_v = \Lambda_v \nabla \theta + \nabla \theta \times \nabla \psi$ may be written in the form

$$\nabla \cdot \underline{B}_v(\underline{\xi} \cdot \nabla \psi) = \Lambda_v \nabla \psi \cdot \text{curl}(\underline{\xi} \times \nabla \theta) + (\nabla \theta \times \nabla \psi) \cdot \nabla(\underline{\xi} \cdot \nabla \psi)$$

Independence of Λ_v is evident for all $\underline{\xi}$ which satisfy eq. (2.5).

Equation (2.5) may be integrated to give

$$\underline{\xi} = R^2 \nabla \theta \times \nabla \varphi + \mu \nabla \theta, \quad (2.6)$$

with

$$\varphi = \varphi(R, z), \quad \mu = \mu(R, \theta, z).$$

Since for any closed curve in an R, z plane we get from eqs. (2.5)

and (2.6)

$$0 = \oint (\underline{\xi} \times \nabla \theta) \cdot d\underline{l} = \oint \nabla \varphi \cdot d\underline{l},$$

φ must be single-valued.

The perturbations (2.6) may be characterized as 'slip motions', since for them the plasma slips through the external toroidal field without changing it.

If we now insert eqs. (2.1), (2.2) and (2.6) in (2.4), we obtain after some manipulations using Gauss's theorem and $\int_0^{2\pi} (\partial \mu / \partial \theta) d\theta = 0$

$$\begin{aligned} \delta W_{Pl} = \frac{1}{2} \int d\tau \{ & [(\underline{e}_\theta \times \nabla \psi) \cdot \nabla \frac{\mu}{R^2}]^2 + (\nabla X)^2 / R^2 \\ & - (g' + R^2 p') [\nabla \theta \times \nabla \varphi] \cdot \nabla X - p' X \nabla \cdot (R^2 \nabla \theta \times \nabla \varphi) \\ & + 4 \cdot \frac{5}{3} p (\partial \varphi / \partial z)^2 + [\frac{5}{3} p + (\nabla \psi)^2] (\frac{\partial \mu}{\partial \theta} \frac{1}{R^2})^2 \} , \end{aligned} \quad (2.7)$$

where

$$\underline{X} = -(\underline{\xi} \cdot \nabla \psi) = R^2 (\nabla \theta \times \nabla \psi) \cdot \nabla \psi$$

and where $\underline{e}_\theta = R \nabla \theta$ is a unit vector in the θ direction.

Since δW_v is independent of μ , $\delta^2 W$ is minimized with respect to μ if

$$\frac{\partial}{\partial \theta} \frac{\mu}{R^2} = 0, \quad (\underline{e}_\theta \times \nabla \psi) \cdot \nabla \frac{\mu}{R^2} = 0$$

i.e. if

$$\mu = R^2 f(\psi).$$

$\mu \nabla \theta$ then represents rigid rotations of the individual flux surfaces, the rotation angle depending on ψ .

This yields

$$\delta W_{\mu} = \frac{1}{2} \int d\tau \left\{ \left(\frac{\nabla X}{R} \right)^2 - (g' + R^2 p') (\nabla \theta \times \nabla \psi) \cdot \nabla X - 2 p' X \frac{\partial \varphi}{\partial z} + 4 \cdot \frac{5}{3} p \left(\frac{\partial \varphi}{\partial z} \right)^2 \right\}. \quad (2.8)$$

Since δW_{μ} is now independent of μ , the class of perturbations left is effectively

$$\underline{\xi} = R^2 (\nabla \theta \times \nabla \psi), \quad (2.9)$$

which is a subclass of the totality of axisymmetric perturbations.

Since $\nabla \cdot \underline{\xi} = 2 \frac{\partial \varphi}{\partial z}$, one obtains for $\varphi = \varphi(R)$ incompressible motions representing R -dependent vertical shifts. All other $\underline{\xi}$ in eq. (2.9) are compressible and become incompressible only in the limit

aspect ratio $A \rightarrow \infty$. For the constant pressure model with surface currents considered in the next section we have $\text{curl}(\xi \times \underline{B}) \sim \text{curl}(\xi \times \nabla \theta) = 0$ inside the plasma, where the slip motions are thus interchanges.

Assuming the usual tokamak ordering $B_p/B_\theta \sim (\rho/B_\theta^2)^{1/2} \sim 1/A$ (B_p = poloidal field, B_θ = toroidal field), we shall now consider the limit $R_0 \sim A \rightarrow \infty$ (R_0 = radial position of magnetic axis). In order to maintain finite quantities, we introduce $R = R_0 + x$ and redefine $\psi(R, z) \rightarrow R_0 \psi(x, z)$, $\varphi(R, z) \rightarrow \varphi(x, z)/R_0$, $\Lambda \rightarrow R_0 \Lambda$ and $X \rightarrow R_0 X$.

