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Some Analytical Properties of the Trapped-ion Transport Equations of Kadomtsev and Pogutse

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IPP 6/144 February 1976

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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Abstract

In support of numerical computations of anomalous diffusion caused by the dissipative trapped-ion instability, several simple analytical properties and solutions of the trapped-ion transport equations of KADOMTSEV and POGUTSE are derived. The scaling of the anomalous diffusion coefficient is investigated by dimensional analysis, and a relation between the electric potential in the plasma and the anomalous diffusion flux is established.

1. Introduction

Theoretical predictions make it probable that in the next tokamak generation the particle transport and the energy transport (of the ions) in the plasma will be dominated by the dissipative trapped-ion instability (DTII) [1 - 8].

Two approaches have been adopted to obtain a quantitative description of the respective transport properties. On the one hand, one merely calculates the unstable parameter range and the growth rates of various modes without worrying about the magnitude of the anomalous transport. On the other, however, one may try to determine the time-asymptotic plasma state due to the DTII in order to calculate the anomalous diffusion or energy flow. This nonlinear problem calls for greater simplification of the physical assumptions than in the linear stability theory first mentioned. The present report deals with this second approach.

The DTII is most simply described by the dissipative transport equations, formulated by KADOMTSEV and POGUTSE (K.P.), for the ions and electrons trapped in the toroidal magnetic field [1-2] (see Sec. 2). These equations can, on the one hand, be numerically treated. It should thereby be borne in mind that the K.P. equations also admit of discontinuous solutions, which, however, are physically unrealistic. Effects omitted in the K.P. equations (finite gyroradii, neoclassical diffusion, Landau damping) lead one to expect smooth solutions. It is therefore a matter of finding a numerical code which does not produce any undesirable singularities, and it has to be verified that the anomalous transport calculated only depends insignificantly on the discretization chosen. Numerical solutions will be reported elsewhere.

In this report we deal with <u>analytical properties</u> of the K.P. equations. These analytical investigations were mainly conducted to check the correctness of the numerical solutions. It is also sometimes assumed here that the solutions are continuous or smooth (see Sec. 8). The K.P. equations are introduced in Sec. 2. In Sec. 3 the boundary conditions are derived. Invariance properties and special exact solutions of the K.P. equations are presented in Secs. 4 and 5. Section 6 deals with bounds for the solutions. Global equations of motion and equations for mean values are derived in Secs. 7 and 8. Section 9 proves the non-existence of special travelling waves (comparison with LaQuey et al., Ref. [9]). The scaling of the anomalous diffusion and a differential relation between the particle flux and mean electric potential are treated in Secs. 10 and 11. A summary is given in Sec. 12. In the Appendix special solution ansatzes are discussed.

Effects neglected by the K.P. equations include the correct influence of the electric field component parallel to B on the trapped and free particles and the influence of $\nabla\delta$ (δ = fraction of trapped particles) [6 - 7]. Both effects are ignored in the following for the sake of simplicity. The same applies to the depletion of trapped particles that may occur at higher temperatures [5, 8] .

2. Equations

The KADOMTSEV-POGUTSE equations [1, 2, 7] are

$$n^i_t + \nu_i (n^i - n^0) + \nabla \cdot (n^i \underset{\sim}{v}) = 0 \, , \tag{2.1}$$

$$n^e_t + \nu_e (n^e - n^0) + \nabla \cdot (n^e \underset{\sim}{v}) = 0 \, , \tag{2.2}$$

$$\Phi = \frac{T}{2eN_0} (n^i - n^e) \, , \tag{2.3}$$

$$\underset{\sim}{v} = - \frac{c}{B} \hat{z} \times \nabla \Phi = \hat{z} \times \nabla \left[A (n^i - n^e) \right] \, , \tag{2.4}$$

$$\nabla \underset{\sim}{B} = 0 \quad ; \quad \nabla \cdot \underset{\sim}{v} = 0 \, , \tag{2.5/6}$$

$$A = \frac{cT}{2eBN^0} \quad ; \quad T = \frac{2T_e T_i}{T_e + T_i} \, . \tag{2.7/8}$$

These equations describe the 2-dimensional $\underset{\sim}{E} \times \underset{\sim}{B}$ drift motion of the trapped ions and electrons in a plasma slab, taking into account collisions with a background of free ions and electrons. The variables n^i, n^e are the trapped particle densities ($n^{i,e} \geqq 0$), the subscript t denotes the derivative with respect to time, $\underset{\sim}{v}$ is the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift velocity, $\nu_i = \nu_{ci} / \delta_0^2$ and $\nu_e = \nu_{ce} / \delta_0^2$ are the effective collision frequencies of the trapped particles, Φ is the electrostatic potential, δ_0 is the equilibrium fraction of trapped particles, and $n^0 = \delta_0 N^P$, $N^0 = (1-\delta_0)N^P$, and N^P are the trapped, untrapped and total particle densities of a species in equilibrium. These parameters depend on the coordinate x only. The remaining notation is standard.

Since $\nabla \cdot \underset{\sim}{v} = 0$, the K.P. equations can also be written in the following <u>Lagrangian representation:</u>

$$\frac{dn^i}{dt} = -\nu_i (n^i - n^0), \tag{2.9}$$

$$\frac{dn^e}{dt} = -\nu_e (n^e - n^0), \tag{2.10}$$

$$\underset{\sim}{v} = \hat{z} \times \nabla\left[A(n^i - n^e)\right], \tag{2.11}$$

with

$$\frac{dn^j}{dt} =: n^j_t + \underset{\sim}{v} \cdot \nabla n^j = n^j_t + \nabla \cdot (n^j \underset{\sim}{v}), \tag{2.12}$$

$$\nabla \cdot (n^i \underset{\sim}{v}) = A(n^i_x n^e_y - n^i_y n^e_x) + A_x \rho\, n^i_y, \tag{2.12a}$$

$$\nabla \cdot (n^e \underset{\sim}{v}) = A(n^i_x n^e_y - n^i_y n^e_x) + A_x \rho\, n^e_y, \tag{2.12b}$$

