New Macroscopic Theory of Anomalous Diffusion Induced by the Dissipative Trapped-Ion Instability.

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Abstract

For an axisymmetric toroidal plasma of the TOKAMAK type a new set of dissipative trapped-fluid equations is established. In addition to $\underline{E} \times \underline{B}$ drifts and collisions of the trapped particles, these equations take full account of the effect of $E_{\parallel}$ (of the trapped ion modes) on free and trapped particles, and of the effect of $\nabla \delta_0$ (δ_0 = equilibrium fraction of trapped particles). From the new equations the linear-mode properties of the dissipative trapped-ion instability and the anomalous diffusion flux of the trapped particles are derived.

1. INTRODUCTION

The paper deals with anomalous diffusion in a plasma torus due to the dissipative trapped-ion instability. Macroscopic theory (trapped-fluid theory) is used throughout.

The purpose of and motivation for the paper are the following. The macroscopic theory by KADOMTSEV and POGUTSE (1970/1971) of anomalous diffusion due to dissipative trapped-ion modes yields the diffusion coefficient

$$\mathcal{D} = \delta_0 \mathcal{D}_t = \delta_0 \frac{v_*^2}{2\,\nu_{eff}}, \qquad (1.1)$$

where δ_0 is the equilibrium fraction of trapped particles, $\nu_{eff} \approx \nu_e/\delta_0^2$ is the effective collision frequency of trapped electrons, and

$$v_* = \frac{cT}{2eBN_p}\,\partial_r n_0 \qquad (1.2)$$

is the trapped diamagnetic velocity, N_p and $n_0 = \delta_0 N_p$ being the total and trapped-particle densities, the remaining notation being standard.

The anomalous transport induced by the dissipative trapped-ion instability, together with other loss mechanisms, has been discussed in connection with planned experiments (FURTH, 1973). It turns out that the diffusion coefficient $\mathcal{D}$ of eq. (1.1) becomes uncomfortably large at fusion temperatures because D increases with the 7/2th power of T. Because of the possible experimental consequences it seems useful to look for effects that could either increase or decrease D above or below the value predicted by KADOMTSEV and POGUTSE (1970) and, thus, to check the validity of the orginial KADOMTSEV - POGUTSE theory. There are many points that can be and have been investigated for this purpose (microscopic effects, impurity effects, geometric effects); but here only the following items are investigated.

As a preliminary, two points are considered. Firstly, in the KADOMTSEV - POGUTSE theory ion collisions have sometimes been neglected (KADOMTSEV and POGUTSE, 1970). It then follows from the equations that the trapped-particle loss would vanish for stationary turbulence. This point of theoretical consistency is discussed and corrected in the present treatment.

Secondly, the method used by KADOMTSEV and POGUTSE in deriving the anomalous diffusion coefficient poses the following question. The linearized theory alone yields, in a certain approximation, a negative diffusion coefficient. The authors argue that, in order to have stationary turbulence, this negative diffusion must be at least compensated by nonlinearities of the equations. Finally, for the effective diffusion coefficient (i.e. the one to observe in experiment) KADOMTSEV and POGUTSE use only the nonlinear, compensating terms and disregard the negative contribution of the linearized equations. However, one could imagine that the effective diffusion arises by a combination of both linear and nonlinear terms. Another shortcoming of the KADOMTSEV-POGUTSE method is that it does not determine to what density gradient (∇N_p, ∇n_0, or whatever) the diffusion flux is proportional, in the sense that reversal of the sign of the first reverses the sign of the second. In order to improve on these points an alternative derivation of the anomalous diffusion coefficient is presented in Section 3. This derivation has the advantages, firstly, that it determines to what density gradient the diffusion flux is proportional and, secondly, that it can also be applied to more general, more correct versions of trapped-fluid equations, contrary to the method of KADOMTSEV and POGUTSE.

However, the main purpose of the paper is to include several effects in the theory of the dissipative trapped-ion instability that have been omitted by KADOMTSEV and POGUTSE. These effects are due to $\nabla \delta_0$ and to the influence of $E_{\parallel}$ (of the instability) on the free and trapped - particle motion and on the collision terms. Inclusion of these effects leads to new trapped-fluid equations and to new diffusion formulas, which will be discussed.

