Finite-ß Stabilization of a Diffuse Helical $\ell = 1$ MHD Equilibrium

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ABSTRACT. The stability of helically symmetric finite - $\beta, \ell = 1$ magnetohydrostatic equilibria with arbitrary pressure profile and vanishing longitudinal current is investigated by means of Mercier´s criterion, a sufficient criterion by Lortz, Rebhan and Spies, and Shafranov´s condition for a high - β magnetic well. The new finite -β effects are that (1) a magnetic well is created throughout the plasma region for $0.2 \leqslant \beta \leqslant 0.95$ and $0.3 \leq a\tau \leq 0.6$ (a plasma radius, τ torsion of the magnetic axis); (2) the mean magnetic well extends out into the vacuum region for $0.75 \leqslant \beta$ and $b\tau \leqslant 0.6$ (b wall radius); (3) with the exception of a very narrow region ($< 0.1\ a\tau$) around the magnetic axis, the Mercier criterion is satisfied for $0.75 \leqslant \beta \leqslant 0.95$ and $0.12 \leqslant a\tau \leqslant 0.3$.

1. INTRODUCTION

The stability of helically symmetric finite - β, $\ell = 1$ magnetohydrostatic equilibria with arbitrary pressure profile is investigated by means of Mercier's criterion [1], a sufficient criterion derived by Lortz, Rebhan, and Spies [2], and Shafranov's condition for a magnetic well [3]. The calculation of the equilibrium is made using a perturbation method whose small parameter ϵ is $\varkappa/\tau$, where $\varkappa$ is the curvature and τ the torsion of the helical magnetic axis. In Sec. 2, a non-dimensional form of the helical equilibrium equation is given. In Sec. 3, we re-write the equilibrium problem in Hamada coordinates because this facilitates evaluation of the stability criteria mentioned above. Section 4 lists suitable forms of the stability criteria for the case of vanishing longitudinal current, which is investigated here. In Sec. 5, we discuss the numerical results using β and the slimness of the configuration as parameters.

2. NON-DIMENSIONAL EQUILIBRIUM EQUATION IN HELICAL SYMMETRY

We start from Mercier's [1] coordinates ρ, φ, s which are associated with the magnetic axis, are a right-handed system, and have the metric tensor

$$\tilde{g}_{\rho\rho} = 1, \quad \tilde{g}_{\varphi\varphi} = \rho^2, \quad \tilde{g}_{ss} = (1 - \varkappa\rho\cos\varphi)^2 + \rho^2\tau^2,$$

$$\tilde{g}_{\rho\varphi} = 0, \quad \tilde{g}_{\rho s} = 0, \quad \tilde{g}_{\varphi s} = -\tau\rho^2, \quad \sqrt{\tilde{g}} = \rho(1 - \varkappa\rho\cos\varphi),$$

where $\varkappa$ and τ are the curvature and torsion of the magnetic axis; $\varkappa$, τ are constants and $\partial/\partial s \equiv 0$ for single-valued functions in helical symmetry. The contravariant and the covariant components of the magnetic field $\vec{B}$ are

$$B^{\rho} = \frac{-1}{\sqrt{g}} \frac{\partial \psi}{\partial \varphi}, \qquad B_{\rho} = B^{\rho},$$

$$B^{\varphi} = \frac{1}{\sqrt{g}} \frac{\partial \psi}{\partial \rho}, \qquad B_{\varphi} = \rho^2 B^{\varphi} - \tau \rho^2 B^{s},$$

$$B^{s} = \left[\bar{f}(\psi) - \frac{\tau}{\sqrt{g}} \rho^2 \frac{\partial \psi}{\partial \rho} \right] / \tilde{g}_{ss}, \qquad B_{s} = \bar{f}(\psi),$$

where $\psi = (\Phi + \chi)/L$, Φ and χ are the longitudinal and transverse fluxes of the magnetic field, L is the helical period, and $\bar{f}(\psi)$ is the second (besides the pressure $\bar{p}(\psi)$) free function in the equilibrium problem. The equilibrium equation is then

$$\frac{\tilde{g}_{ss}}{\sqrt{g}} \left\{ \frac{\partial}{\partial \rho} \left(\frac{\sqrt{g}}{\tilde{g}_{ss}} \frac{\partial \psi}{\partial \rho} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\sqrt{g}} \frac{\partial \psi}{\partial \varphi} \right) \right\} = \frac{2\tau}{\tilde{g}_{ss}} \bar{f} - \bar{f} \frac{d\bar{f}}{d\psi} - \tilde{g}_{ss} \frac{d\bar{p}}{d\psi}.$$

Introducing a non-dimensional flux function T, a radial coordinate x, $f(T)$ and $p(T)$ by

$$\psi = \frac{B_0}{2\tau^{\circ}} (1 + \iota) T, \quad x = \rho\tau, \qquad \bar{p}(\psi) = B_0^2 \; p(T)$$

$$\bar{f}(\psi) = B_0 f(T), \quad (f(0) = 1),$$

where B_0 is the magnetic field on the magnetic axis and ι the rotational transform per period on the magnetic axis, we obtain the non-dimensional equilibrium equation

$$\frac{\tilde{g}_{ss}}{\lambda} \left\{ \frac{\partial}{\partial x} \left(\frac{\lambda}{\tilde{g}_{ss}} \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\lambda} \frac{\partial T}{\partial \varphi} \right) \right\}$$

$$= \frac{4}{(1+\iota)} \left\{ \frac{f}{\tilde{g}_{ss}} - \frac{f\, df/dT}{(1+\iota)} - \frac{\tilde{g}_{ss}\; dp/dT}{(1+\iota)} \right\}, \qquad (1)$$

$$\tilde{g}_{ss} = (1 - \varepsilon x \cos\varphi)^2 + x^2,$$

$$\lambda = x(1 - \varepsilon x \cos\varphi)$$

with $\varepsilon = \varkappa/\tau = R_h h$, where R_h is the radius of the cylinder on which the helical magnetic axis lies and h is related to the period L by $h = 2\pi/L$.

Thus, we see that the equilibrium equation contains only two non-dimensional parameters characterizing the configuration: ϵ describes the effect of the curvature of the magnetic axis, and ι is related to the longitudinal current. In the case of vanishing longitudinal current ($J(T) \equiv 0$), we have $\iota = (1+\epsilon^2)^{-\frac{1}{2}} - 1$ for an $\ell = 1$ equilibrium, so that, in this case, it is convenient to omit the factor $(1+\iota)$ in the normalization of T, $\psi = B_o T/2\tau$, and use the equilibrium equation in the form

$$\frac{\tilde{g}_{\varsigma\varsigma}}{\lambda}\left\{\frac{\partial}{\partial x}\left(\frac{\lambda}{\tilde{g}_{\varsigma\varsigma}}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial \varphi}\left(\frac{1}{\lambda}\frac{\partial T}{\partial \varphi}\right)\right\} = 4\left(\frac{f}{\tilde{g}_{\varsigma\varsigma}} - f\frac{df}{dT} - \tilde{g}_{\varsigma\varsigma}\frac{dp}{dT}\right)$$

which contains only ϵ as non-dimensional parameter. From Eq. (1), we see that one possible approximate treatment of helical equilibria is obtained by using ϵ as the only small parameter. This expansion in ϵ is refered to as old Scyllac ordering [5]. (If one rescales the radial coordinate by introducing $\tilde{x} = x/\epsilon$ and keeps the plasma-β finite; the so-called new Scyllac ordering [4] is obtained. This ordering leads to finite distortions from the circular form of the flux surfaces and is not considered here .)