$$\rho = n^i - n^e. \tag{2.12c}$$

It may be of advantage to consider the K.P. equations in the approximation $\partial A/\partial x = 0$. This condition can, on the one hand, be interpreted as a special equilibrium with T/N^0 = const. Of course, the assumption $\nu_{i,e}$ = const, i.e. $N^0/T^{3/2}\delta_0^2$ = const, then generally assumes the character of an approximation. On the other hand, attention can be restricted to perturbations of equilibrium with

$$|\partial_x \ln(T/N^0)| \ll |\partial_x \ln \rho| \tag{2.12d}$$

which for $|K_x a| \gg 1$ at any rate is guaranteed on the average. The approximation $\partial_x A = 0$ allows the parameter $\eta = d \ln T / d \ln N^0$ to be ignored. For the resulting specialized equations the drift velocity $\underset{\sim}{v}$ is tangential to the instantaneous ρ = const contours, where $\rho = n^i - n^e$ is the trapped charge density. One thus has $\underset{\sim}{v}\cdot\nabla\rho = 0$ and therefore

$$\underset{\sim}{v}\cdot\nabla n^{i,e} = A[\nabla n^i \times \nabla n^e] = A(n_x^i n_y^e - n_y^i n_x^e). \tag{2.13}$$

The subscripts x and y denote spatial derivatives. The following form of the K.P. equations is obtained:

$$n_t^i + \nu_i(n^i - n^0) + A(n_x^i n_y^e - n_y^i n_x^e) = 0, \tag{2.14}$$

$$n_t^e + \nu_e(n^e - n^0) + A(n_x^i n_y^e - n_y^i n_x^e) = 0, \tag{2.15}$$

with

$$\rho_t + \nu_i(n^i - n^0) + \nu_e(n_e - n^0) = 0 \tag{2.16}$$

and

$$\underset{\sim}{v} = A\hat{z} \times \nabla\rho. \tag{2.17}$$

This system of two first-order equations may also be transformed into one second-order equation:

$$\sigma_{tt} + 2\nu_1 \sigma_t + \nu_e \nu_i \sigma - 2\nu_2 v_0 \sigma_y$$

$$+ 2A(\sigma_x \sigma_{yt} - \sigma_y \sigma_{xt}) = 0, \tag{2.18}$$

where σ is half the trapped charge density: $\sigma = \rho/2 = (n^i - n^e)/2.$

The remaining quantities are

$$\nu_1 = \frac{\nu_e + \nu_i}{2}, \quad \nu_2 = \frac{\nu_e - \nu_i}{2}, \tag{2.19}$$

$$v_0 = A n_x^0 . \tag{2.20}$$

The diamagnetic drift velocity v_0 is often assumed to be constant, but it is generally a function of x. From σ and σ_t, n^i and n^e are reconstructed by means of

$$n^i = n^0 + \frac{\sigma_t}{\nu_2} + \frac{\nu_1 + \nu_2}{\nu_2} \sigma, \tag{2.21}$$

$$n^e = n^0 + \frac{\sigma_t}{\nu_2} + \frac{\nu_1 - \nu_2}{\nu_2} \sigma . \tag{2.22}$$

It is readily seen, e.g. from eqs. (2.18), (2.21), (2.22), that the K.P. equations in the approximation $A_x = 0$ admit of only a single time-independent solution, namely the equilibrium $n^i = n^e = n^0$, $\sigma = \sigma_t = 0$. In particular, there is no stationary convection in the laboratory system. As has already been shown [7], the following dispersion relation is valid for small perturbations of equilibrium:

$$(-i\omega+\gamma)^2 + 2\nu_1(-i\omega+\gamma) - 2i\omega_0\nu_2 + \nu_e\nu_i = 0, \qquad (2.23)$$

where ω is the real part of the frequency, γ the growth rate, $\omega_0 = K_y v_0$,

and the instability condition is

$$\omega_0^2 > \nu_i\nu_e\left(\frac{\nu_2}{\nu_1}\right)^2 . \qquad (2.24)$$

3. Boundary conditions

Previous authors have either not introduced boundary conditions at all or else only partially (for y) because the equations have only been intended for estimates or because crude approximations have been made. Boundary conditions are indispensable, however, for numerical or analytical calculations of solutions of the K.P. equations.

The slab coordinates x and y represent the coordinates r and $r(\theta - q\zeta)$ in the plasma torus [9] because the surface considered has to be aligned perpendicular to $\underset{\sim}{B}$. Here r is the distance from the magnetic axis, θ and ζ are the poloidal and toroidal angles, and q(r) is the "safety factor". A surface in the torus (or cylinder) everywhere perpendicular to $\underset{\sim}{B}$, with $q \neq \infty$, is a surface that spirals around the magnetic axis. As the trapped-ion modes are flute-like, it is possible to consider separately a section of the spiral surface which spans an angle $\Delta(\theta - q\zeta) = 2\pi$ and impose periodic boundary conditions in $r(\theta - q\zeta)$ or y. One thus has the intervals $[0,a]$ for r, and $[0, 2\pi r]$ for $r(\theta - q\zeta)$. In the slab model one accordingly chooses the intervals $[0,a]$ for x, and $[0,b]$ for y, with $b = 2\pi a$ or $b = \pi a$, respectively. It now remains to discuss the boundary conditions for x = 0,a. The minor axis, r = 0, is replaced by a reflecting wall (zero particle fluxes) at x = 0. The boundary conditions at x = 0 in the approximation $\partial A/\partial x = 0$ are then (since $n_y^0 = 0$):

$$v^x = -A\varrho_y = A(n_y^e - n_y^i) = 0 \,, \tag{3.1}$$

$$v_t^x = -A\varrho_{yt} = A(\nu_i n_y^i - \nu_e n_y^e) = 0 . \tag{3.2}$$

This yields ($\nu_i \neq \nu_e$!) for x = 0

$$n_y^i = n_y^e = 0, \tag{3.3}$$

and, since the nonlinear term of the K.P. equations vanishes at x = 0, it also follows for x = 0 that

$$n_t^i + \nu_i (n^i - n^0) = 0, \tag{3.4}$$

$$n_t^e + \nu_e (n^e - n^0) = 0, \tag{3.5}$$

or

$$n^i = n^0 + (n^{i0} - n^0)\, e^{-\nu_i t} , \tag{3.6}$$

$$n^e = n^0 + (n^{e0} - n^0)\, e^{-\nu_e t} , \tag{3.7}$$

with

$$n_y^{i0} = n_y^{e0} = 0 . \tag{3.8}$$

The K.P. equations can thus be solved separately at the reflecting wall and yield a solution that relaxes towards equilibrium.