In Section 2 an extended version of the trapped-particle fluid equations of KADOMTSEV and POGUTSE is introduced and briefly discussed. In Section 3 the original KADOMTSEV - POGUTSE diffusion formula, up to a small numerical factor, is rederived from a critical-mode, mixing-length model for small-scale turbulence. For large-scale turbulence a new diffusion formula is derived, with $D \propto B^{-\frac{1}{2}} T^{-\frac{1}{4}}$. In Section 4 the theory is further extended to include the effect of the spatial gradient of the loss-cone angle. In Section 5 it is shown that several electrostatic effects are lacking in the KADOMTSEV - POGUTSE theory; these effects are calculated, and new, corrected trapped-fluid equations are derived. In Section 6 the new, corrected dispersion equation and critical-mode properties are derived. Finally, in Section 7, the new diffusion formula (for the case of small-scale turbulence) is presented. Section 8 gives a brief summary.

2. EQUATIONS OF THE ORIGINAL KADOMTSEV - POGUTSE THEORY

We introduce the following version of the trapped-particle fluid equations used by KADOMTSEV and POGUTSE:

$$\frac{dn_i}{dt} \equiv \frac{\partial n_i}{\partial t} + \underline{v} \cdot \nabla n_i = -\nu_{iff}(n_i - n_o) \tag{2.1}$$

$$\frac{dn_e}{dt} \equiv \frac{\partial n_e}{\partial t} + \underline{v} \cdot \nabla n_e = -\nu_{eff}(n_e - n_o) \tag{2.2}$$

$$\underline{v} = A \underline{V}(n_i - n_e) \tag{2.3}$$

$$A = \frac{cT}{2eBN_p}. \tag{2.4}$$

The version chosen takes account of both electron and ion collisions. Equations (2.1) and (2.2) are the continuity equations for the trapped-particle densities n_i and n_e in two-dimensional space. The right-hand sides describe particle transitions (free $\longleftrightarrow$ trapped) due to collisions. The velocity $\underline{v}$ is a pure $\underline{E} \times \underline{B}$ drift, given by eq. (2.3). The electric potential ϕ is derived from the quasineutrality condition, viz.

$$\phi = \frac{T}{2eN_p}(n_i - n_e). \tag{2.5}$$

The quantity n_0 is the trapped-particle equilibrium density, determined by field geometry and by the total particle equilibrium density $N_p = n_o + N_o$ where N_0 is the free-particle density in equilibrium; ν_{iff} and ν_{eff} are the effective collision frequencies for trapped ions and electrons,

B is the magnetic field strength, and $T = 2T_e T_i/(T_e+T_i)$. The operator $\underline{V}$ is defined as

$$\underline{V} = \hat{z} \times \nabla = -\hat{x}\,\partial_y + \hat{y}\,\partial_x , \tag{2.6}$$

where x, y are Cartesian coordinates in the two-dimensional space considered, and z spans the third dimension, over which some short of average has been taken in order to arrive at the above equations. The gradients of B and of (T/N_p) are neglected, hence $\nabla \cdot \underline{v} = 0$. At one place KADOMTSEV and POGUTSE use $N_0 = N_p - n_0$ instead of N_p in their equations, see KADOMTSEV and POGUTSE (1971). We shall see in Sections 5-7 that actually neither choice is correct, but for the present we choose the version employing N_p.

KADOMTSEV and POGUTSE use a slab model in which the equilibrium quantities T, N_p, n_0 depend on the Cartesian coordinate x only. Another simple possibility would be to use a cylindrical plasma with B = const, where T, N_p, n_0 only depend on the radius r. A more realistic cylindrical model would take into account $B \neq$ const and the dependence of n_0 on both r and θ. In this paper we shall use the Cartesian slab model with $\nu_{iff}, \nu_{eff} = \mathrm{const}$. In a more realistic model the dependence of the collision frequencies on T, N_p, n_0 should also be taken into account.

A different form of eqs. (2.1) to (2.3) reads

$$\dot{n} + v \cdot \nabla n = -\nu_s (n - n_0) + \nu_d \rho \tag{2.7}$$

$$\dot{\rho} = 4 \nu_d (n - n_0) - \nu_s \rho \tag{2.8}$$

$$\underline{v} = A \underline{V} \rho \tag{2.9}$$

with $\nu_s = \frac{1}{2}(\nu_{eff} + \nu_{iff})$, $\nu_d = \frac{1}{4}(\nu_{eff} - \nu_{iff})$, $n = \frac{1}{2}(n_i + n_e)$, $\rho = n_i - n_e$. This form shows explicitly that the $\underline{E} \times \underline{B}$ drift is directed along the lines ρ = const, and it has the advantages that eq. (2.8) is <u>a linear equation</u> in ρ and $(n - n_0)$. <u>A more explicit form</u> of eqs. (2.1) to (2.3) is obtained by introducing the quantities $\tilde{n}_i = n_i - n_0$, $\tilde{n}_e = n_e - n_0$, $\tilde{\rho} = \rho$, and by inserting $\underline{v}$ from eq. (2.3) in eqs. (2.1), (2.2), viz.:

$$\partial_t \tilde{n}_i - v_0 \partial_y \tilde{\rho} + A\{\tilde{n}_i, \tilde{n}_e\} + \nu_{iff} \tilde{n}_i = 0 \tag{2.10}$$

$$\partial_t \tilde{n}_e - v_0 \partial_y \tilde{\rho} + A\{\tilde{n}_i, \tilde{n}_e\} + \nu_{eff} \tilde{n}_e = 0. \tag{2.11}$$