According to equ. (2.2) the toroidal current density j_θ becomes a function of ψ alone, and in equ. (2.8) only the first two terms contribute to δW_{PI} , yielding

$$\delta W_{PI} = \frac{1}{2} \int_{PI} d\tau \{ (\nabla X)^2 + j_\theta (\underline{e}_\theta \times \nabla \varphi) \cdot \nabla X \}, \quad (2.10)$$

where

$$X = (\underline{e}_\theta \times \nabla \psi) \cdot \nabla \varphi.$$

Using the identity $\nabla \cdot (j_\theta X \underline{e}_\theta \times \nabla \varphi) = j_\theta (\underline{e}_\theta \times \nabla \varphi) \cdot \nabla X - X^2 \frac{dj_\theta}{d\psi}$, we finally obtain

$$\begin{aligned} \delta W_{PI} = \frac{1}{2} \int_{PI} d\tau \left[(\nabla X)^2 + \frac{dj_\theta}{d\psi} X^2 \right] \\ + \frac{1}{2} \int_S d\sigma X j_\theta \left(\frac{\nabla \psi}{|\nabla \psi|} \times \underline{e}_\theta \right) \cdot \nabla \varphi. \end{aligned} \quad (2.11)$$

This expression is identical with the expression found previously

[1], [2], [3] to be minimizing among all axisymmetric

perturbations in the limit considered here, and thus in this

case the exact stability boundaries may be derived from a slip motion

analysis. In the case of finite aspect ratio such an analysis should still yield a good approximation. Since for slip motions minimization is thereby required only with respect to the scalar quantity φ , an enormous simplification relative to the full axisymmetric problem is achieved.

As an application, we shall now carry out a detailed slip motion analysis for a constant pressure plasma model with surface currents, where the minimization with respect to φ can be done analytically.

3. Constant pressure model with surface currents - equilibrium.

For a detailed description of the model considered here the reader is referred to the literature ([6], [7]). Using dimensionless variables, we consider plasma surfaces S of the form

$$F(R, \theta, z) = e^2(R-1)^2 + (1+\tau_3^2)z^2 - 2A\tau_3(R-1)z^2 - A^2\tau_4(R-1)^2z^2 = \frac{e^2}{A^2} \quad (3.1)$$

R and z are normalized in such a way that the circle $R-1 = z = 0$ is inside the plasma. The aspect ratio is defined by

$$A = \frac{R_1 + R_2}{R_2 - R_1} \quad (3.2)$$

(see Fig. 1), the axis ratio

$$e = \frac{z_2 - z_1}{R_2 - R_1} \quad (3.3)$$

is a measure of ellipticity, while τ_3 and τ_4 describe triangular

and rectangular deformations of the plasma cross-section respectively. We shall only consider the case $\tau_3 \neq 0, \tau_4 = 0$ with shapes like those shown in Fig.6a and the case $\tau_3 = 0, \tau_4 \neq 0$ with shapes like those shown in Fig.6b. Since in a $\theta = \text{const}$ plane the curves given by eq. (3.1) should be closed, we get the restrictions

$$\begin{aligned} -1 &\leq \tau_3 \leq 1, \\ -\infty &< \tau_4 \leq 1. \end{aligned} \quad (3.4)$$

The magnetic field inside the plasma is purely toroidal

$$\underline{B}_p = \frac{\Lambda_p}{R} \underline{e}_\theta, \quad (3.5)$$

its values on the vacuum side of S are

$$\underline{B}_v = \frac{\Lambda_v}{R} \underline{e}_\theta + B \underline{t}, \quad (3.6)$$

where

$$\underline{t} = \underline{n} \times \underline{e}_\theta, \quad (3.7)$$

and $\underline{n}$ is the normal vector on S . Λ_p and Λ_v are constants, whereas B varies on S . In using dimensionless variables, we choose the normalization

$$B_0 = B(R=1, z=e/A) = 1. \quad (3.8)$$

Defining beta poloidal by

$$\beta_p = 2p/B_0^2, \quad (3.9)$$

the equilibrium relation

$$p + \frac{1}{2} \underline{B}_{Pl}^2 = \frac{1}{2} \underline{B}_v^2$$

may be written

$$\beta_p + \Lambda_p^2 / R^2 = \Lambda_v^2 / R^2 + B^2. \quad (3.10)$$

Combining eqs. (3.8) and (3.10), we finally obtain

$$\Lambda_p^2 - \Lambda_v^2 = 1 - \beta_p \quad (3.11)$$

and

$$B = \frac{1}{R} [1 + \beta_p (R^2 - 1)]^{1/2}. \quad (3.12)$$

On the surfaces considered we have $R \geq 1 - \frac{1}{A}$ and hence for B to be real the restriction

$$\beta_p \leq \beta_p^* = \frac{A^2}{2A - 1} \quad (3.13)$$

must be observed.