At r = a or x = a an <u>absorbing wall</u> that is supposed to absorb the anomalously diffusing particle currents is assumed. The boundary conditions there first take the form

$$n^i v^x \geqq 0 \quad ; \quad n^e v^x \geqq 0 . \tag{3.9}$$

Since $n^{i,e} \geqq 0$, this can be rearranged to

$$\left. \begin{array}{ll} v^x \geqq 0 & \text{for} \quad n^i + n^e > 0 , \\ v^x \ \text{arbitrary for} & n^i = n^e = 0 . \end{array} \right\} \tag{3.10}$$

On the other hand, owing to the definition of $\underset{\sim}{v}$ and the periodicity in y it holds for every constant x at every time t that

$$\int_0^b v^x \, dy \equiv 0 \tag{3.11}$$

(thus, specially for $x = a$). If v^x is continuous, only y intervals contribute to the integral, either with $n^i + n^e > 0$ or with $n^i = n^e = 0$. Since, however, $v^x = A(n^e_y - n^i_y)$, it holds for such intervals that

$$v^x \geqq 0 \quad \text{for} \quad n^i + n^e > 0 , \tag{3.12}$$

$$v^x = 0 \quad \text{for} \quad n^i = n^e = 0 . \tag{3.13}$$

The integral thus vanishes at x = a for all t only if

$$v^x \equiv 0 \quad \text{for} \quad x = a , \tag{3.14}$$

$$v^x_t \equiv 0 \quad \text{for} \quad x = a . \tag{3.15}$$

These boundary conditions are identical with those of the reflecting wall (see above).

One thus has the, perhaps, paradox result that the K.P. equations do not allow wall losses if v^x and hence ρ_y are assumed to be continuous.

However, the collisional interaction of trapped with free particles allows one to have a mean particle flux in the x direction in the volume, for instance at $x = a/2$, since, for example, trapped particles are produced for small x and annihilated at large x. In so far as the presumed turbulence is a quasilocal effect, the anomalous diffusion in the volume (instead of at the wall) can be calculated in the hope of estimating the corresponding wall losses of a real plasma. Other remedies might be either to make the equilibrium density n^o dependent on y as well or to admit additional transport mechanisms, e.g. neoclassical diffusion.

4. Invariance properties

The nonlinear K.P. equations are not only invariant to translations of y and t, they also remain unchanged with respect to "translations with shear" of the form $y \to y + \eta(x)$, $\eta(x)$ arbitrary, together with the boundary conditions. Thus, with every solution $n^{i,e}(x,y,t)$ one also has a solution $n^{i,e}(x, y+\eta(x), t+\tau)$. This is valid for the complete K.P. equations with arbitrary $A(x)$, $\nabla A \not\equiv 0$. This invariance is immediately verified by substitution in eqs. (2.1) to (2.6). The invariance can be used for checking numerical solutions. By substituting a sum of Heaviside step functions for $\eta(x)$ a discontinuous solution is produced from an arbitrary, smooth solution. This proves that the K.P. equations have discontinuous solutions, which have to be eliminated for physical reasons (see above), e.g. by suitable numerical methods or by adding diffusion terms. It can be shown that the K.P. equations as modified by LaQuey et al. (Ref. [9] , eqs. (3) and (4)) also have this invariance property. The same applies to the modified K.P. equations derived by one of the authors (Ref. [7] , eqs. (5.1) to (5.3)).

In the special case $A_x = 0$, $\partial_x n^0$ = const there is a further invariance, viz. with respect to pseudo-reflection at the straight line x = a/2; i.e. for every solution $\sigma(x,y,t)$ of eq. (2.18) satisfying the boundary conditions, $[-\sigma(a-x, y, t)]$ is also a solution satisfying the boundary conditions. Antisymmetric initial conditions,

$\sigma(x, y, 0) + \sigma(a-x, y, 0) = 0$, and $\sigma_t(x, y, 0) + \sigma_t(a-x, y, 0) = 0$, yield antisymmetric solutions for all times.

It would be desirable to derive constants of the motion from the invariance properties. This is not possible for want of a Lagrangian density producing the K.P. equations. It can in fact be readily proved that no polynomial function constructed from $n^{i,e}$ and first derivatives can be a Lagrangian density for the K.P. equations.

5. Exact solutions

For general $A(x)$, $\nabla A \not\equiv 0$, the K.P. equations have the following special solutions, which can easily be verified by substitution in eqs. (2.1) to (2.6):

$$\left.\begin{aligned} \sigma &= \sigma_1(x)\, e^{-\nu_i t} + \sigma_2(x)\, e^{-\nu_e t}, \\ n^i &= 2\sigma_1(x)\, e^{-\nu_i t} + n^0(x), \\ n^e &= -2\sigma_2(x)\, e^{-\nu_e t} + n^0(x), \end{aligned}\right\} \tag{5.1}$$

where $\sigma_1(x)$ and $\sigma_2(x)$ can be chosen arbitrarily as long as $n^{i,e} \geqq 0$ and the boundary conditions are satisfied. These solutions make the non-linear terms of the K.P. equations vanish separately. In particular, $\sigma_1(x)$ and $\sigma_2(x)$ can again be chosen as sums of Heaviside step functions.

Trying out special ansatzes (see Appendix) did not yield any further exact solutions.

6. Time asymptotic and absolute bounds for the solutions

Consideration of the K.P. equations in the form (eqs. (2.9), (2.10))

$$\frac{dn^{i,e}}{dt} = -\nu_{i,e}\left(n^{i,e} - n^{o}\right) \tag{6.1}$$

for general $A(x)$, $\nabla A \not\equiv 0$, shows that the property $n^{i,e} \geqq 0$ is preserved in time. Furthermore, for asymptotically large times

$$n^{i,e} \lesssim n^{o}_{max} \tag{6.2}$$

$$|\varrho| = |n^{i} - n^{e}| \lesssim n^{o}_{max} , \tag{6.3}$$

i.e. <u>the densities remain bounded</u> (saturation of the instability?).