3. EQUILIBRIUM DESCRIPTION IN HAMADA COORDINATES

We describe the equilibrium using Hamada coordinates [6] V, θ, ς as independent variables because this is advantageous for evaluating the stability criteria. The coordinates $\theta + \varsigma$ are angle-like coordinates increasing by unity once around the short way and per period, respectively.

Therefore, $\theta + \varsigma$ is a single-valued function on the lines φ= constant for a left-hand helical magnetic axis. Since in helical symmetry any single-valued function is constant along the lines φ = constant, $\theta + \varsigma$ is constant along these lines. Thus, in helical symmetry, all single-valued functions are functions of $\theta+\zeta$ alone $(\partial/\partial\theta = \partial/\partial\zeta)$. Introducing the non-dimensional arc length $\sigma = 2\pi s/L$ and volume $V = \bar{V}\tau^2/L\pi$, we can therefore express Mercier's coordinates as functions of the Hamada coordinates as follows

$$
\begin{aligned}
x &= \tilde{x}(V, \theta+\zeta), \\
\varphi &= 2\pi(\theta+\zeta) + \tilde{\varphi}(V, \theta+\zeta), \qquad (2)\\
\sigma &= 2\pi\zeta + \tilde{\sigma}(V, \theta+\zeta),
\end{aligned}
$$

where $\tilde{x}, \tilde{\varphi}, \tilde{\sigma}$ are periodic functions in $\theta+\zeta$. Using the expansion introduced in Sec. 2 and observing that it is sufficient to calculate first-order quantities to evaluate the stability criteria in leading order (see Sec. 4), we specialize Eqs. (2) to

$$
\begin{aligned}
x &= \overset{0}{x} + \epsilon \overset{1}{x}(\overset{0}{x}) \cos 2\pi(\theta+\zeta) + O(\epsilon^2), \\
\varphi &= 2\pi(\theta+\zeta) + \epsilon \overset{1}{\varphi}(\overset{0}{x}) \sin 2\pi(\theta+\zeta) + O(\epsilon^2), \\
\sigma &= 2\pi\zeta + \epsilon\overset{1}{\sigma}(\overset{0}{x}) \sin 2\pi(\theta+\zeta) + O(\epsilon^2),
\end{aligned}
$$

where $\overset{0}{x}$ is used as independent radial coordinate instead of V. The equilibirium is then a pure $\ell = 1$ equilibrium in leading order with flux surfaces symmetric to the osculating plane of the magnetic axis.

The first-order equilibrium equations are (for the derivation, see Appendix)

$$
\begin{aligned}
&\overset{1}{x}{}' + \overset{1}{x}/\overset{0}{x} + \overset{1}{\varphi} = \overset{0}{x}, \\
&\overset{1}{\sigma}{}' - 2 + \overset{1}{x} + 2K'(\overset{1}{\sigma} - \overset{0}{x}) = 0, \qquad (3)\\
&\overset{1}{\sigma}(1+\overset{0}{x}{}^2) - \overset{0}{x}{}^2\, \overset{1}{\varphi} = 2\overset{0}{x},
\end{aligned}
$$

where $' = d/d\overset{0}{x}$ and

$$K' = -\frac{p'}{2(1-p)} .$$

We choose a simple two-parameter pressure profile

$$p = \beta \exp \left[- \alpha \overset{0}{x}{}^2 - (\alpha \overset{0}{x}{}^2)^4 \right]$$

where $\beta = 2 p(0) / B^2_{vac}$ and α determines the half-width of the pressure profile. The first-order equilibrium solution is obtained by numerically solving Eqs. (3) with the boundary conditions at $\overset{0}{x} = 0$: $\overset{1}{x}(0) = \overset{1}{\sigma}(0) = 0$.

For the vacuum case ($K' \equiv 0$), the solution $\overset{1}{x}$ and $\overset{1}{\sigma}$ can be given analytically in terms of Bessel functions $I_0(\overset{0}{x})$, $I_1(\overset{0}{x})$

$$\overset{1}{x}(\overset{0}{x}) = 1 - 2 I_0(\overset{0}{x}) + \frac{2 I_1(\overset{0}{x})}{\overset{0}{x}} , \qquad \overset{1}{\sigma}(\overset{0}{x}) = \overset{0}{x} + 2 I_1(\overset{0}{x}) .$$

The solutions $\overset{1}{x}(\overset{0}{x})$ and $\overset{1}{\sigma}(\overset{0}{x})$ are shown in Figs. 1 - 2 for various pressure profiles and various β-values ($\beta = 0.15$: $\alpha = 3$ and $\alpha = 81$; $\beta = 0.95$: $\alpha = 3$ and $\alpha = 81$). Two different effects are obviously observed, namely the strong dependence of the functions $\overset{1}{x}$ and $\overset{1}{\sigma}$ on β and on the finite periodicity number of the helical magnetic axis. The function $\overset{1}{x}(\overset{0}{x})$ represents the displacement of the magnetic surfaces with respect to the magnetic axis. For small β-values ($\beta = 0.15$, $\alpha = 3$) the displacement is small and opposite to the direction of the normal of the magnetic axis; for intermediate β - values, the magnetic surfaces near the axis are displaced in the direction of the normal and the remote surfaces in the opposite direction; for high β - values ($\beta = 0.95$), the displacement is large and in direction of the normal.