Here $v_0 = A \partial_x n_0$, and the curly bracket is a commutator, viz.

$$\{\varphi, \psi\} = \underline{V}\varphi \cdot \nabla\psi = \partial_x \varphi \cdot \partial_y \psi - \partial_y \varphi \cdot \partial_x \psi . \tag{2.12}$$

Equations (2.10), (2.11) have the advantage of showing explicitly which terms are linear or nonlinear in the perturbations. From eqs. (2.10), (2.11) the following linear equations follow:

$$(\partial_t + \nu_{eff})\tilde{\rho} = (\nu_{eff} - \nu_{iff})\,\tilde{n}_i \tag{2.13}$$

$$(\partial_t + \nu_{iff})\tilde{\rho} = (\nu_{eff} - \nu_{iff})\,\tilde{n}_e \tag{2.14}$$

which relate $\tilde{\rho}$ to $\tilde{n}_i$ or $\tilde{n}_e$. From eq. (2.8) or from eqs. (2.13) and (2.14) the following relation for time-averaged quantities follows:

$$\nu_{iff}\,(\bar{n}_i - n_o) = \nu_{eff}\,(\bar{n}_e - n_o). \tag{2.15}$$

The equilibrium solution of the trapped-particle fluid equation is $n_j = n_o$, $\tilde{n}_j = \rho = \tilde{\rho} = 0$. The dispersion equation for linear perturbations proportional to exp $\left[(-i\omega + \gamma)t + iK_y y\right]$ is

$$-i\omega + \gamma = -\nu_s \left\{ 1 \pm \left(1 + \frac{4i\omega_0 \nu_d - \nu_{eff}\nu_{iff}}{\nu_s^2} \right)^{\frac{1}{2}} \right\} \tag{2.16}$$

with $\omega_0 = K_y v_0$. It can be shown that the growth rate γ of the unstable mode is a monotonically increasing function of $|\omega_o|$ or $|K_y|$. In the limit $|\omega_0| \ll \nu_{eff}$ one has the unstable modes

$$-i\omega + \gamma \approx i\omega_o + \left(\frac{\omega_0^2}{\nu_{eff}} - \nu_{iff} \right) \tag{2.17}$$

and the damped modes:

$$-i\omega + \gamma \approx -i\omega_o - \left(\frac{\omega_0^2}{\nu_{eff}} + \nu_{eff} \right), \tag{2.18}$$

while for $|\omega_0| \gg \nu_{eff}$

$$-i\omega + \gamma \approx \pm \left(\tfrac{1}{2} |\omega_0| \nu_{eff} \right)^{\frac{1}{2}} \left(1 + i \,\mathrm{sign}\, \omega_0 \right) \tag{2.19}$$

is obtained. The stability condition is given by

$$\nu_{iff}\, \nu_{eff} \gtrapprox \left(\frac{\nu_{eff} - \nu_{iff}}{\nu_{eff} + \nu_{iff}} \right)^2 \omega_0^2 \approx \omega_0^2 \, . \tag{2.20}$$

Thus, in the hypothetical case $\nu_{iff} = \nu_{eff}$ the instability is eliminated. KADOMTSEV and POGUTSE (1970) have derived the macroscopic dispersion equation only for the case $\nu_{iff} = 0$ and $|K_y|$ small, i.e. $|\omega_0| \ll \nu_{eff}$. In addition these authors give the dispersion equation following from microscopic theory (KADOMTSEV and POGUTSE, 1970, 1971).