In addition, it holds that <u>stationary values</u> of n^{i}, $\nabla n^{i} = 0$, or n^{e} $(\nabla n^{e} = 0)$, particularly extrema of these quantities, always relax towards the equilibrium n^{o}. This may involve stationary points or curves, and the proper velocities $\underset{\sim}{W}^{i}$ and $\underset{\sim}{W}^{e}$ of such stationary points and curves, respectively, have to be taken into account. These velocities generally differ from the drift velocity $\underset{\sim}{v}$. Let us consider, for example, a stationary point $\nabla n^{i} = 0$. Substitution in eqs. (2.1), (2.12a) yields

$$n^{i}_{t} + \nu_{i}\left(n^{i} - n^{o}\right) = 0. \tag{6.4}$$

The same equation is satisfied for the total time derivative moving with $\underset{\sim}{W}^{i}$:

$$\frac{\delta n^{i}}{\delta t} =: n^{i}_{t} + \underset{\sim}{W}^{i} \cdot \nabla n^{i} = n^{i}_{t} \tag{6.5}$$

since $\nabla n^{i} = 0$. This demonstrates the validity of the assertion. The same applies to stationary values of $\mathcal{N}^e$. There is, of course, also the possibility that a stationary point (e.g. an extremum) vanishes altogether as such before it has relaxed to the vicinity of the equilibrium.

7. Global equations of motion

If the trapped particle numbers are defined as

$$Z^{i,e}(t) =: \int_0^a dx \int_0^b dy \; n^{i,e}(x,y,t), \tag{7.1}$$

$$Z^{o} =: \int_0^a dx \int_0^b dy \; n^{o}(x), \tag{7.2}$$

one obtains from eqs. (2.1), (2.2) with allowance for the boundary condition $v^x = 0$ at x = 0 and a the equations of motion

$$Z^i_t + \nu_i \left(Z^i - Z^o\right) = 0, \tag{7.3}$$

$$Z^e_t + \nu_e \left(Z^e - Z^o\right) = 0. \tag{7.4}$$

The solutions are trivial :

$$Z^i = Z^o + \left(Z^{io} - Z^o\right) e^{-\nu_i t}, \tag{7.5}$$

$$Z^e = Z^o + \left(Z^{eo} - Z^o\right) e^{-\nu_e t}. \tag{7.6}$$

The total charge of the trapped particles, $Q =: Z^i - Z^e$, also relaxes: $Q \to 0$ for $t \to \infty$.

8. Mean values

Equations (2.1) and (2.2) are multiplied by the functions $f_i(n^i)$ and $f_e(n^e)$ and integrated over x and y. Let the mean value $M_2[\varphi]$ be defined as

$$M_2[\varphi] =: \frac{1}{ab}\int_0^a dx \int_0^b dy\ \varphi(t,x,y). \tag{8.1}$$

Because of the boundary conditions the nonlinear terms of the K.P. equations disappear. This yields for the ions

$$M_2\left[f_i(n^i)\cdot\left\{n^i_t + \nu_i(n^i - n^0)\right\}\right] = 0, \tag{8.2}$$

and an analogous relation is obtained for the electrons. We specialize for $f_i = 1$ and $f_i = n^i$, the result being

$$M_2\left[n^i_t + \nu_i(n^i - n^0)\right] = 0, \tag{8.3}$$

$$M_2\left[\tfrac{1}{2}\{(n^i)^2\}_t + \nu_i n^i(n^i - n^0)\right] = 0. \tag{8.4}$$

Equation (8.3) is essentially identical with eq. (7.3) for the number of trapped ions. The mean values of the densities accordingly relax to zero, and eq. (8.4) gives the time asymptotic boundedness of the mean quadratic densities. The most detailed statement is obtained by choosing $f_i(n^i)$ as a differentiable function such that it is essentially non-zero only in the interval $I_i =: [\hat{n}^i - \Delta n^i,\ \hat{n}^i + \Delta n^i]$.

It then follows that

$$\langle n^i_t \rangle_{n^i \in I_i} = -\nu_i\left[\hat{n}^i - \langle n^0 \rangle_{n^i \in I_i}\right], \tag{8.5}$$

where $\langle\ \rangle$ denotes the corresponding mean value. For time periodic or quasi-periodic (turbulent) solutions, as expected for the time asymptotic state, one may also integrate over the time t. Instead of eqs. (8.2) to (8.5) one obtains corresponding relations which, however, no longer contain derivatives with respect to time, M_2 being replaced by the mean value

$$M_3 =: \frac{1}{abT} \int_0^a dx \int_0^b dy \int_0^T dt . \tag{8.6}$$

In the approximation $\partial A/\partial x = 0$, furthermore, eq. (2.18) is valid for $\sigma = (n^i - n^e)/2$. Mean value relations are obtained by multiplying by $f(\sigma)$ and integrating over x and y or x, y and t. In the first case one obtains the relations

$$M_2 \left[f(\sigma) \left\{ \sigma_{tt} + 2\nu_1 \sigma_t + \nu_i \nu_e \sigma \right\} \right] = 0, \tag{8.7}$$

i.e. for the special cases f = 1 and σ :

$$Q_{tt} + 2\nu_1 Q_t + \nu_i \nu_e Q = 0 \tag{8.8}$$

and

$$M_2 \left[\tfrac{1}{2} (\sigma^2)_{tt} + \nu_1 (\sigma^2)_t \right] = M_2 \left[(\sigma_t)^2 - \nu_i \nu_e \sigma^2 \right]. \tag{8.9}$$

The most detailed statement is obtained for $f(\sigma) \neq 0$ only in the interval $I_\sigma =: [\hat{\sigma} - \Delta\sigma , \hat{\sigma} + \Delta\sigma]$, namely

$$\langle \sigma_{tt} + 2\nu_1 \sigma_t \rangle_{\sigma \in I_\sigma} + \nu_i \nu_e \hat{\sigma} = 0 , \tag{8.10}$$

where $\langle\ \rangle$ denotes the corresponding mean value.