4. STABILITY CRITERIA

4.1. Sufficient criterion.

A sufficient criterion for stabilty has been obtained by Lortz et al. [2] . In terms of A, ($\cdot \equiv d/dV$)

$$A := \frac{2}{|\nabla V|^6} \; (\vec{j} \times \nabla V) \cdot (\vec{B} \cdot \nabla) \, \nabla V$$

$$\equiv \frac{1}{|\nabla V|^2} \left\{ \frac{\vec{j}^{\,2}}{|\nabla V|^2} + \dot{I} \ddot{\Phi} - \dot{J} \ddot{\chi} - \nabla \cdot \left[\vec{B} \left(I g^{V\zeta} - J g^{V\theta} \right) \frac{1}{g^{VV}} \right] \right\},$$

the sufficient criterion reads: Stability holds if a single-valued function Λ exists which satisfies the inequality

$$\frac{1}{|\nabla V|^2} \; \vec{B} \cdot \nabla \Lambda - \Lambda^2 - A \geq 0 \quad . \tag{4}$$

Since our equilibrium solution is expandable in a small parameter, A is expandable: A starts with a non-vanishing first-order term $A^{(1)}$ $(A = A^{(1)} + \mathcal{O}(\varepsilon^2))$, whose mean value is of second order. Under these conditions, the solubility condition of the inequality (4) reduces to

$$\frac{1}{\dot{\Phi}^2} \langle |\nabla V|^6 \; (\int_0^{\zeta} A^{(1)} \; (\zeta^*) \, d\zeta^*)^2 \; \rangle \; + \langle |\nabla V|^2 A \rangle \;\; \leq 0 \; . \tag{5}$$

Here, because of the helical symmetry, the brackets $\langle \dots \rangle$ indicate $\int (\dots) \, d\theta \, d\zeta$. Thus, the mean value of A has to have an upper negative bound related to the first-order term of A. Inequality (5) can only be satisfied for vanishing longitudinal current on the magnetic axis.

4.2. Necessary criterion.

The necessary criterion of Mercier [1] is, of course, less restrictive than the sufficient criterion. For vanishing longitudinal current $(J(V) \equiv 0)$, it reads

$$\dot{I} \ddot{\Phi} + \langle \vec{j}^{\,2} / |\nabla V|^2 \rangle = \langle \; |\nabla V|^2 \; A \rangle \quad \leq 0 \; . \tag{6}$$

4.3. Magnetic well condition

Shafranov's magnetic well condition [3] for finite-β configurations is given by

$$\langle 2p + B^2 \rangle^{\cdot} - \dot{p} \left[1 - \langle B^2 \rangle \; \langle 1/B^2 \rangle \; \right] \; > 0. \tag{7}$$

For equilibria with vanishing longitudinal current, it can be shown that inequality (7) is less restrictive than inequality (6) (for the proof, see Appendix).

5. RESULTS

By means of the first-order equations (3), the criteria (5) - (7) can be reduced to the following form

$$C_L = C_M - \frac{K'}{\overset{0}{x}} (1 - \overset{1}{x})^2 > 0 \ , \qquad (8)$$

$$C_M = \frac{1}{2}\left[\frac{1}{\overset{0}{x}} \left(\overset{1}{x} + \frac{\overset{1}{\sigma}}{\overset{0}{x}} - 2\right)\right]^2 - \frac{1}{\overset{0}{x}} (\overset{1}{\sigma} - \overset{0}{x})(1 - \overset{1}{x}) > 0 \ , \qquad (9)$$

$$C_{Sh} = C_M + \frac{K'}{\overset{0}{x}} \left(\frac{\overset{1}{\sigma}}{\overset{0}{x}} - 1\right)^2 > 0 \ , \qquad (10)$$

(for the derivation , see Appendix). Note that Mercier's criterion is reduced to a form without derivatives and without explicit pressure dependence. Since $K' > 0$, the sufficient criterion is more restrictive and Shafranov's condition less restrictive than Mercier's criterion.

Near the magnetic axis, the solutions are of the form

$$\overset{1}{\sigma} = 2\overset{0}{x} + O(\overset{0}{x}{}^3), \quad \overset{1}{x} = O(\overset{0}{x}{}^2), \quad \overset{1}{\varphi} = O(\overset{0}{x})$$

which shows that Mercier's criterion is violated on the magnetic axis[1)], whereas the magnetic well condition is satisfied for

$$\frac{\alpha\beta}{(1-\beta)} > 1 .$$

Figures 3 - 6 show the stable and unstable regions according to the criteria (8) - (10) for various pressure profiles. The results may be described in terms of the three parameters β, $a\tau$, and $b\tau$, where a and b are the plasma and wall radii, respectively. For low β- values, smaller than or of the order of 0.1 , and not too

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1) As has been shown by Correa [7], and Mikhailovskii and Shafranov [10], it is possible to satisfy Mercier's criterion in the neighbourhood of the magnetic axis for helical equilibria with finite $\epsilon = \varkappa / \tau$.

small values of $a\tau$ (≥ 0.3), there is hardly any effect on the vacuum result, which for all of these criteria is just the unfavourable value $\ddot{\Phi} < 0$ of the helical $\ell = 1$ vacuum field. Figure 3 shows the result for $a\tau \approx 0.3$ and a somewhat higher β-value: $\beta = 0.15$. We see that Mercier's criterion is hardly affected, whereas the magnetic well condition is already satisfied throughout the plasma region, i.e. the plasma creates its own magnetic well. This property holds for all β-values above 0.15 and $a\tau$ of the order of 0.3 (for $\beta = 0.15$) to 0.64 (for $\beta = 0.95$). Figure 4 ($\beta = 0.55$, $\alpha = 27$) shows a tendency for Mercier's criterion to be satisfied in the region of maximum pressure gradient. Figure 5 ($\beta = 0.75$, $\alpha = 81$) and Fig. 6 ($\beta = 0.95$, $\alpha = 81$) show that the Mercier unstable region has shrunk to a quite small neighbourhood ($< 0.1\ a\tau$) around the magnetic axis [11]. They also show a second finite-β effect: the magnetic well extends out into the vacuum region. This means that, if one puts a wall at a point $\overset{o}{x} = b\tau$ where the magnetic well vanishes ($\ddot{\Phi}(b\tau) = 0$), the plasma is then surrounded by a region of increasing mean value of B^2 which extends to that wall. Since this effect scales

$$\left(\frac{b}{a}\right)^4 = \frac{\beta}{(b\tau)^2\,(1-\beta/2)}$$

for small b and flat pressure profile, similarly to the wall stabilization term of the $m = 1$ gross mode [4], we presume that the wall stabilization is connected to the vacuum magnetic well confining the plasma. The experimental values proposed [9] for the wall-stabilized high-beta stellarator in Garching are $a\,h \simeq a\tau \simeq 0.125$, $\beta \simeq 0.75$ (0.5, ..., 0.9) and $(b/a)_{exp.} \simeq 2.5$. This situation is approximately represented in Fig. 5 ($\beta = 0.75$) (and Fig. 6 for $\beta = 0.95$) which yields a ratio $b/a \simeq 2.0$.