Although not needed in this paper, for completeness we add relations for the safety factor q and the total beta. From eq. (3.6) one readily obtains

$$q = \Lambda_v q_s / A, \quad (3.14)$$

where

$$q_0 = \frac{A}{2\pi} \oint \frac{dl}{R^2 B}, \quad (3.15)$$

and where the line integration extends along S in the R, z plane.

Defining

$$\beta = p / \left(p + \frac{1}{2} B_{PL}^2 |_{R=1} \right) \quad (3.16)$$

we obtain from eq. (3.16) with eqs. (3.5), (3.8), (3.9), (3.11) and (3.14)

$$\beta = \beta_p / (1 + A^2 q^2 / q_0^2) \quad (3.17)$$

If $\beta_p > 1$, the restriction

$$q \geq \frac{q_0}{A} (\beta_p - 1)^{1/2} \quad (3.18)$$

is obtained from eq. (3.17) in order to have $\beta \leq 1$ in keeping with the definition (3.16).

4. Constant pressure model with surface currents - stability.

In the presence of surface currents the expression for $\int W^2$ must be supplemented with a surface contribution, whereas the constancy of pressure leads to a simplification of the plasma contribution:

$$\delta^2 W = \delta W_{Pl} + \delta W_S + \delta W_v, \quad (4.1)$$

$$\delta W_{Pl} = \frac{1}{2} \int_{Pl} d\tau \left\{ \text{curl}^2 (\underline{\xi} \times \underline{B}_{Pl}) + \frac{5}{3} p (\nabla \cdot \underline{\xi})^2 \right\}, \quad (4.2)$$

$$\delta W_S = \frac{1}{2} \int_S d\sigma \left\{ (\underline{\xi} \cdot \underline{n})^2 \underline{n} \cdot \nabla \left(\frac{1}{2} \underline{B}_v^2 - \frac{1}{2} \underline{B}_{Pl}^2 \right) \right\}, \quad (4.3)$$

$$\delta W_v = \frac{1}{2} \int_S d\sigma (\underline{B}_v \cdot \delta \underline{B}_v) (\underline{\xi} \cdot \underline{n}). \quad (4.4)$$

According to eqs. (3.5) - (3.6) δW_S depends on Λ_v only through the combination $\Lambda_p^2 - \Lambda_v^2 = 1 - \beta_p$ (eq. (3.11)), where β_p may be chosen independently of Λ_v , and, therefore, the general statements on slip motions in Section 2 apply without change.

The procedure of minimization will be performed according to ref. [4].

Keeping prescribed values f_n of $(\underline{\xi} \cdot \underline{n})$ on S , we shall first minimize δW_{Pl} with respect to the perturbations (2.9) and relate δW_v to the values f_n . Then, we shall minimize $\delta^2 W$ with respect to f_n , thereby observing the side condition

$$\oint \frac{f_n}{R} dl = 0, \quad (4.5)$$

which is a consequence of eq. (2.9) and the single-valuedness of φ .

Since the equilibria considered are symmetric with respect to the plane $z = 0$, it is useful to split f_n into symmetric and antisymmetric

contributions :

$$\xi_n = \xi_n^s + \xi_n^a, \quad (4.6)$$

where on S with $z = z(R)$

$$\begin{aligned} \xi_n^s &= \frac{1}{2} [\xi_n(R, z) + \xi_n(R, -z)], \\ \xi_n^a &= \frac{1}{2} [\xi_n(R, z) - \xi_n(R, -z)]. \end{aligned} \quad (4.7)$$

Obviously, the side condition (4.5) reduces to

$$\oint \frac{\xi_n^s}{R} dl = 0. \quad (4.8)$$

a) Minimization of δW_{Pl} .

For the perturbations (2.9) we obtain from eq.(4.2) with eqs.(3.8)

and (3.9) after performing the Θ - integration

$$\delta W_{Pl} = \frac{10}{3} \pi \beta_p \iint_{Pl} R \varphi_z^2 dR dz. \quad (4.9)$$

At φ the subscript z means partial differentiation with respect to z .