In the second case it follows for a time periodic or quasi-periodic final state that

$$M_3 \left[f'(\sigma)(\sigma_t)^2 - \nu_i \nu_e \sigma f(\sigma) \right] = 0 \tag{8.11}$$

i.e. for the special cases f = 1 and σ :

$$M_3 [\sigma] = 0, \tag{8.12}$$

$$M_3 \left[(\sigma_t)^2 - \nu_i \nu_e \sigma^2 \right] = 0. \tag{8.13}$$

For $f(\sigma) \neq 0$ only in the interval I_σ it follows that

$$\langle \sigma_{tt} \rangle_{\sigma \in I_\sigma} = -\nu_i \nu_e \hat{\sigma}, \tag{8.14}$$

where $\langle\ \rangle$ denotes the corresponding three-dimensional mean value. Equations (8.13) and (8.14) are particularly interesting since they provide information independent of n^0 on the magnitude of the time derivatives. The main importance of the relations (8.2) to (8.14) then appears, however, to be that numerical results can be checked.

Equation (8.14) suggests looking for <u>special time periodic solutions</u> of the form

$$\sigma_{tt} = -\nu_i \nu_e \sigma. \tag{8.15}$$

The boundary conditions at x = 0 and a are then $\sigma = \sigma_t = 0$. It can be shown, however, that there are no such solutions (Appendix). This is demonstrated by substituting eq. (8.15) in eq. (2.18), which shows that σ would have to be of the form

$$\sigma = \sigma_0(x) \cdot \sin\left[\sqrt{\nu_i \nu_e}\left(t + \frac{\nu_1}{\nu_2 v_0} y\right) + \alpha(x)\right]; \qquad (8.16)$$

on the other hand, this function would have to make the nonlinear term in eq. (2.18) zero, which is only the case for $\sigma_0 = 0$ or $d\sigma_0/dx = 0$ with $\sigma_0 \neq 0$. The former would be a trivial solution, while the second would violate the boundary conditions at x = 0 and a.

9. Nonlinear travelling waves

Let us consider whether there exist solutions of the K.P. equations in the form of travelling waves:

$$\sigma = \sigma(x, y - wt), \qquad w = \text{const}, \tag{9.1}$$

i.e. propagating in the y direction. Here only the special case $\partial A/\partial x = 0$ is considered. Equation (2.18) transforms to

$$w^2 \sigma_{yy} - 2K\sigma_y + \nu_e \nu_i \sigma$$
$$+ 2Aw(\sigma_y \sigma_{xy} - \sigma_x \sigma_{yy}) = 0, \tag{9.2}$$

with

$$K = \nu_1 w + \nu_2 v_0, \tag{9.3}$$

and the boundary conditions at x = 0 and x = a, eqs. (3.6), (3.7), yield $\sigma = 0$ at these boundaries ($w \neq \infty$).

Multiplying eq. (9.2) by σ_y yields

$$\frac{1}{2} w^2 (\sigma_y^2)_y - 2K\sigma_y^2 + \frac{1}{2}\nu_e \nu_i (\sigma^2)_y$$
$$+ Aw\left[(\sigma_y^3)_x - (\sigma_y^2 \sigma_x)_y\right] = 0. \tag{9.4}$$

It follows by integration over the domain of definition that

$$\int_0^a dx \int_0^b dy \; 2K\sigma_y^2 = 0. \tag{9.5}$$

This yields for K $\neq$ 0 with allowance for eq. (9.2) only the trivial solution (= equilibrium configuration) $\sigma \equiv \sigma_y \equiv 0$.

The case K = 0, i.e. $W = -\nu_2 v_0/\nu_1$, has to be treated separately. Multiplying eq. (9.2) by σ yields

$$W^2(\sigma\sigma_y)_y - W^2\sigma_y^2 + \nu_e\nu_i\sigma^2$$

$$+ 2AW\left[(\sigma\sigma_y^2)_x - (\sigma\sigma_x\sigma_y)_y\right] = 0. \qquad (9.6)$$

By integrating it follows in agreement with eq. (8.9) that

$$\int_0^a dx \int_0^b dy \left\{\nu_e\nu_i\sigma^2 - W^2\sigma_y^2\right\} = 0. \qquad (9.7)$$

If one takes $\nu_i = 0$, the trivial solution $\sigma \equiv \sigma_y \equiv 0$ is again obtained. It follows that nonlinear waves are only possible for K = 0, $\nu_e\nu_i \neq 0$ at most. It was not possible to give a proof of existence or construct solutions for this case. This result is in contrast with the disparate case of LaQuey et al. [9] . Their equations differ essentially from the K.P. equations used here in that they include Landau damping and $A_x \neq 0$ and use various approximations, and they do allow nonlinear travelling waves with $W = -v_0$ in the case $\nu_i = 0$.

10. Dimensional analysis: scaling of the anomalous diffusion coefficient

The variables t, x, y, σ in eq. (2.18) will be made dimensionless. Examination shows that the following natural units occur

$$\left.\begin{array}{ll} \text{For } t: & \nu_e^{-1}, \quad (\nu_e \nu_i)^{-\frac{1}{2}}, \quad \nu_i^{-1}, \\ \text{for } x: & a, \quad v_0 t_0, \\ \text{for } y: & b, \quad v_0 t_0, \\ \text{for } \sigma: & n^0, \quad a n_x^0 \end{array}\right\} \tag{10.1}$$

where t_o denotes any of the units of t. The following choice of units

$$\left.\begin{array}{l} t_0 = (\nu_e \nu_i)^{-\frac{1}{2}}, \\ x_0 = y_0 = v_0 / \sqrt{\nu_e \nu_i}, \\ \sigma_0 = x_0 \, n_x^0, \end{array}\right\} \tag{10.2}$$

yields the following dimensionless form of eq. (2.18)

$$\sigma'_{t't'} + 2\sqrt{\frac{\nu_e}{\nu_i}}\,\sigma'_{t'} + \sigma' - 2\sigma'_{y'}$$

$$+ 2\left(\sigma'_{x'}\,\sigma'_{y't'} - \sigma'_{y'}\,\sigma'_{x't'}\right) = 0, \tag{10.3}$$

where the primed symbols denote the dimensionless quantities.