Consider a slab model with periodic boundary conditions for, say, $y = 0$ and $y = b$, and with a reflecting boundary, say, at $x = 0$ an absorbing boundary, say, at $x = a$. The equilibrium quantities depend only on x , not on y . It follows from the fluid equations that the trapped current in the x -direction vanishes identically in x and t :

$$I_x = \int_0^b dy\, \rho\, v_x = - \int_0^b dy\, \rho\, A \frac{\partial \rho}{\partial y} \equiv 0 . \tag{2.21}$$

This means that trapped particle transport is ambipolar. Provided the symmetry of the equilibrium is not broken by the turbulent processes, even the time average of the local trapped-current density in the x -direction vanishes identically in stationary turbulence:

$$\langle j_x \rangle = \langle \rho v_x \rangle = \frac{1}{b} \langle I_x \rangle \equiv 0 . \tag{2.22}$$

The same relations hold, of course, for the average velocity in the x -direction, i.e.

$$\int_0^b v_x \, dy \equiv 0 \tag{2.23}$$

and

$$\langle v_x \rangle \equiv 0 . \tag{2.24}$$

If we introduce the trapped-particle fluxes in the x-direction:

$$\Gamma_x^{(j)} = \int_0^b dy \, n_j \, v_x \quad , \quad j = i, e \tag{2.25}$$

then in stationary turbulence the time averages obey the relations

$$\langle \Gamma_x^{(j)} \rangle = - \int_0^x dx' \int_0^b dy \, \nu_{jff} \left(\bar{n}_j - n_0 \right) . \tag{2.26}$$

In the case of conserved equilibrium symmetry the local relation reads

$$\langle n_j v_x \rangle = - \int_0^x dx' \, \nu_{jff} \left(\bar{n}_j - n_0 \right) . \tag{2.27}$$

Hence it is seen that putting $\nu_{iff} = 0$, as done by KADOMTSEV and POGUTSE (1970) would lead to the exact result $\langle \Gamma_x^{(j)} \rangle = \langle n_j v_x \rangle = 0$ for the case of stationary turbulence.

3. REDERIVATION OF THE KADOMTSEV - POGUTSE ANOMALOUS DIFFUSION COEFFICIENT

We consider the average particle flux densities in the x-direction, $\gamma_x^{(j)} \equiv \langle n_j v_x \rangle$. Because of eq. (2.22), $\gamma_x^{(i)} = \gamma_x^{(e)} = \gamma_x$, and, because of eq. (2.23),

$$\gamma_x = -A \langle \tilde{n}_j \, \partial_y \tilde{\rho} \rangle . \tag{3.1}$$

Inserting eq. (2.13) or (2.14) yields

$$\gamma_x \approx -\frac{A}{\nu_{eff}} \langle \partial_t \tilde{\rho} \cdot \partial_y \tilde{\rho} \rangle , \tag{3.2}$$

since $\langle \tilde{\rho} \, \partial_y \tilde{\rho} \rangle = 0$. On introducing for $\tilde{\rho}$ a mode representation

$$\tilde{\rho} = \frac{1}{\sqrt{2}} \sum_k \rho_k (x,t) \exp(-i\omega_k t + i k_y y) + c.c. , \tag{3.3}$$

with a slow time variation of the amplitudes $\rho_k(x,t)$ assumed, the <u>random phase approximation</u> yields

$$\gamma_x \approx \frac{A}{\nu_{eff}} \sum_k k_y \, \omega_k \langle |\rho_k|^2 \rangle . \tag{3.4}$$

By using eq. (2.13) the square amplitude of ρ_k can be replaced by that of n_{ik}:

$$\gamma_x \approx A \nu_{eff} \sum_k \frac{k_y \omega_k}{\nu_{eff}^2 + \omega_k^2} \langle |n_{ik}|^2 \rangle . \tag{3.5}$$

We now employ a critical-mode hypothesis; i.e. a certain number of modes with similar values of K_y and ω_k are assumed to contribute overwhelmingly to γ_x. Hence

$$\gamma_x \approx Z_k \frac{A\,\nu_{eff}\,K_y\,\omega_k}{\nu_{eff}^2 + \omega_k^2} \left\langle |n_{ik}|^2 \right\rangle , \tag{3.6}$$

where K_y and ω_k are now the critical wave number and frequency, to be determined later, and Z_k is the number of critical modes taken into account. The square amplitude of the critical mode is assumed to obey the mixing-length hypothesis, viz.

$$Z_k\, K_x^2 \left\langle |n_{ik}|^2 \right\rangle \approx |\nabla n_0|^2 . \tag{3.7}$$

If in addition, isotropic strong turbulence is assumed, we may put $K_x^2 \approx K_y^2$ in eq. (3.7). Then

$$\gamma_x \approx \frac{A\,\nu_{eff}\,\omega_k}{K_y\left(\nu_{eff}^2 + \omega_k^2\right)} |\nabla n_0|^2 , \tag{3.8}$$

where the number Z_k of critical modes has dropped out. Equation (3.7) provides that the root-mean-square velocity amplitude is of the order of the phase velocity of the small-wavelength modes.