Variation with respect to φ yields the Euler equation

$$\varphi_{zz} = 0 \quad (4.10)$$

with the general solution¹⁾

$$\varphi = f(R)z + g(R). \quad (4.11)$$

For this, δW_{Pl} assumes a true minimum, as is easily seen when

1)

If for $z \gg 0$ or $z \leq 0$ the plasma boundary $z(R)$ is multi-valued as in doublet configurations, $f(R)$ and $g(R)$ become multi-valued functions. This case will be treated elsewhere.

for fixed R the one-dimensional minimization with respect to z is done first. The functions $f(R)$ and $g(R)$ are determined by the boundary condition

$$\underline{n} \cdot R^2 (\nabla \theta \times \nabla \varphi) = f_n(l), \quad (4.12)$$

which expresses the prescription of $\underline{f} \cdot \underline{n}$ on S . The variable l is the arc-length of the plasma boundary in the R, z plane, orientation and origin being chosen according to Fig. 1. From equ. (4.12) we obtain

$$\varphi(R, z(R)) = \int_0^{l(R)} \frac{f_n^s(l')}{R(l')} dl' + \int_0^{l(R)} \frac{f_n^a(l')}{R(l')} dl' + \text{const}, \quad (4.13)$$

and, using obvious symmetry properties together with the condition (4.8), a comparison of eqs. (4.11) and (4.13) yields

$$\left. \begin{aligned} f(R) &= \int_0^{l(R)} \frac{f_n^s(l')}{R(l')} dl' / z(R), \\ g(R) &= \int_0^{l(R)} \frac{f_n^a(l')}{R(l')} dl' + \text{const}. \end{aligned} \right\} \quad (4.14)$$

From eqs. (2.9), (4.11) and (4.14) for any $f_n(l)$ the minimizing $\underline{f}$ may be obtained.

According to eqs. (4.9) and (4.11) δW_{PI} depends only on $f(R)$.

Using eq. (4.14) and $dR = -n_z dl$, where n_z is the z -component of $\underline{n}$, we easily obtain

$$\delta W_{PI} = -\frac{10}{3} \pi \beta_p \oint dl \frac{R n_z}{z} \left(\int_0^l \frac{f_n^s(l')}{R(l')} dl' \right)^2. \quad (4.15)$$

We note that δW_{pl} has now become independent of f_n^a .

b) Calculation of δW_s .

In eq. (3.6) both contributions to $\underline{B}_v$ are vacuum fields and hence

$$\nabla \frac{B^2}{2} = B \underline{t} \cdot \nabla B \underline{t} = B^2 \underline{t} \cdot \nabla \underline{t} + \underline{t} (\underline{t} \cdot \nabla \frac{B^2}{2}) .$$

Since $-\underline{n} \cdot (\underline{t} \cdot \nabla \underline{t}) = \kappa$ is the poloidal curvature of $\mathcal{S}$, we obtain from eqs. (3.5), (3.6), (3.11) and (4.3)

$$\delta W_s = \frac{1}{2} \int_S d\sigma f_n^2 \left(\frac{1-\beta_p}{R^3} n_R - \kappa B^2 \right) ,$$

where n_R is the R -component of $\underline{n}$. With the splitting (4.6) and carrying out the θ -integration, we obtain

$$\delta W_s = \pi \oint dl [(\underline{f}_n^s)^2 + (\underline{f}_n^a)^2] \left(\frac{1-\beta_p}{R^3} n_R - \kappa B^2 \right) R . \quad (4.16)$$

c) Calculation of δW_v .

As already mentioned in the introduction, we shall not consider wall stabilization or any effect of the external conductors on stability. For axisymmetric perturbations the vacuum field perturbation $\delta \underline{B}_v$ is purely poloidal because of the boundary conditions at infinity, and thus from eqs. (3.6) and (4.4) we obtain

$$\delta W_v = \pi \oint dl R B f_n y , \quad (4.17)$$

where

$$y = (\underline{n} \times \underline{e}_\theta) \cdot \delta \underline{B}_v . \quad (4.18)$$

Splitting $y = y^s + y^a$ by analogy with eqs. (4.6) - (4.7), we obtain

$$\delta W_v = \delta W_v^s + \delta W_v^a, \quad (4.19)$$

$$\delta W_v^s = \pi \oint dl R B f_n^s y^s, \quad \delta W_v^a = \pi \oint dl R B f_n^a y^a.$$

To determine y , it is necessary, in principle, to solve the so-called electrical boundary value problem for the vectorial Poisson equation in the whole vacuum region V surrounding the plasma:

$$\begin{aligned} \Delta \underline{A} &= 0 & \text{in } V, \\ \underline{n} \times \underline{A} &= -f_n \underline{B}_v & \text{on } S, \end{aligned} \quad (4.20)$$

where

$$\delta \underline{B}_v = \text{curl } \underline{A}. \quad (4.21)$$

However, according to eqs. (4.17), (4.18) and (4.20) we need only the tangential components of $\text{curl } \underline{A}$ on S , and these may be determined directly from an integral equation on S with reference to a method by Kress [8].