APPENDIX

Here we prove that inequality (7) is less restrictive than inequality (6) and reduce the criteria (5) - (7) to the form (8) - (10). In case of helical symmetry ($\partial/\partial\theta = \partial/\partial\zeta$) and for vanishing longitudinal current ($J(V) \equiv 0$), the equilibrium equations written in Hamada coordinates (see e.g. [8]) reduce to

($\bullet \mathrel{\widehat{=}} d/dV$)

$\sqrt{g} = 1$,

$$\dot{\Phi}\,(g_{\zeta\zeta} - g_{\theta\zeta}) + \dot{\chi}\,(g_{\theta\zeta} - g_{\theta\theta}) = -I \, , \qquad \text{(A.1)}$$

$$\frac{\partial}{\partial V}\,(\dot{\Phi}\, g_{\theta\zeta} + \dot{\chi}\, g_{\theta\theta}) - \frac{\partial}{\partial\theta}\,(\dot{\Phi}\, g_{V\zeta} + \dot{\chi}\, g_{V\theta}) = 0 \, . \qquad \text{(A.2)}$$

The mean value equations ($\int(\ldots)\, d\theta\, d\zeta = \langle\ldots\rangle$)

$$\dot{\Phi}\,\langle g_{\zeta\zeta}\rangle + \dot{\chi}\,\langle g_{\theta\zeta}\rangle = -I \, ,$$

$$\dot{\Phi}\,\langle g_{\theta\zeta}\rangle + \dot{\chi}\,\langle g_{\theta\theta}\rangle = 0$$

entail

$$\langle B^2\rangle = -I\dot{\Phi}, \quad \langle B^2\rangle^{\cdot} = \frac{\ddot{\Phi}}{\dot{\Phi}}\,\langle B^2\rangle - \dot{p} \, .$$

Thus, Shafranov's condition (7) becomes

$$\ddot{\Phi}\,\dot{\Phi} + \dot{\Phi}^2\,\dot{p}\,\langle 1/B^2\rangle > 0. \qquad \text{(A.3)}$$

On the other hand, Mercier's criterion (6) is equivalent to

$$\ddot{\Phi}\,\dot{\Phi} + \dot{p}\,\langle\,(g_{\zeta\zeta} - g^2_{\theta\zeta}/g_{\theta\theta})^{-1}\rangle > 0. \qquad \text{(A.4)}$$

Because of

$$\frac{B^2}{\dot{\Phi}^2} = g_{\zeta\zeta} - 2\,g_{\theta\zeta}\,\frac{\langle g_{\theta\zeta}\rangle}{\langle g_{\theta\theta}\rangle} + g_{\theta\theta}\,\frac{\langle g_{\theta\zeta}\rangle^2}{\langle g_{\theta\theta}\rangle^2}$$

the proof is completed by observing

$$\frac{g_{\theta\zeta}^2}{g_{\theta\theta}} - 2\, g_{\theta\zeta} \frac{\langle g_{\theta\zeta} \rangle}{\langle g_{\theta\theta} \rangle} + g_{\theta\theta} \frac{\langle g_{\theta\zeta} \rangle^2}{\langle g_{\theta\theta} \rangle^2}$$

$$= \left[\frac{g_{\theta\zeta}}{\sqrt{g_{\theta\theta}}} - \frac{\langle g_{\theta\zeta} \rangle}{\langle g_{\theta\theta} \rangle} \sqrt{g_{\theta\theta}} \right]^2 > 0.$$

The reduction of the criteria (4), (6), (7) to (8) - (10) starts from the following expansions and first order relations

$$\dot{\Phi} = \dot{\Phi}^{(0)} + \dot{\Phi}^{(2)} + O(\epsilon^4), \quad \dot{\chi} = \dot{\chi}^{(2)} + O(\epsilon^4),$$

$$g_{V\theta} = g_{V\theta}^{(1)} + O(\epsilon^2), \quad g_{V\zeta} = g_{V\zeta}^{(1)} + O(\epsilon^2), \quad g_{\theta\zeta} = g_{\theta\zeta}^{(1)} + O(\epsilon^2),$$

$$g_{\theta\theta} = g_{\theta\theta}^{(0)} + O(\epsilon), \quad g_{\zeta\zeta} = g_{\zeta\zeta}^{(0)} + g_{\zeta\zeta}^{(1)} + g_{\zeta\zeta}^{(2)} + O(\epsilon^3),$$

$$\langle g_{\theta\zeta}^{(1)} \rangle = \langle g_{\zeta\zeta}^{(1)} \rangle = 0,$$

$$\langle g_{\zeta\zeta}^{(0)} \rangle = g_{\zeta\zeta}^{(0)}, \quad \langle g_{\zeta\zeta}^{(0)} \rangle^{\cdot} = 0,$$

$$g_{\theta\zeta}^{(1)} = g_{\zeta\zeta}^{(1)}, \quad (\dot{\Phi}^{(0)} g_{\zeta\zeta}^{(1)})^{\cdot} = \dot{\Phi}^{(0)} \frac{\partial}{\partial \zeta} g_{V\zeta}^{(1)}, \quad \sqrt{g}^{(1)} = 0$$

where the order is indicated by upper indices in parentheses.

Mercier's criterion (A.4) then becomes

$$-(\dot{\Phi}^{(0)})^2 \langle g_{\zeta\zeta}^{(2)} \rangle^{\cdot} + \frac{\dot{p} \langle (g_{\zeta\zeta}^{(1)})^2 \rangle}{g_{\zeta\zeta}^{(0)}} \left[\frac{1}{g_{\theta\theta}^{(0)}} + \frac{1}{g_{\zeta\zeta}^{(0)}} \right] > 0$$

and Shafranov's condition (A.3) reads

$$-(\dot{\Phi}^{(0)})^2 \langle g_{\zeta\zeta}^{(2)} \rangle^{\cdot} + \dot{p} \langle (g_{\zeta\zeta}^{(1)})^2 \rangle / (g_{\zeta\zeta}^{(0)})^2 > 0.$$

The quantity $A^{(1)}$ appearing in the sufficient criterion (4) is given by

$$(|\nabla V|^2)^{(0)} A^{(1)} = \frac{\dot{p}}{g_{\zeta\zeta}^{(0)}} \frac{\partial}{\partial \zeta} g_{V\zeta}^{(1)} - (\dot{I}^{(0)})^2 g_{\zeta\zeta}^{(1)} / (g_{\zeta\zeta}^{(0)})^2$$

$$= \frac{\dot{I}^{(0)}}{\dot{\Phi}^{(0)} g_{\zeta\zeta}^{(0)}} \left[(\dot{\Phi}^{(0)})^2 g_{\zeta\zeta}^{(1)} \right]^{\cdot}.$$