We are interested in the scaling of the anomalous particle flux or diffusion coefficient in the time-asymptotic fluctuating state or states ($t \rightarrow \infty$). The particle flux in the x direction, averaged over y and the time t, is given by

$$\gamma^{x} =: \langle n^{i,e} v^{x} \rangle = -\frac{2A}{\nu_e - \nu_i} \langle \sigma_y \sigma_t \rangle$$

$$\approx -\frac{2A\,\sigma_0^2}{\nu_e\, t_0\, y_0} \langle \sigma'_{y'} \sigma'_{t'} \rangle . \qquad (10.4)$$

From (10.1) and eqs. (10.2) to (10.4) it follows that

$$\gamma^{x}(x) = -\frac{2v_0^2}{\nu_e} n_x^0 \cdot g\left(\sqrt{\frac{\nu_i}{\nu_e}}, \frac{v_0}{b\sqrt{\nu_i \nu_e}}, \frac{b}{a}, x', \frac{a n_x^0}{n^0}\right) \qquad (10.5)$$

where g remains undetermined. This expression can be simplified since, in general, one has $a n_x^0 / n^0 \sim 1$, and because eq. (10.5) can be averaged over x:

$$\overline{\gamma^{x}} = -\frac{2v_0^2}{\nu_e} n_x^0 \cdot g_1\left(\sqrt{\frac{\nu_i}{\nu_e}}, \frac{v_0}{b\sqrt{\nu_i \nu_e}}, \frac{b}{a}\right) \qquad (10.6)$$

A KADOMTSEV-POGUTSE-type scaling [1] is possibly obtained for small marginal wavelengths, $v_0 / \sqrt{\nu_i \nu_e} \ll b$, with b/a fixed, e.g. $\frac{b}{a} \sim \pi$, and by putting $v_0 / b\sqrt{\nu_i \nu_e}$ equal to zero, namely:

$$\overline{\gamma^{x}} = -\frac{2v_0^2}{\nu_e} n_x^0 \cdot g_2\left(\sqrt{\frac{\nu_i}{\nu_e}}\right) . \qquad (10.7)$$

This, of course, can only be expected if a "local turbulence" develops in this limiting case for which the boundary conditions in x and y become insignificant. The boundary conditions in x and y play, however, an important role at higher temperatures when the marginal wavelength becomes large and depletion of the trapped particles can occur, and so eq. (10.6) has again to be used. It is interesting that eq. (10.6) can also be written in the form

$$\overline{\gamma^x} = -2\nu_i a^2 n_x^0 \cdot g_3\left(\sqrt{\frac{\nu_i}{\nu_e}}, \frac{v_0}{b\sqrt{\nu_i \nu_e}}, \frac{b}{a}\right), \tag{10.8}$$

a form which emphasizes the importance of the replenishment of trapped ions by collisions with the free particle background. Putting again $b \sim \pi a$ and substituting $\partial_x \to 1/a$, the following particular scaling laws in powers of the quantities a, N^P, B, T, δ_0 can be derived from eq. (10.8) for the trapped diffusion coefficient D_t :

$$D_t = const. \frac{N^P a^2}{T^{3/2} \delta_0^2} \cdot \left[\frac{\delta_0^3 T^{5/2}}{a^2 B N^P (1-\delta_0)}\right]^{\alpha}, \tag{10.9}$$

where $T_e = T_i = T$ has been used for simplicity. For $\alpha = 1$ a BOHM-type diffusion obtains while for $\alpha = 2$ KADOMTSEV-POGUTSE scaling is recovered.

11. Differential relation for the particle flux in the x direction

The particle flux in the x direction, averaged over y and t, for late times ($t \to \infty$),

$$\gamma^x(x) =: \langle n^{i,e} v^x \rangle , \tag{11.1}$$

satisfies a simple differential relation which may serve for checking numerical calculations. Averaging eqs. (2.1) and (2.2) over y and t yields for the x-derivative of the particle flux:

$$\gamma^x_x = \nu_i \langle n^0 - n^i \rangle = \nu_e \langle n^0 - n^e \rangle . \tag{11.2}$$

For $A_x = 0$ the right-hand side can be expressed according to eqs. (2.21) and (2.22) in terms of half the trapped charge density σ, the result being

$$\gamma^x_x = - \frac{2 \nu_e \nu_i}{\nu_e - \nu_i} \langle \sigma \rangle , \tag{11.3}$$

since $\langle \sigma_t \rangle = 0$. In the case n^0_x = const an analogous relation is obtained for the diffusion coefficient D_t of the trapped particles:

$$D_t = - \gamma^x / n^0_x . \tag{11.4}$$

Equation (11.3) can also be derived from eqs. (2.18) and (10.4). By averaging over y and t one obtains from eq. (2.18) the relation

$$\nu_e \nu_i \langle \sigma \rangle - 2A \langle \sigma_y \sigma_t \rangle_x = 0 , \tag{11.5}$$

since the nonlinear term of eq. (2.18) can be written as follows:

$$2A\left(\sigma_x \sigma_{yt} - \sigma_y \sigma_{xt}\right) = 2A\left[(\sigma_x \sigma_t)_y - (\sigma_y \sigma_t)_x\right]. \qquad (11.6)$$

On the other hand, one has according to eq. (10.4)

$$\nu^* = -\frac{2A}{\nu_e - \nu_i} \langle \sigma_y \sigma_t \rangle . \qquad (11.7)$$

Equation (11.3) can be used to determine the electric potential Φ, averaged over y and t, from the anomalous particle flux (cf. eq. (2.3)). The trapped charge density and the potential are negative in the interior and positive outside if $\nu^* > 0$:

$$\langle \phi \rangle = -\frac{T}{eN^0} \cdot \frac{\nu_e - \nu_i}{2\nu_e \nu_i} \cdot \nu^*_x . \qquad (11.8)$$