The critical mode parameters can now be determined. We shall assume that ω_k obeys the linear dispersion equation, $\omega_k = \omega(K_y)$. Then the right-hand side of eq. (3.8) is a decreasing function of $|K_y|$, and the maximum contributions come from small $|K_y|$.

However, ambipolar diffusion can be carried only by modes with $|\omega_k| > \nu_{eff}$.

The reason is that for $|\omega_k| < \nu_{eff}$ the trapped electron density relaxes towards the equilibrium density n_0 , but because of eqs. (2.23) or (2.24) only density deviations from equilibrium contribute to diffusion. This relaxation argument is supported by the relation of the density amplitudes as derived from eqs. (2.13), (2.14), viz.

$$\langle |n_{ek}|^2 \rangle \approx \frac{\omega_k^2}{\omega_k^2 + \nu_{eff}^2} \langle |n_{ik}|^2 \rangle . \tag{3.9}$$

Hence $|\omega_k| = \nu_{eff}$ is chosen for the critical modes. The critical wave number follows from eq. (2.16), viz. $|\omega_0| \equiv |K_y v_0| = \sqrt{5}\,\nu_{eff}$, with $\omega_k / K_y < 0$ for the unstable modes and $\omega_k / K_y > 0$ for the damped modes. It follows that a definite result for the diffusion flux can be obtained with this method only if the damped modes can be assumed to have negligible amplitudes. On making this assumption one finally obtains the flux

$$\gamma_x \approx -\frac{1}{2\sqrt{5}} \frac{A v_0}{\nu_{eff}} |\nabla n_0|^2 = -\frac{1}{2\sqrt{5}} \frac{v_0^2}{\nu_{eff}} \partial_x n_0 \tag{3.10}$$

and the trapped diffusion coefficient

$$D_t \approx \frac{1}{2\sqrt{5}} \frac{v_0^2}{\nu_{eff}} = \frac{1}{2\sqrt{5}} \frac{1}{\nu_{eff}} \left(\frac{c T \nabla n_0}{2 e B N_p} \right)^2 . \tag{3.11}$$

The result differs only by a factor $1/\sqrt{5}$ from the original KADOMTSEV - POGUTSE result. The derivation is valid if the critical wavelength is smaller than the plasma radius (case of small-scale turbulence).

It is amusing to observe that, according to the above derivation, a negative diffusion coefficient could be produced in principle, if the damped modes could be sufficiently overpopulated by pumping.

Next, we consider the case of large-scale turbulence. Then the wave-length of the critical mode is of the order of the plasma diameter, i.e. $\pi/|K_y| \approx a$, say. It suffices to consider the limiting case $|\omega| >> \nu_{eff}$. Then from eq. (3.8)

$$\gamma_x \approx \frac{A\,\nu_{eff}}{K_y\,\omega_K}\,|\nabla n_0|^2 . \qquad (3.12)$$

By using eq. (2.19) one obtains

$$\gamma_x = -\,\partial_x n_0 \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left(2\nu_{eff}\,|v_0|\right)^{\frac{1}{2}} \qquad (3.13)$$

or

$$D_t = \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left(\frac{c\,T\,\nu_{eff}\,|\nabla n_0|}{e\,B\,N_p}\right)^{\frac{1}{2}} \propto B^{-\frac{1}{2}}\,T^{-\frac{1}{4}} . \qquad (3.14)$$

As in the case of small-scale turbulence the trapped-particle diffusion coefficient D_t is of the order of γ / K_y^2, taken for the critical mode. This case of large-scale turbulence occurs at higher temperatures than the small-scale turbulence case. Whether this form of diffusion is observable depends on whether in this temperature range also the collisionless trapped-particle mode occurs and whether it does in fact lead to BOHM diffusion, as forecast by KADOMTSEV and POGUTSE (1970).

4. EFFECT OF SPATIAL DEPENDENCE OF THE LOSS-CONE ANGLE

If the loss-cone angle and, hence, the quantity $\delta_0 = n_0/N_p$ depend on space coordinates, then the KADOMTSEV - POGUTSE equations, eqs. (2.1) to (2.3), are not correct. In this case, when the trapped particles move around as a result of $\underline{E} \times \underline{B}$ drift, some of them become untrapped, and drifting free particles may become trapped, by virtue of the spatial dependence of the loss-cone angle and of δ_0. This implies that the $\underline{E} \times \underline{B}$ drift of those few free particles that get trapped by the $\nabla\delta_0$-effect is now also taken into account. For the bulk of free particles the $\underline{E} \times \underline{B}$ drift is omitted as before. We shall take this $\nabla\delta_0$-effect into account in the linear approximation only. This effect will change the continuity equations, the dispersion equations, and the diffusion formulas.