With the help of the first order results

$$\sqrt{g}^{(1)} = \varepsilon \left(\overset{1}{x}{}' + \frac{\overset{1}{x}}{\overset{0}{x}} + \overset{1}{\varphi} - \overset{0}{x} \right) \cos 2\pi (\theta + \zeta) \, ,$$

$$g_{\zeta\zeta}^{(1)} = 2 \varepsilon L^2 \left(\overset{1}{\sigma} - \overset{0}{x} \right) \cos 2\pi (\theta + \zeta) \, ,$$

$$g_{\theta\zeta}^{(1)} = \varepsilon L^2 \left[\overset{0}{x}{}^2 \overset{1}{\varphi} + \overset{1}{\sigma} \left(1 - \overset{0}{x}{}^2 \right) \right] \cos 2\pi (\theta + \zeta) \, ,$$

$$g_{V\zeta}^{(1)} = \frac{\varepsilon}{\overset{0}{x} L} \left[- \overset{1}{x} + \overset{1}{\sigma}{}' \right] \sin 2\pi (\theta + \zeta) \, ,$$

the equations (3) and the terms modifying the Mercier expression in Eqs. (8) and (10) can readily be varified. Finally, the second order result

$$\langle g_{\zeta\zeta} \rangle = L^2 \left\{ 1 + \varepsilon^2 \left[\frac{\overset{1}{x}{}^2}{2} + \frac{\overset{0}{x}{}^2}{2} \left(1 + \overset{1}{\varphi}{}^2 + \overset{1}{\sigma}{}^2 - 2 \overset{1}{\varphi} \overset{1}{\sigma} \right) + \overset{0}{x} \overset{1}{\varphi} - \overset{1}{x} - 2 \overset{0}{x} \overset{1}{\sigma} + \frac{\overset{1}{\sigma}{}^2}{2} \right] + O(\varepsilon^4) \right\}$$

leads to the expression in inequality (9).

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FIGURE CAPTIONS

Fig. 1. Displacement $\overset{1}{x}(\overset{o}{x})$ of the magnetic surfaces vs $\overset{o}{x}$ (labelling coordinate) for various β-values and pressure profiles: $\beta = 0.15$: $\alpha = 3$ and $\alpha = 81$; $\beta = 0.95$: $\alpha = 3$ and $\alpha = 81$.

Fig. 2. First-order quantity $\overset{1}{\sigma}$ ($\overset{o}{x}$) vs $\overset{o}{x}$($\overset{o}{x}$ labelling coordinate of magnetic surfaces) for various β-values and pressure profiles : $\beta = 0.15$: $\alpha = 3$ and $\alpha = 81$; $\beta = 0.95$: $\alpha = 3$ and $\alpha = 81$.

Fig. 3. Pressure profile (——), Mercier criterion (—·—), sufficient criterion (—) and Shafranov's magnetic well condition (– – –) vs $\overset{o}{x}$ for $\beta = 0.15$ and pressure profile $\alpha = 9$. A magnetic well exists throughout the plasma.

Fig. 4. Pressure profile (—), Mercier criterion (—·—), sufficient criterion (—), and Shafranov's magnetic well condition (– – –) vs $\overset{o}{x}$ for $\beta = 0.55$ and $\alpha = 27$. The Mercier criterion is satisfied in the boundary region.

Fig. 5. Pressure profile (—), Mercier criterion (—·—), sufficient criterion (—), and Shafranov's magnetic well condition (– – –) vs $\overset{o}{x}$ for a high $\beta = 0.75$ and a steep pressure profile ($\alpha = 81$). There is vacuum magnetic well surrounding the high-β plasma column.

Fig. 6. Pressure profile (—), Mercier criterion (—·—), sufficient criterion (—), and Shafranov's magnetic well condition (– – –) vs $\overset{o}{x}$ for $\beta = 0.95$ and $\alpha = 81$.

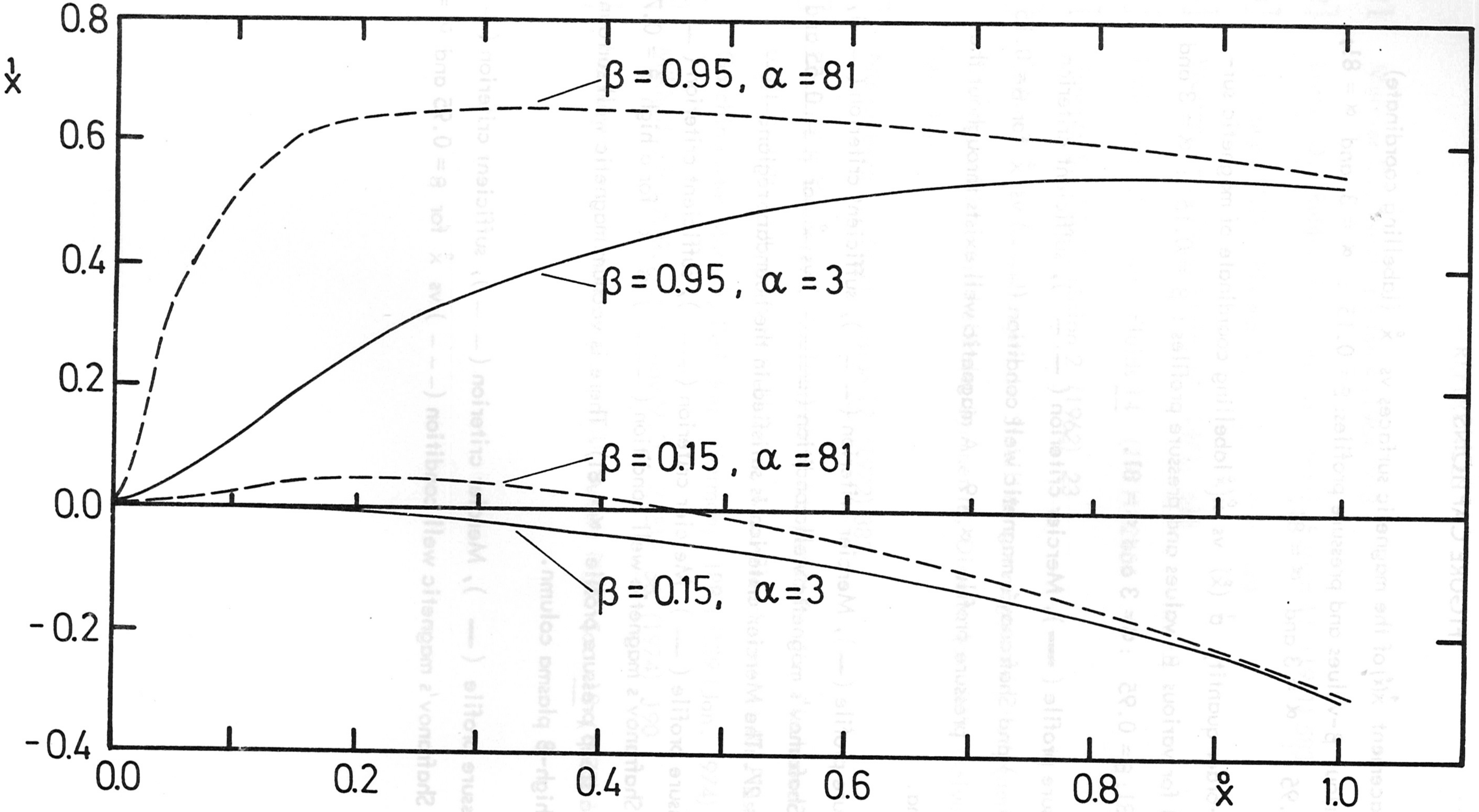

β = 0.95 , α = 81
β = 0.95 , α = 3
β = 0.15 , α = 81
β = 0.15, α = 3
0.8
0.6
0.4
0.2
0.0
- 0.2
- 0.4
0.0
0.2
0.4
0.6
0.8
1.0