We shall proceed here as in ref. [5] and make use of the equations (43) - (47) from there. Note that $\nabla\psi/|\nabla\psi| = \underline{n}$, and that the definitions of y, δ and $L(\ell, \ell')$ are here slightly modified. The equation for $\text{Div } \underline{\gamma}^*$ with $\underline{\gamma}^* = -f_n \underline{B}_v$ is adapted by setting $|\nabla\psi| = RB$, and with $\partial f_n / \partial \theta = 0$ we have here

$$\operatorname{Div} \underline{y}^* = -\frac{1}{RB} \operatorname{div} \underline{B}_v \{ \underline{f}_n RB \} = -\frac{1}{R} \frac{d}{d\ell} (\{ \underline{f}_n RB \}).$$

For $\underline{y}$ we obtain the integral equation

$$\underline{y} = \frac{1}{2\pi} \oint K(\ell, \ell') \underline{y}' d\ell' + \frac{1}{2\pi} \oint L(\ell, \ell') \sigma' d\ell', \quad (4.22)$$

where

$$K(\ell, \ell') = \frac{4R'}{r_0^3(1-k^2)} \left\{ -\eta_R R' E + \frac{1}{k^2} [\eta_R R + \eta_z(z-z')] [(2-k^2)E - 2(1-k^2)K] \right\},$$

$$L(\ell, \ell') = \frac{4R'}{r_0^3(1-k^2)} \left\{ [\eta_z R - \eta_R(z-z')] E - \frac{\eta_z R'}{k^2} [(2-k^2)E - 2(1-k^2)K] \right\},$$

$$\sigma = \frac{1}{R} \frac{d}{d\ell} (RB \{ \underline{f}_n \}),$$

$$k = \left(\frac{4RR'}{(R+R')^2 + (z-z')^2} \right)^{1/2}, \quad r_0 = \frac{2}{k} (RR')^{1/2},$$

$$E = E(k) = \int_0^{\pi/2} (1-k^2 \sin^2 \varphi)^{1/2} d\varphi, \quad K = K(k) = \int_0^{\pi/2} (1-k^2 \sin^2 \varphi)^{-1/2} d\varphi.$$

With ℓ and $\bar{\ell}$ belonging to the points $R, z(R)$ and $R, -z(R)$ respectively,

it follows from the definitions above that

$$K(\ell, \ell') = K(\bar{\ell}, \bar{\ell}') \quad \text{and} \quad L(\ell, \ell') = -L(\bar{\ell}, \bar{\ell}').$$

Splitting $\sigma = \sigma^s + \sigma^a$, we get

$$\sigma^s = \frac{1}{R} \frac{d}{d\ell} (RB \{ \underline{f}_n^s \}), \quad \sigma^a = \frac{1}{R} \frac{d}{d\ell} (RB \{ \underline{f}_n^a \}), \quad (4.23)$$

and it is easily seen that $\oint K(\ell, \ell') \underline{y}^s d\ell'$ and $\oint L(\ell, \ell') \sigma^a d\ell'$ are symmetric with respect to the $z=0$ plane, whereas $\oint K(\ell, \ell') \underline{y}^a d\ell'$

and $\oint L(e, e') \delta^{s'} dl'$ are antisymmetric. Consequently, eq. (4.22) separates into

$$y^s = \frac{1}{2\pi} \oint K(e, e') y^{s'} dl' + \frac{1}{2\pi} \oint L(e, e') \delta^{a'} dl', \quad (4.24a)$$

$$y^a = \frac{1}{2\pi} \oint K(e, e') y^{a'} dl' + \frac{1}{2\pi} \oint L(e, e') \delta^{s'} dl'. \quad (4.24b)$$

Since the homogeneous integral equation belonging to eq. (4.22) has a solution (it corresponds to a poloidal magnetic field which is singularity-free in the whole vacuum region V and is tangential on S), according to ref. [8] a side condition must be added to eq. (4.22) in order to restore the uniqueness of solutions [9] of the original problem (4.20).

In this paper we shall use a much simpler side condition than that in ref. [8]. It is derived in the Appendix and results in

$$\oint dl [R v y^s + R B f_n^s \underline{t} \cdot \text{curl } \underline{v}] = 0, \quad (4.25)$$

where v and $\underline{t} \cdot \text{curl } \underline{v}$ are given by eqs. (A6) - (A8) of the Appendix. The equivalence of our side condition to that of ref. [8] is shown elsewhere [10], certain generalizations being included.

Since the side condition (4.25) is independent of y^a , only eq. (4.24a) is affected by it, and according to ref. [8] eq. (4.24b) must have a unique antisymmetric solution without side condition. This behaviour may be attributed to the fact that the solution of the homogeneous

equation belonging to eq. (4.22) is symmetric.

d) Summary.

