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Cole's Ansatz and Extensions of Burgers' Equation

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Abstract

A sequence of nonlinear partial differential equations is constructed. It contains all equations whose solutions can be obtained from applying the Cole-Hopf transformation to linear partial differential equations. An exemple is $u_t = (u^3)_x + \frac{3}{2} (u^2)_{xx} + u_{xxx}$.

It has been suggested in a recent paper [1] that Cole's ansatz be considered as part of a Bäcklund transformation for reducing the solution of nonlinear equations to the solutions of linear ones. The authors of [1] consider extensions of Burgers' equation involving spatial derivatives of even order (see [1] p. 295).

In this short contribution it is shown that it is also possible to find equations having odd and even spatial derivatives simultaneously. This can help to construct solvable models for physical problems having nonlinearities, dissipation and dispersion [2].

In accordance with [1] let $u(x,t)$ be the dependent variable of some evolution equation. Let $v(x,t)$ be defined for each time:

$$v_x = u(x,t)\, v \;, \tag{1}$$

than

$$v_t = A\; v \;, \tag{2}$$

where A can be an operator or a functional of u.

There is a compatibility condition for eqs. (1) and (2):

$$u_t = \left[A, u \right] + A_x \tag{3}$$

For $A = u$ one obtains

$$v_t = uv = v_x$$

and

$$u_t = u_x \ .$$

For $A = u^2 + u_x$ one obtains

$$v_t = (u^2 + u_x)\, v = v_{xx} \ ,$$

$$u_t = (u^2)_x + u_{xx} \ , \quad \text{which is Burgers' equation.}$$

For $A = (u^2 + u_x)\, u + (u^2 + u_x)_x$,

one obtains

$$v_t = (u^2 + u_x) v_x + (u^2 + u_x)_x v = v_{xxx} \ , \tag{4}$$

$$u_t = (u^3)_x + \frac{3}{2} (u^2)_{xx} + u_{xxx} \ . \tag{5}$$

This evolution equation (5) for u is a new one and contains non-linearities, dissipation and dispersion. Its solutions can be reduced to the solutions of eq. (4) by Cole's ansatz.

One can by induction construct an infinite set of equations for which Cole's ansatz means a reduction to linear equations. Let us consider the sequence

$$A_{n+1} = \alpha_{n+1}(x,t) \left(A_n u + \frac{\partial A_n}{\partial x} \right) \tag{6}$$

and $A_1 = \alpha_1(x,t)\, u\,,$ (7)

where α_i are given functions of x and t.

We then apply

$$A_{n+1} v = \alpha_{n+1} \left(A_n uv + \frac{\partial A_n}{\partial x} v\right)$$

$$= \alpha_{n+1} \left(A_n v_x + \frac{\partial A_n}{\partial x} v\right) = \alpha_{n+1} (A_n v)_x \,.$$

This can be repeated for $A_n v$ until $A_1 v = \alpha_1 v_x$.

From the sequence (6) it is possible to construct evolutions equations of the type

$$u_t = \sum_{n=1}^{N} \frac{\partial A_n}{\partial x}\,, \tag{8}$$

for which the corresponding linearized equations are

$$v_t = \sum_{n=1}^{N} A_n v\,, \tag{9}$$

where N can be any natural number.

Burgers' equation and eq. (5) are examples of the set given by eqs. (6), (7), (8).

Equations (9) can be solved by Fourier transformation if the α_i are independent of x . For constant α_i , equations (8) cannot have traveling soliton-like solutions because of the trigonometrical or exponential character of the stationary solutions of eqs. (9). Periodic and shock-like traveling solutions exist.

Equation (8) with $N = 3$ could be an interesting model for physical problems involving nonlinearities, dispersion and dissipation.

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References

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