Fig. 1

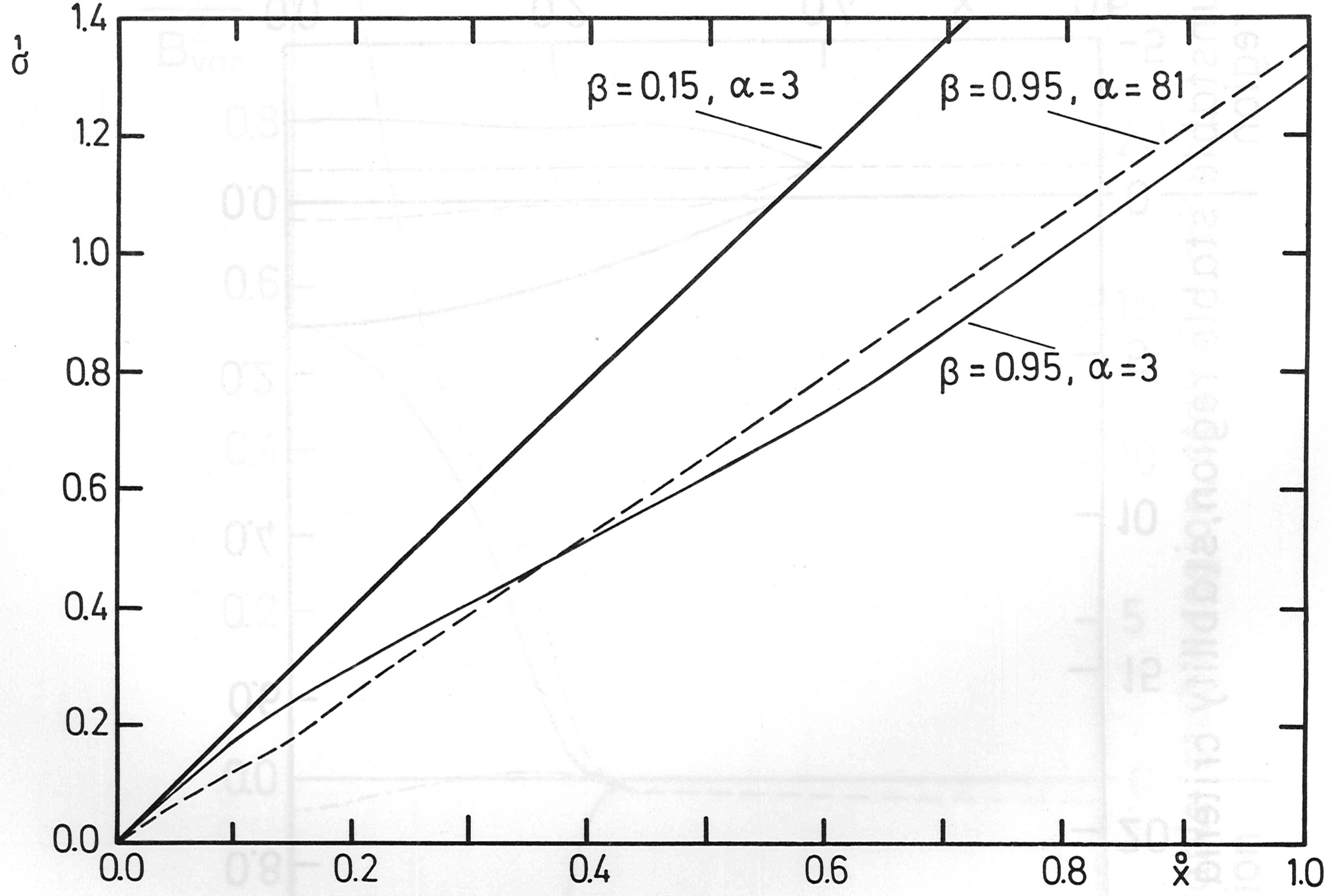