Differentiating again with respect to x yields the averaged E field $\langle E_x \rangle$. For the order of magnitude it holds that

$$\langle \phi \rangle \sim \frac{\delta_0 T}{2e\nu_i a^2} \cdot D_t . \qquad (11.9)$$

For $D_t \sim 2\nu_i a^2$ (see eq. (10.8)) it follows that

$$\langle \phi \rangle \sim \frac{\delta_0 T}{e} , \quad \langle E_x \rangle \sim -\frac{\delta_0 T}{a e} , \qquad (11.10)$$

whereas for KADOMTSEV-POGUTSE scaling [1] with $D_t \sim 2 v_0^2 / \nu_e$ it follows that

$$\langle \phi \rangle \sim \frac{\delta_0 T}{e} \cdot \frac{v_0^2}{\nu_e \nu_i a^2} , \qquad (11.11)$$

$$\langle E_x \rangle \sim -\frac{\delta_0 T}{ea} \cdot \frac{v_0^2}{\nu_e \nu_i a^2} \quad . \tag{11.12}$$

12. Summary

In support of numerical calculations we investigated a few simple analytical properties of the trapped-ion transport equations of KADOMTSEV and POGUTSE [1]. First we derived consistent boundary conditions which state that the flow velocity of trapped particles at material walls must be tangential to the surface. This means that the K.P. equations are only compatible with zero particle loss; it is nevertheless possible to define an anomalous diffusion coefficient in the volume. Other investigations were concerned with the invariance of the equations and boundary conditions with respect to "translations with shear" and reflection at the straight line $x = a/2$, with time asymptotic bounds for the densities, equations for global quantities (particle numbers) and mean values. Special nonlinear solutions (relaxing with time) of the K.P. equations were found, and the non-existence of smooth travelling wave solutions except in the special case $\nu_e \nu_i \neq 0$, $W = -\nu_2 v_0 / \nu_1$, was proved, this being of interest with respect to Ref. [9]. Furthermore, the anomalous particle flux in the x direction (= direction of the equilibrium gradient) was investigated. By dimensional analysis a scaling law for the particle flux was found which generally still contains a free function of three dimensionless parameters and, in particular, is compatible with the KADOMTSEV-POGUTSE scaling [1]. In addition, we derived a simple differential relation which allows the electric potential averaged over y and t to be calculated from the gradient of the particle flux. If the particle flux is directed outwards, the plasma is negatively charged in the interior and positively outside.

In the Appendix some special solution ansatzes are discussed.

The authors are grateful to Dr. D. Lortz for a stimulating discussion.

Appendix: Discussion of special solution ansatzes

A 1. Product ansatz 1: We put

$$\sigma = \sigma_1(t) \cdot \sigma_2(x,y). \tag{A 1.1}$$

The nonlinear term in eq. (2.18) disappears and we are left with

$$\frac{\sigma_{1tt}}{\sigma_1} + \frac{2\nu_1 \sigma_{1t}}{\sigma_1} + \nu_i \nu_e = 2\nu_2 v_0 \frac{\sigma_{2y}}{\sigma_2} = \mathrm{const.} \tag{A 1.2}$$

Periodicity in y yields $\sigma_{2y} = 0$ and

$$\sigma_{1tt} + 2\nu_1 \sigma_{1t} + \nu_i \nu_e \sigma_1 = 0, \tag{A 1.3}$$

with the solutions

$$\sigma_1 = \sigma_{11}\, e^{-\nu_i t} + \sigma_{12}\, e^{-\nu_e t}. \tag{A 1.4}$$

The general solution of eq. (A 1.1) relaxes with time in accordance with eq. (5.1). In particular, it is proved that there are no time periodic or unstable solutions of the form (A 1.1).

A 2. Product ansatz 2: We put

$$\sigma = f(x) \cdot g(y,t). \tag{A 2.1}$$

One obtains the equation

$$g_{tt} + 2\nu_1 g_t + \nu_i \nu_e g - 2\nu_2 v_0 g_y$$

$$+ 2A f_x \left[g\, g_{yt} - g_y g_t \right] = 0; \tag{A 2.2}$$

this yields f_x = const,

$$f = f_0 + f_1 x, \tag{A 2.3}$$

this being a violation of the boundary conditions $\sigma_y = \sigma_{yt} = 0$

for x = 0 and a for all non-trivial g(y,t), except when the non-linear term vanishes identically. In this case it holds that

$$g(y,t) = \lambda(t) \cdot \mu(y), \tag{A 2.4}$$

which is a special case of case A 1.

A 3. Product ansatz 3: We put

$$\sigma = f(y) \cdot g(x,t). \tag{A 3.1}$$

One obtains the equation

$$g_{tt} + 2\nu_1 g_t + \nu_i \nu_e g - 2\nu_2 v_0 \frac{f_y g}{f}$$

$$+ 2A f_y \left[g_x g_t - g g_{xt} \right] = 0. \tag{A 3.2}$$

Averaging over y yields

$$g_{tt} + 2\nu_1 g_t + \nu_i \nu_e g = 0, \tag{A 3.3}$$

with relaxing solutions in accordance with eq. (5.1). This leaves

$$f_y \left\{ -2\nu_2 v_0 \frac{g}{f} + 2A \left[g_x g_t - g g_{xt} \right] \right\} = 0, \tag{A 3.4}$$

which can only be solved with $f_y = 0$ for $g \neq 0$. This means that σ is also again given by eq. (5.1).

A 4. Solutions with $\sigma_y \equiv 0$

Of eq. (2.18) we are left with

$$\sigma_{tt} + 2\nu_1 \sigma_t + \nu_e \nu_i \sigma = 0 , \tag{A 4.1}$$

with the solution

$$\sigma = \sigma_1(x)\, e^{-\nu_i t} + \sigma_2(x)\, e^{-\nu_e t} , \tag{A 4.2}$$

in accordance with eq. (5.1).