Summarizing the results of Section 4a) - c), we have

$$\delta^2 W = \delta W^s + \delta W^a, \quad (4.26)$$

where δW^s depends only on f_n^s , and δW^a only on f_n^a . Thus, the minimization with respect to f_n still pending may be carried out for f_n^s and f_n^a independently. The plasma and surface contributions to δW^s and δW^a , given by eqs. (4.15) - (4.16) with $\delta W_{pl}^a = 0$, are explicit expressions of f_n^s and f_n^a respectively. The functions y^s and y^a entering the vacuum contributions (4.19) are functionals of f_n^s and f_n^a respectively, determined by eq. (4.22a) plus side condition (4.23) and eq. (4.22b) respectively. In the minimization procedure normalizations of the form

$$\oint g(\ell) (f_n^s)^2 d\ell = 1 \quad \text{and} \quad \oint g(\ell) (f_n^a)^2 d\ell = 1 \quad (4.27)$$

are here the most convenient ones. (We recall that the minimum value of $\delta^2 W$ and the minimizing f_n depend on this choice of normalization and on the choice of $g(\ell)$. However, the boundary between stable and unstable is unaffected.) In minimizing δW^s with respect to f_n^s , the side condition (4.8) must be observed in addition to (4.27).

Obviously, the problem of minimization is now just one-dimensional

since only differentiations and integrations with respect to ℓ are involved. This part of the problem was carried out numerically.

5. Numerical procedure

For solving the integral equations (4.24a) with side condition (4.25) and (4.24b) we applied methods used by Martensen [11] in solving Neumann's problem for toroidal boundaries.

A complication is provided by the singularity of the kernels

$K(\ell, \ell')$ and $L(\ell, \ell')$ at $\ell = \ell'$. This singularity may be removed from eq. (4.22) and eqs. (4.24a,b) by using the relations $\frac{1}{2\pi} \oint K(\ell, \ell') d\ell - 1 = \oint L(\ell, \ell') d\ell = 0$, yielding

$$\frac{1}{2\pi} \oint [K(\ell', \ell) y - K(\ell, \ell') y'] d\ell' = \frac{1}{2\pi} \oint [L(\ell, \ell') \delta' - L(\ell', \ell) \delta] d\ell'. \quad (5.1)$$

Since the integrands in eq. (5.1) may still become large for small $(\ell - \ell')$, the kernels are each decomposed into a regular contribution and into a singular one. Changing from ℓ to the polar angle u with centre at $R = 1$, $z = 0$, all nonsingular functions $F(u)$ including the regular parts of the kernels are now approximated by finite Fourier series $S_{2N}(u)$ ($2N$ Fourier coefficients) such that at $2N$ equally distributed grid points u_i one has

$$S_{2N}(u_i) = F(u_i) = F_i, \quad i = 1, \dots, 2N$$

(trigonometric interpolation). With this and the proper decomposition of the kernels, all contributions in eq. (5.1) including the singular ones may be analytically transformed into finite sums. By this procedure a system of $2N+1$ equations for the $2N$ unknown values of y at the gridpoints (splitting into N unknown values of y^s and y^a respectively for a gridpoint distribution which is symmetric with respect to the $z=0$ plane) is obtained from eqs. (4.24a), (4.24b) and (4.25). Since summation of the N equations coming from eq. (4.24a) yields an identity, any one of them may be omitted and replaced by the equation coming from the side condition (4.25). By inverting the matrix of these equations the integral equation problem is finally solved.

Evaluating, again by trigonometric interpolation, the integrals which make up δW^s and δW^a , expressions of the form

$$\delta W^s = \sum_{i,k=1}^N W_{ik}^s f_{ni}^s f_{nk}^s, \quad \delta W^a = \sum_{i,k=1}^N W_{ik}^a f_{ni}^a f_{nk}^a \quad (5.2)$$

are obtained for minimization, an equation coming from eq. (4.8) thereby having to be observed as a side condition as far as δW^s is concerned. However, the minimum of the expressions (5.2) was found not to converge with increasing N , the minimizing f_{ni} changing sign in each grid point. Since for a reasonable approximation there should be several grid points between two consecutive zeroes of any interpolated function, we consequently minimized only in the restricted class of perturbations

$$f_n(u) = \sum_{k=0}^{N^*-1} a_k \cos ku + \sum_{k=0}^{N^*} b_k \sin ku \quad (5.3)$$

with

$$N^* < N.$$

In our calculations with $N^* = \frac{2N}{3}$ numerical convergence was obtained in all cases considered, the minimizing f_n now representing rather smooth functions (see Figs. 6a, b). For circular or almost circular plasma cross-sections sufficient accuracy was obtained with $N = 12$, while for the considered equilibria with stronger deviation from circularity (see, for example, Figs. 6a, b) $N = 24$ was still sufficient. The computer used for our calculations was an IBM 360/91 at IPP in Garching. The computing time for calculating the minimum of $\delta^2 W$ for one specific equilibrium was ≈ 2.4 seconds for $N = 24$.