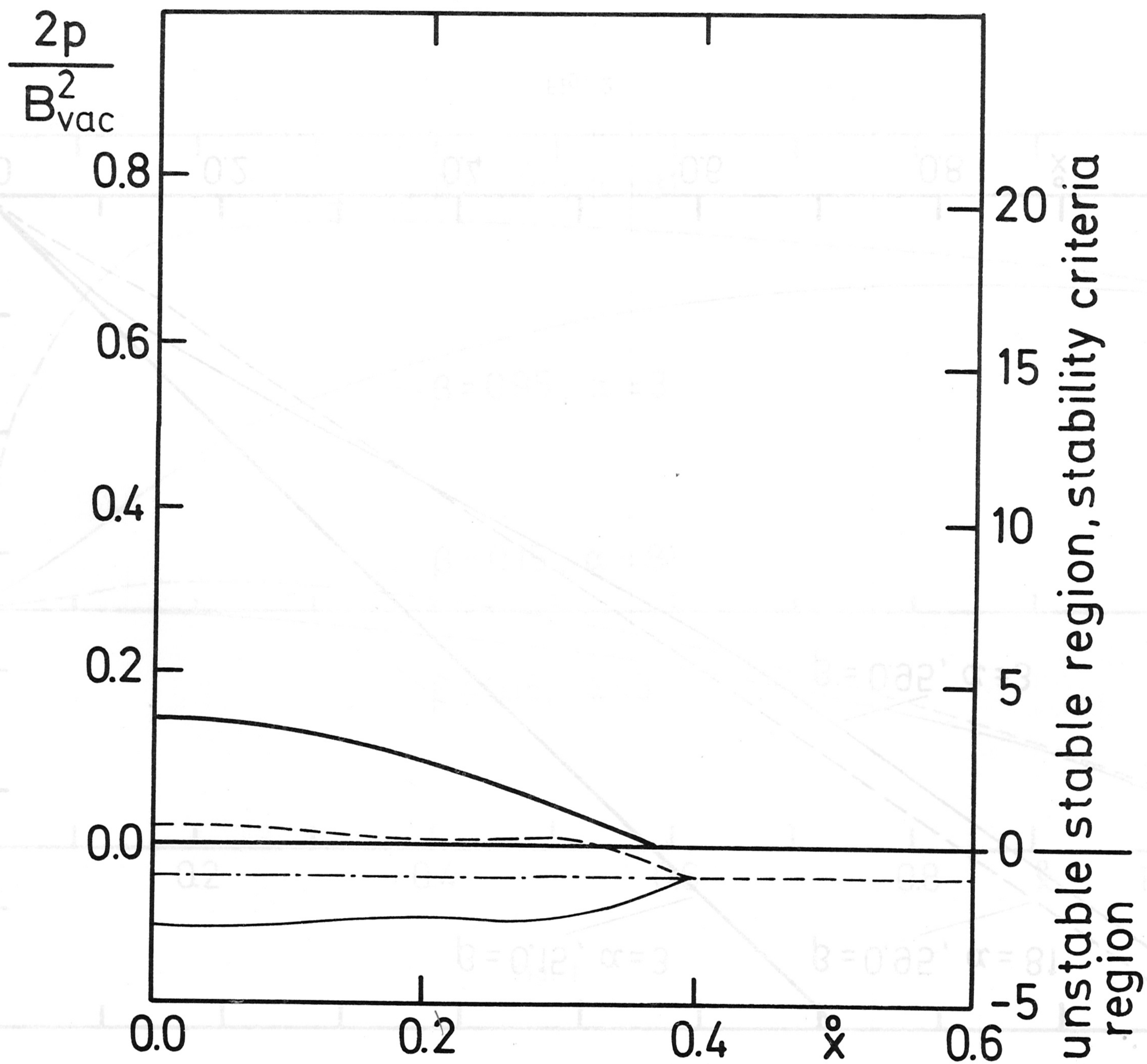
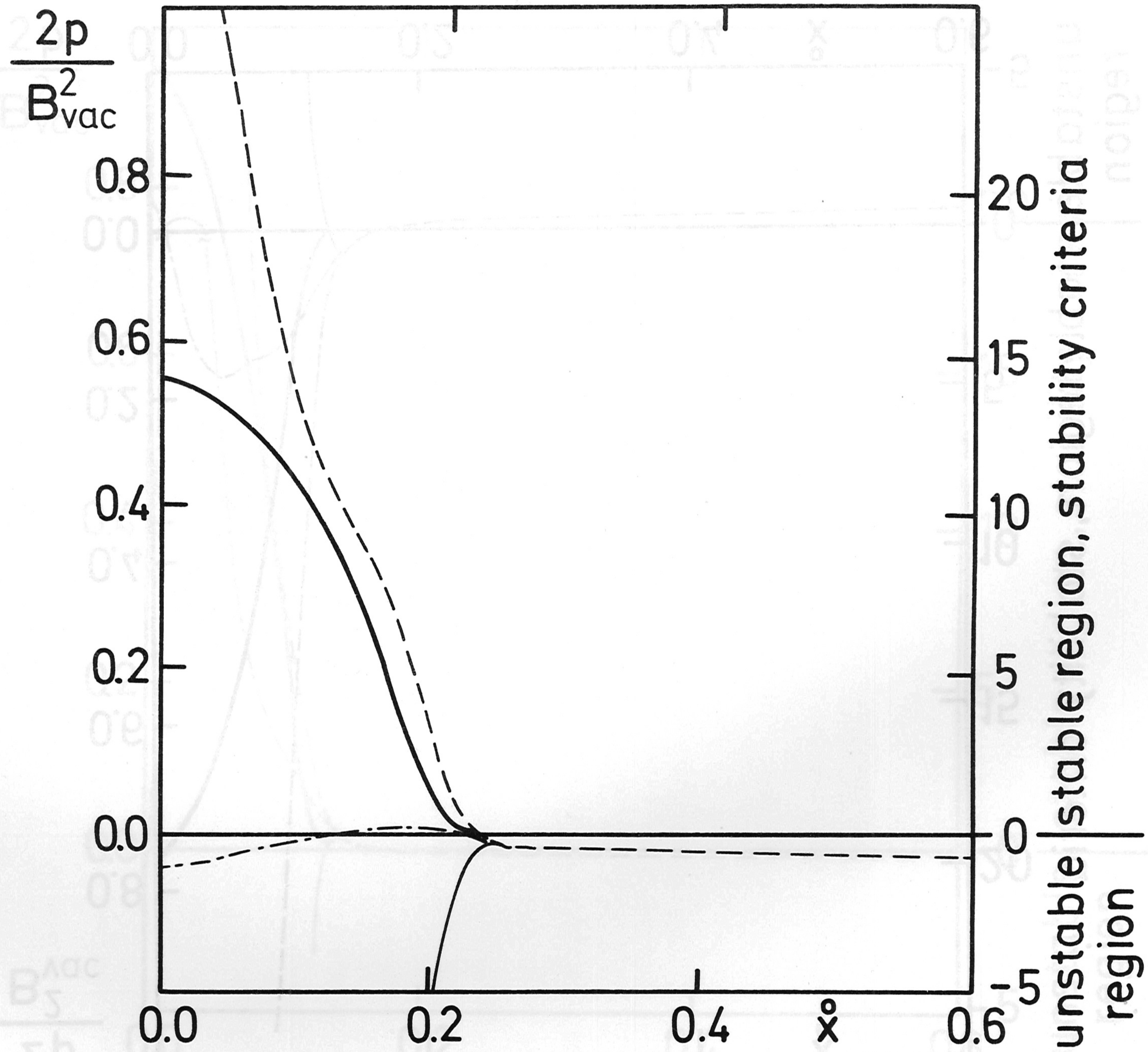