A 5. Solutions with $\sigma_x \equiv 0$

Of eq. (2.18) we are left with

$$\sigma_{tt} + 2\nu_1 \sigma_t + \nu_e \nu_i \sigma - 2\nu_2 v_0 \sigma_y = 0 \tag{A 5.1}$$

Although this equation agrees with the linearized version of the K.P. equations, there are no unstable solutions of this equation since $\sigma_x \equiv 0$ and owing to the boundary conditions at x = 0 and a. This is because it holds at the boundaries that $\sigma_y = 0$; since $\sigma_x \equiv 0$ it thus holds that $\sigma_y = 0$ in the entire volume, and the case A 5 is a special case of A 4.

A 6. Solutions with $n^i = n^i(n^e)$

From eqs. (2.14), (2.15) it follows that the only solutions relax with time, being again of the form of eq. (5.1) or (A 4.2).

A 7. Ansatz $\tilde{n}^i = f(\tilde{n}^e)$

with the definition

$$\tilde{n}^i = n^i - n^0 \,, \quad \tilde{n}^e = n^e - n^0 . \tag{A 7.1}$$

This only has the trivial solution $\tilde{n}^i \equiv \tilde{n}^e \equiv 0$ since substitution of the ansatz in eq. (2.14) and (2.15) yields (with $\rho = n^i - n^e$):

$$\tilde{n}^i_t = -\nu_i \tilde{n}^i - v_0 \rho_y \,, \tag{A 7.2}$$

$$\tilde{n}^e_t = -\nu_e \tilde{n}^e - v_0 \rho_y \,, \tag{A 7.3}$$

or

$$f' \tilde{n}^e_t = -\nu_i f - v_0 (f' - 1) \tilde{n}^e_y \,, \tag{A 7.4}$$

$$\tilde{n}^e_t = -\nu_e \tilde{n}^e - v_0 (f' - 1) \tilde{n}^e_y \,. \tag{A 7.5}$$

From this it follows that

$$\tilde{n}^e_t = f_1(\tilde{n}^e) =: \frac{\nu_e \tilde{n}^e - \nu_i f}{f' - 1} \,, \tag{A 7.6}$$

$$\tilde{n}^e_y = f_2(\tilde{n}^e) =: \frac{\nu_e f' \tilde{n}^e - \nu_i f}{v_0 (f'-1)^2}, \qquad \text{(A 7.7)}$$

and $\tilde{n}^e_{ty} = \tilde{n}^e_{yt}$ yields $f_1 = \text{const} \cdot f_2$, i.e.

$$\tilde{n}^e_t = - W \tilde{n}^e_y, \quad W = \text{const.} \qquad \text{(A 7.8)}$$

Substitution in eqs. (A 7.4) and (A 7.5) finally yields

$$f(\tilde{n}^e) = \frac{\nu_e}{\nu_i} \cdot \frac{W - v_0}{v_0} \cdot \tilde{n}^e \qquad \text{(A 7.9)}$$

and

$$\tilde{n}^e_y = \frac{\nu_e (W - v_0)}{W^2} \tilde{n}^e . \qquad \text{(A 7.10)}$$

Because of the periodicity in y it follows for non-trivial $\tilde{n}^e \not\equiv 0$ that $W = v_0$, $\tilde{n}^e_y \equiv f \equiv \tilde{n}^e_t \equiv 0$. This means that according to eqs. (A 7.4) and (A 7.5) only the trivial solution $\tilde{n}^i \equiv \tilde{n}^e \equiv 0$ is compatible.

A 8. Time periodic solution ansatz

According to eq. (8.14) it is obvious to postulate

$$\sigma_{tt} = - \nu_i \nu_e \sigma \qquad \text{(A 8.1)}$$

for this purpose. Substitution in eq. (2.18) yields

$$\nu_1 \sigma_t - \nu_2 v_0 \sigma_y + A\left[(\sigma_x \sigma_t)_y - (\sigma_y \sigma_t)_x\right] = 0. \qquad \text{(A 8.2)}$$

It follows that

$$\sigma_{yt} = -\frac{\nu_1 \nu_i \nu_e}{\nu_2 v_0}\,\sigma, \tag{A 8.3}$$

and by integration that

$$\nu_1 \sigma_t = \nu_2 v_0 \sigma_y + f(x,y), \tag{A 8.4}$$

with the general solution

$$\sigma = g\left(x, y + \frac{\nu_2 v_0}{\nu_1} t\right) + \ell(x,y). \tag{A 8.5}$$

Here f and g are still arbitrary for the time being, with the exception of the boundary conditions in x and y, and ℓ satisfies the equation

$$f(x,y) = -\nu_2 v_0\, \ell_y (x,y). \tag{A 8.6}$$

On the other hand, eq. (A 8.1) has the general solution

$$\sigma = \sigma_0(x,y)\cdot \sin\left(\sqrt{\nu_i \nu_e}\, t + \alpha(x,y)\right), \tag{A 8.7}$$

or

$$\sigma = \sigma_1(x,y)\cdot \sin\left(\sqrt{\nu_i \nu_e}\, t\right)$$

$$+ \sigma_2(x,y)\cdot \cos\left(\sqrt{\nu_i \nu_e}\, t\right). \tag{A 8.8}$$

A comparison with eq. (A 8.3) yields

$$\sigma_{i,yy} = -\left(\frac{\nu_1}{\nu_2 v_0}\right)^2 \nu_i \nu_e \sigma_i \quad , \quad i = 1, 2. \qquad \text{(A 8.9)}$$

The periodicity in y is thus ensured at most for

$$\frac{\nu_1 b}{\nu_2 v_0} \sqrt{\nu_i \nu_e} = 2\pi m, \quad m = \text{entire}. \qquad \text{(A 8.10)}$$

Comparison of eqs. (A 8.5) and (A 8.8) yields

$$f(x,y) \equiv \ell(x,y) \equiv 0 \qquad \text{(A 8.11)}$$

and

$$\sigma = \sigma_0(x) \cdot \sin\left[\sqrt{\nu_i \nu_e}\, t + \frac{\nu_1}{\nu_2 v_0} \sqrt{\nu_i \nu_e}\, y + \alpha(x)\right]. \qquad \text{(A 8.12)}$$

In the special case α = const a solution of this form does not exist, as was proved in A 1 and A 2. But such a solution does not exist either for arbitrary $\alpha(x) \neq$ const, as can readily be shown by substitution in eq. (A 8.2).

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