Assuming that $\delta_0 = \delta_0(x)$, but still B = const, we may write the corrected version of the trapped-fluid equations as:

$$\frac{\partial n_i}{\partial t} + \underline{v} \cdot \nabla n_i = -\nu_{iff}(n_i - n_0) + N_p\, \underline{v} \cdot \nabla\delta_0 \tag{4.1}$$

$$\frac{\partial n_e}{\partial t} + \underline{v} \cdot \nabla n_e = -\nu_{eff}(n_e - n_0) + N_p\, \underline{v} \cdot \nabla\delta_0 \tag{4.2}$$

or

$$\partial_t n_i + \delta_0\, \underline{v} \cdot \nabla N_p + \underline{v} \cdot \nabla\tilde{n}_i + \nu_{iff}\,\tilde{n}_i = 0 \tag{4.3}$$

$$\partial_t n_e + \delta_0 \underline{v} \cdot \nabla N_p + \underline{v} \cdot \nabla \tilde{n}_e + \nu_{eff} \tilde{n}_e = 0. \tag{4.4}$$

In order to obtain the new dispersion equation and the new diffusion formula the quantities v_0 and ω_0 of Section 2 must now be replaced by

$$v_1 = A \delta_0 \partial_x N_p = \frac{cT\delta_0}{2eB N_p} \partial_x N_p \tag{4.5}$$

and $\omega_1 = K_y v_1$ in the formulas of Sections 2 and 3.

In the case of small-scale turbulence the new diffusion formula is:

$$\gamma_x = -\frac{1}{2\sqrt{5}} \frac{A v_1}{\nu_{eff}} |\nabla n_0|^2 = \frac{-\delta_0}{2\sqrt{5}} \frac{A^2 |\nabla n_0|^2}{\nu_{eff}} \partial_x N_p$$

$$= -\frac{1}{2\sqrt{5}} \frac{v_0^2 \delta_0}{\nu_{eff}} \partial_x N_p \, . \tag{4.6}$$

It is seen that the effect of $\nabla\delta_0$ is important. The diffusion formula of eq. (4.6) contains $\partial_x N_p$, while eq. (3.10) contains $\partial_x n_0$. While eq. (3.10) would predict inward diffusion of trapped particles near the magnetic axis, the correct formula, eq. (4.6), predicts outward diffusion, if $\partial_x N_p < 0$. At the same time, comparison of eqs. (3.10) and (4.6) shows that the present method of deriving an anomalous diffusion flux does indeed determine to what density gradient the diffusion flux is proportional, contrary to the method used by KADOMTSEV and POGUTSE (1970).

In the case of large-scale turbulence the new result for $|\omega| \gg \nu_{eff}$ is now:

$$\gamma_x = -\left(\frac{a}{\pi|\nabla N_p|}\right)^{\frac{3}{2}} |\nabla n_0|^2 \left(\frac{cT\nu_{eff}}{eBN_p\delta_0}\right)^{\frac{1}{2}} \partial_x N_p \tag{4.7}$$

instead of eq. (3.13). Again $\gamma_x \propto B^{-\frac{1}{2}}\, T^{-\frac{1}{4}}$.

5. NEW TRAPPED-FLUID EQUATIONS WITH ELECTROSTATIC CORRECTIONS

The KADOMTSEV - POGUTSE equations do not account correctly for all the effects of the electrostatic potential ϕ of the trapped-particle modes, even in linear approximation. Hence we now consider new fluid equations that include these "ϕ - effects" correctly to linear order in ϕ . This order suffices for deriving again the dispersion equation for the modes and the anomalous diffusion coefficient, by way of the mixing-length model.

The electrostatic effects to be included are now the following: $\underline{E} \times \underline{B}$ drifts of trapped particles as before ($\underline{E} \times \underline{B}$ drifts of free particles are, in the main, again omitted); perturbation of ion and electron distribution functions $f_{i,e}$ by $E_{\parallel}$ (the component of $\underline{E}$ parallel to $\underline{B}$); change of the loss-cone angles of ions and electrons by $E_{\parallel}$; and change of the (instantaneous) collision terms by $E_{\parallel}$ and by $\underline{E} \times \underline{B}$ drift. In linear approximation these effects enter the new trapped-fluid equations in the following way:

$$\frac{\partial n_i}{\partial t} + \underline{v} \cdot \nabla n_i = -\nu_{iff}(n_i - n_{io}) + N_p \underline{v} \cdot \nabla \delta_o + \left(\frac{\partial n_i}{\partial t}\right)_{\phi} \tag{5.1}$$

$$\frac{\partial n_e}{\partial t} + \underline{v} \cdot \nabla n_e = -\nu_{eff}(n_e - n_{eo}) + N_p \underline{v} \cdot \nabla \delta_o + \left(\frac{\partial n_e}{\partial t}\right)_{\phi} \tag{5.2}$$

$$N_i(\phi) + n_i = N_e(\phi) + n_e \equiv N_p'(t). \tag{5.3}$$

Here $N_{i,e}(\phi)$ are the instantaneous free-particle densities. The potential ϕ influences $N_{i,e}$ via $E_{\parallel}$ by changing $f_{i,e}$ ("f-effect") and the loss-cone factors, $\delta_0 \to \delta_{i,e}(\phi)$, ("$\delta$-effect"). The same holds for the ϕ-corrections to $\partial n_{i,e}/\partial t$ on the right-hand sides of eqs. (5.1), (5.2). In the collision terms there are now ϕ-dependent instantaneous equilibrium densities of trapped ions and electrons, $n_{io}(\phi)$ and $n_{eo}(\phi)$. They come about by variation of the plasma density, $N_p \to N_p'(t)$, eq. (5.3), and by variation of the loss-cone factors $\delta_{i,e}$. We shall give formulas for all of these terms below.

The modified trapped-fluid equations proposed by HORTON et al. (1974) and LAQUEY et al. (1975) do not agree with eqs. (5.1) to (5.3). The reason is that these authors omit the terms $N_p \underline{v} \cdot \nabla \delta_0$ and $(\partial n_\nu/\partial t)_\phi$ altogether and incorrectly evaluate the terms $N_\nu(\phi)$ and $n_{\nu o}(\phi)$, as will be explained below.

Let us first consider the loss-cone factors $\delta_{i,e}$. In the absence of ϕ one has the well-known relations for the local $\delta(r,\theta)$ in a tokamak geometry:

$$\delta \equiv \left|\frac{v_{\parallel}}{v}\right|_{crit} = \left(1 - \frac{B}{B_{max}}\right)^{\frac{1}{2}} \approx \left(1 - \frac{R-r}{R+r\cos\theta}\right)^{\frac{1}{2}}, \tag{5.4}$$

where locally $\frac{n}{N_p} = \delta$ for any isotropic distribution function. As an approximate average over θ, usually $\delta_0 \sim \sqrt{r/R}$ is used. In the presence of an electrostatic perturbation ϕ , the critical value of $|v_{\parallel}/v|$ at the boundary between trapped and free particles becomes dependent on ϕ , particle charge, and energy. An elementary calculation, using energy conservation, and $\phi = 0$ at $B = B_{max}$, yields

$$\delta_{i,e} \equiv \left|\frac{v_{\parallel}}{v}\right|_{crit} = \left(\frac{\delta^2 - \frac{2q\phi}{m v_0^2}}{1 - \frac{2q\phi}{m v_0^2}}\right)^{\frac{1}{2}}, \tag{5.5}$$

with $v_0^2 = v^2 + 2q\phi/m$, and $q = \pm e$ being the charge of ions or electrons. The assumption $\phi = 0$ at $B = B_{max}$ follows from microscopic theory (see COPPI and REWOLDT, 1974) and is consistent with the new quasineutrality condition in its final form [see below eq. (5.22)] . In linear approximation in ϕ one has

$$\delta_{i,e} = \delta \mp \frac{e\phi}{m_{i,e} v^2} \cdot \frac{1-\delta^2}{\delta}, \tag{5.6}$$

an approximation that formally breaks down for $\delta \to 0$. This will not be of concern, however, because for $\delta \to 0$ the contribution to anomalous transport vanishes, in a fashion such that the validity condition for eq. (5.6) remains satisfied by the mixing length hypothesis, for $\delta \to 0$.