Fig. 2



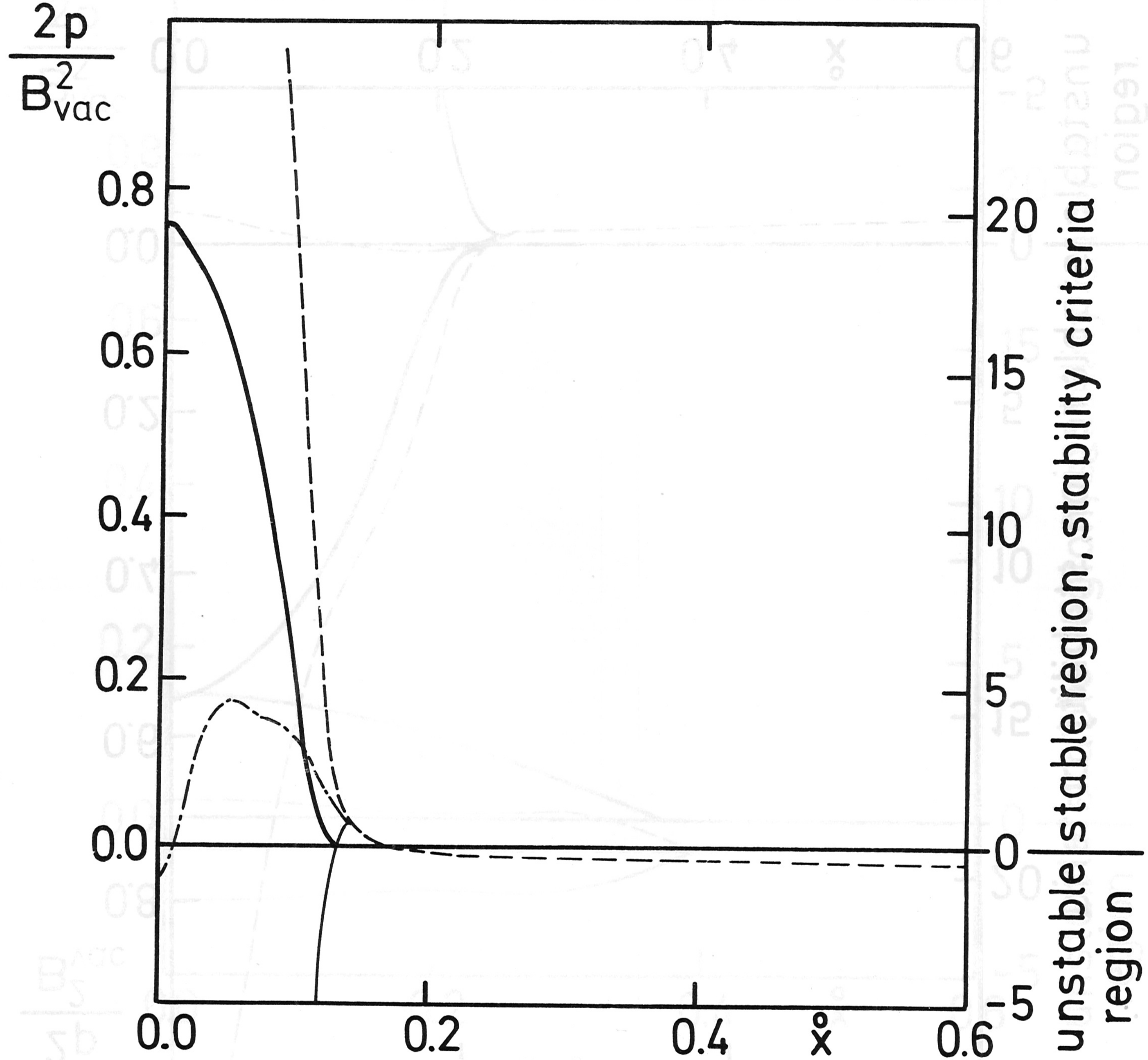
2p / B²vac
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
x̊
0.6
20
15
10
5
0
-5
unstable region
stable region, stability criteria

Fig. 3



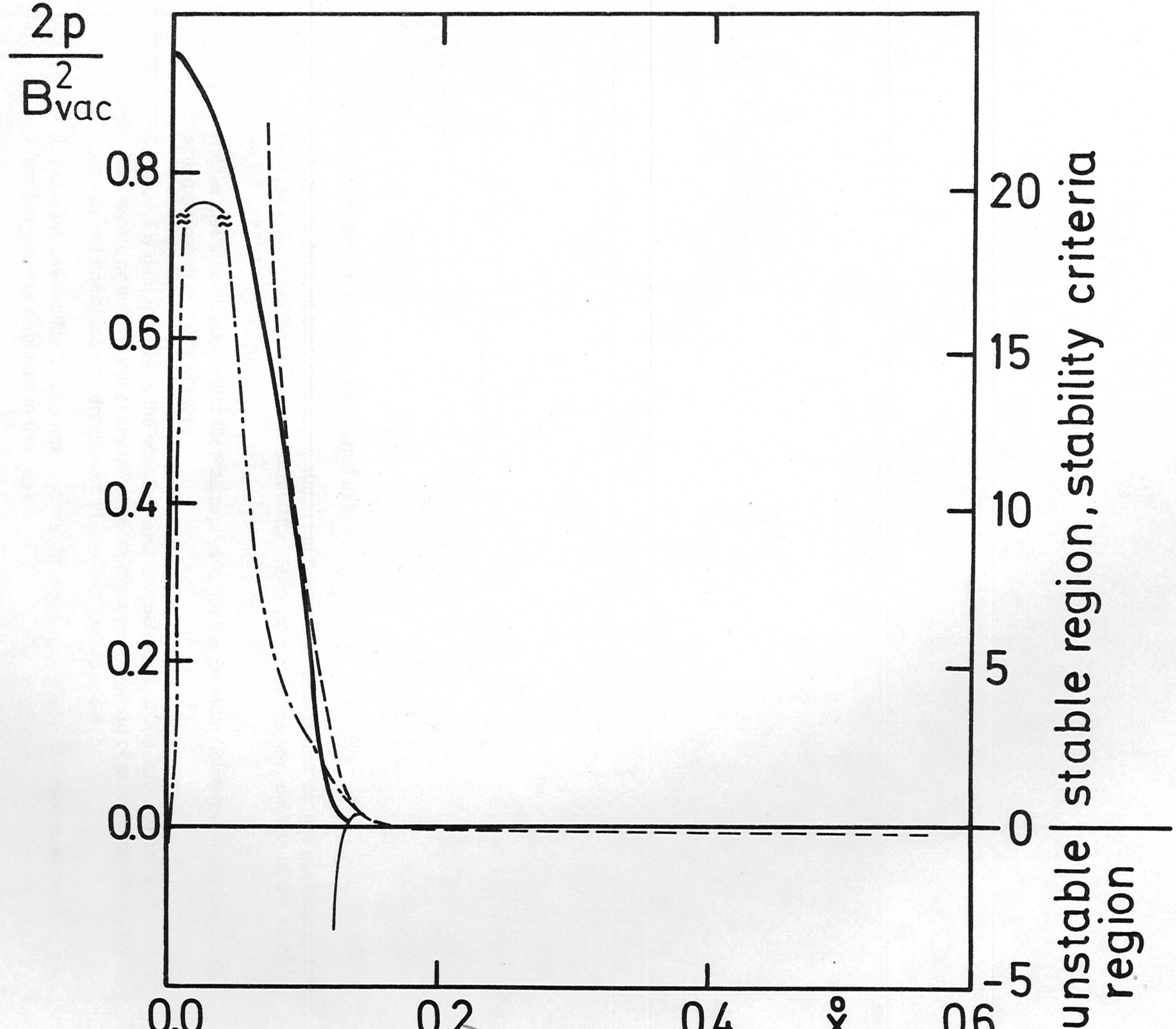
2p
B^2_{vac}
0.8
0.6
0.4
0.2
0.0
20
15
10
5
0
-5
stable region, stability criteria
unstable region
0.0
0.2
0.4
$\mathring{x}$
0.6

Fig. 4



$\frac{2p}{B_{vac}^2}$
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
$\mathring{x}$
0.6
20
15
10
5
0
-5
unstable region
stable region, stability criteria

Fig. 5



2p / B²vac
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
x̊
0.6
20
15
10
5
0
-5
stable region, stability criteria
unstable region

Fig. 6