6. Results⁺) and conclusions.

Considering the limit $A \rightarrow \infty$ with usual tokamak ordering, tokamaks with circular cross-sections are marginally stable with respect to axisymmetric perturbations [13], [14], whereas for cross-sections with arbitrary elliptic shape [1] - [3] or with small rectangular deformation [15] instability with respect to certain axisymmetric perturbations was found. With these results as a starting point, our interest was mainly in the effect of smaller aspect ratio and,

⁺)

The following results were partly reported at the 7th Europ. Conf. on Plasma Physics and Contr. Fusion [12]. Owing to a mistake the curves presented there deviate rather slightly from the corresponding corrected curves in this paper. In particular we point out that the stable region may extend up to the boundary $\beta_p = \beta_p^*$ determined by the existence of equilibrium (see Figs. 4a, b).

furthermore, of β_p on stability.

In Fig. 2a for $A=4$, $\beta_p = 1/3$ and $\tau_4 = 0$ the stability boundaries $\delta W^s = 0$ (curve W_s) and $\delta W^a = 0$ (curve W_a) are shown in a τ_3, e plane. In the eye-shaped shaded region the plasma is stable, while outside either the symmetric ($\xi_n = \xi_n^s$) or the antisymmetric ($\xi_n = \xi_n^a$) minimizing perturbation or both are unstable. Also shown is the stability boundary for rigid vertical displacements W_r (stable region below), which shows good qualitative agreement with related curves for plasma with volume currents [5]. Thus, for nonlocalized axisymmetric gross motions such as considered here the constant pressure surface current model appears to be a reasonable approximation. As is seen, except for almost elliptical cross-sections the boundary W_a is much more pessimistic than W_r . (The two curves do not touch exactly, W_r passing slightly above W_a .) In Fig. 2b the analogous boundaries are shown in a τ_4, e plane for $\tau_3 = 0$. In Figs. 3a and 3b it is shown for different values of A how the stable region increases with decreasing A . Thus, a clear stabilization of the $A \rightarrow \infty$ instabilities by finite aspect ratio is demonstrated. On the other hand, the result is recovered that for $A \rightarrow \infty$ and β_p chosen in accordance with usual tokamak ordering only circular cross-sections are (marginally) stable.

Next we consider the influence of β_p on stability. In Fig. 4b for $A=4$, $\tau_3 = 0$ and for the three values $\tau_4 = -0.2, 0$ and 0.2 the boundaries $\delta W^a = 0$ (curves on the right) and $\delta W^s = 0$ (curves on the left) are shown in an e, β_p plane (stable region in the middle). In all cases the stable region increases with increasing β_p , the limitation $\beta_p \leq \beta_p^*$ being determined by the existence of equilibria. In

Fig. 5b widening of the stable region with increasing β_p is demonstrated for the whole range of possible τ_4 values.

The situation is somewhat different for triangular cross-sections ($\tau_4 = 0$), as shown in Figs. 4a and 5a. Here, only for $\tau_3 \lesssim 0$ (D-shape) does the stable region widen everywhere with increasing β_p , whereas for $\tau_3 \gtrsim 0$ only the boundary W_s moves on uniquely as β_p increases.

Now, we present some examples of what the minimizing perturbations look like in unstable situations. As was mentioned in Section 4d, the result depends somewhat on the normalization chosen, and an eigenmode analysis yielding per se exact growth rates is, of course, more adequate for predicting which instabilities would be observed in an experiment. Nevertheless, the most important unstable eigenmodes should be contained in the minimizing perturbations and thus still some valuable information is obtained. In Fig. 6a and 6b for $A = 4$ and $\beta_p = 1/3$ ten different unstable cross-sections are shown. From the minimizing $\underline{f}_n$ the full perturbation $\underline{f}$ was calculated using eqs. (2.9), (4.11) and (4.14). Its value and direction on S are indicated by arrows, the resulting shift and / or deformation of the plasma being indicated by dotted curves.

In the case with $e = 1.3$ the minimizing perturbation is an R -dependent vertical displacement, while in the case with $e = 0.8$ it is a z -dependent radial shift with a vertical stretching or shortening or both being

superimposed. Roughly speaking, it is characteristic of all cases shown that the perturbation is largest where the poloidal curvature of the plasma surface is largest. There is, however, no monotonic relation between the two.

If a plasma with smooth pressure profile and current distribution is approximated by one of the step profiles considered here, it appears reasonable to place the pressure jump of the step profile on an interior magnetic surface of the continuous profile enclosing the bulk of the plasma rather than let it coincide with the boundary. Since this interior surface has, on the one hand, a smaller aspect ratio and, on the other, smaller elongation and less triangularity or rectangularity than the boundary, there are two competitive effects of this way of approximation according to our results. For very elongated or strongly non-circular boundaries stabilization by the transition to the more circular interior surface is expected to dominate, whereas for rather circular boundaries the destabilizing effect of a decrease in aspect ratio is likely to prevail.

This section is concluded with a few comments on stabilization since non-circular cross-sections exhibiting instabilities of the kind discussed here have gained increasing interest because of several advantages. (See, for example, [16] and [17].) With an infinitely conductive wall placed close enough to the plasma surface, stabilization of these instabilities will certainly be possible. However, in future tokamaks with longer confinement times an external copper shell can no longer be considered as infinitely conductive.

Growth rates may then only be reduced by an outer wall, the stability boundaries being unaltered [18]. Moreover, for technical reasons a certain minimum distance between the plasma surface and an outer wall will have to be maintained, and then stabilization of modes with shorter wavelength can no longer be expected even from an infinitely conductive wall. Feedback stabilization appears as a possible alternative which will, however, be limited to situations where only one or few eigenmodes are unstable. It would therefore be of great importance to carry out a detailed eigenmode analysis of the instabilities considered in this paper. This will shortly become possible with the large stability codes being developed at present.

Appendix: Derivation of a side condition.

We consider the vector potential

$$\underline{v}(\underline{x}) = \int_G \frac{\underline{e}'_0}{|\underline{x} - \underline{x}'|} d\theta' \quad (A1)$$

of a constant current flowing along the circle C at $R=1, z=0$.

For both $\underline{v}$ and $\underline{A}$, the vector potential of $\delta \underline{B}_r$, we have

$$\text{curl curl } \underline{v} = \text{curl curl } \underline{A} = 0 \quad \text{in } V.$$

With

$$\nabla \cdot [(\text{curl } \underline{A}) \times \underline{v} + \underline{A} \times \text{curl } \underline{v}] = 0 \quad \text{in } V-S \quad (A2)$$

and since

$$|\text{curl} \underline{A} \times \underline{v}| \sim |\underline{A} \times \text{curl} \underline{v}| \sim \frac{1}{|x|^3} \quad \text{for } |x| \rightarrow \infty,$$

we obtain from (A2) by integration over V

$$\int_S d\Omega (\text{curl} \underline{A} \times \underline{v} + \underline{A} \times \text{curl} \underline{v}) \cdot \underline{n} = 0. \quad (\text{A3})$$

From eq. (A1) it follows that

$$\underline{v} = v(R, z) \underline{e}_\theta. \quad (\text{A4})$$

With this, eqs. (3.7) - (3.8), (4.18) and (4.20) we obtain from eq. (A3) after carrying out the θ -integration

$$\oint (R v_y + R B f_n \underline{t} \cdot \text{curl} \underline{v}) dl = 0. \quad (\text{A5})$$

$\underline{v}$ and $\text{curl} \underline{v}$ may be expressed in a known manner in terms of elliptical integrals:

$$v = 4 [(2 - k_c^2) K - 2E] / (k_c^2 r_c), \quad (\text{A6})$$

$$\begin{aligned} \underline{t} \cdot \text{curl} \underline{v} = & \frac{4}{r_c^3 (1 - k_c^2)} \left\{ n_R E \right. \\ & \left. - \frac{n_R R + n_z z}{k_c^2} [(2 - k_c^2) E - 2(1 - k_c^2) K] \right\}, \end{aligned} \quad (\text{A7})$$

where

$$\left. \begin{aligned} K &= K(k_c), & E &= E(k_c), \\ k_c &= 2\sqrt{R} [(1+R)^2 + z^2]^{-1/2}, & r_c &= 2\sqrt{R}/k_c. \end{aligned} \right\} \quad (\text{A8})$$

Symmetric functions with respect to the $z=0$ plane are $k_c, \tau_c, K, E, B, n_R, R$ and v ; antisymmetric functions are z, η_z . From this it follows that only y^s and f_n^s enter the side condition (A5).

The derivation presented here proves only the validity of the side condition (A5). It is shown in [10] that for the system (4.22), (A5) and for the system (4.24a), (4.25) respectively a solution exists which is unique.

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Fig. 1: Coordinate system.

Fig. 2: Stability boundaries w_s , w_α and w_r for symmetric, antisymmetric and rigid - vertical displacements respectively, a) for triangular and b) for rectangular cross - sections; shaded region stable.

Fig. 3: Stable domains in the e, τ_3 plane (Fig.a) and e, τ_4 plane (Fig.b) for several values of aspect ratio A .

a)

b)

Fig. 4: Stability boundaries W_s and W_a in an e, β_p plane for three values of τ_3 (Fig.a) and τ_4 (Fig.b); β_p^* = equilibrium limit; symbols indicate plasma shape.

Fig. 5: Stability domains in the e, τ_3 plane (Fig.a) and e, τ_4 plane (Fig.b) for several values of β_p .

Fig. 6: Examples of unstable cross - sections with minimizing displacements.