Next we consider the linear <u>perturbation</u> of the ion and electron <u>distribution functions</u> by $E_{\parallel}$ alone; with collisions, etc., neglected. The linear 1-dimensional Vlasov equation (in the limit of vanishing gyroradius and without $\underline{E} \times \underline{B}$ drift) reads:

$$\frac{\partial f_1}{\partial t} + v_{\parallel} \frac{\partial f_1}{\partial z} + \dot{W} \frac{\partial f_0}{\partial W} = 0 \tag{5.7}$$

where the magnetic moment μ is constant in time, and $W = v^2$ obeys

$$\dot{W} \equiv (v^2)^{\boldsymbol{\cdot}} = -\frac{2q}{m} v_{\parallel} \frac{\partial \phi}{\partial z} . \tag{5.8}$$

In the limit of slow time variations, i.e. for $|\omega| \ll K_{\parallel} v_{\parallel}$ and for $f_0 =$ Maxwellian, the solution is $f_1 = -f_0 q_{\nu} \phi / T_{\nu}$, $\nu = i, e$; hence

$$\left.\begin{aligned} f_i &\approx f_0 \left(1 - \frac{e\phi}{T_i}\right) \\ f_e &\approx f_0 \left(1 + \frac{e\phi}{T_e}\right) \end{aligned}\right\} \tag{5.9}$$

which agrees with the linear approximation of the Boltzmann factor.

The <u>free particle density</u> can now be calculated, viz.

$$N_{i,e} = \pi \int_0^{\infty} dW \int_0^{1-\delta_{i,e}^2} d\lambda \left(\frac{W}{1-\lambda}\right)^{\frac{1}{2}} f_{i,e} \ , \tag{5.10}$$

with $W = v^2$, $\lambda = (v_{\perp}/v)^2$; δ_{ν} and f_{ν} being given by eqs. (5.6) and (5.9), respectively. The result is

$$N_i(\phi) = N_0 + \frac{1-\delta}{\delta} \frac{eN_p\phi}{T_i} = N_0 - \frac{T}{2T_i} \rho \tag{5.11}$$

$$N_e(\phi) = N_0 - \frac{1-\delta}{\delta} \frac{eN_p\phi}{T_e} = N_0 + \frac{T}{2T_e} \rho . \tag{5.12}$$

Notice that the δ-effect predominates over the f-effect, so that the sign of the perturbation is the reverse one compared with the linear approximation of the Boltzmann factor. The elimination of Φ by ϱ in eqs. (5.11), (5.12) follows from the quasineutrality condition, eq. (5.3), and $n_i - n_e = \varrho$. The free-particle densities used in the quasineutrality condition of HORTON et al. (1974) and LAQUEY et al. (1975) do not agree with eqs. (5.11) and (5.12) because these authors neglect the predominant δ-effect.

In a similar manner the Φ-contribution to $\partial n_\nu/\partial t$, $\nu = i, e$, are calculated from

$$\left(\frac{\partial n_\nu}{\partial t}\right)_\Phi = \frac{\partial}{\partial t}\pi\int_0^\infty dW \int_{1-\delta_\nu^2}^1 d\lambda \left(\frac{W}{1-\lambda}\right)^{\frac{1}{2}} f_\nu . \tag{5.13}$$

The result is:

$$\left(\frac{\partial n_i}{\partial t}\right)_\phi = -\frac{eN_p}{T_i\delta}\frac{\partial\phi}{\partial t} = +\frac{T}{2T_i}\frac{1}{1-\delta}\frac{\partial\varrho}{\partial t} \tag{5.14}$$

$$\left(\frac{\partial n_e}{\partial t}\right)_\phi = +\frac{eN_p}{T_e\delta}\frac{\partial\phi}{\partial t} = -\frac{T}{2T_e}\frac{1}{1-\delta}\frac{\partial\varrho}{\partial t} . \tag{5.15}$$

Next the variation of the collision term with Φ is considered. As mentioned above, the instantaneous equilibrium densities of the trapped particles are influenced by the varying plasma density and the varying loss-cone factors. The equilibrium densities are obtained by

$$n_{0\nu}(\Phi) = \pi \int_0^\infty dW \int_{1-\delta_\nu^2(\Phi)}^{1} d\lambda \left(\frac{W}{1-\lambda}\right)^{\frac{1}{2}} f_0(W) \frac{N_p'(t)}{N_p} \tag{5.16}$$

where $f_0 N_p' / N_p$ represents the isotropized true distribution function (ν = i,e). In linear approximation one obtains:

$$n_{0\nu}(\Phi) = \delta N_p'(t) + \pi \int_0^\infty dW \int_{1-\delta_\nu^2}^{1-\delta^2} d\lambda \left(\frac{W}{1-\lambda}\right)^{\frac{1}{2}} f_0 \,. \tag{5.17}$$

By evaluating this expression the following corrected collision terms are obtained:

$$\nu_{iff}(n_i - n_{i0}) = \nu_{iff}(1-\delta)\tilde{n}_i - \nu_{iff}\frac{T}{2T_i}\rho \tag{5.18}$$

$$\nu_{eff}(n_e - n_{e0}) = \nu_{eff}(1-\delta)\tilde{n}_e + \nu_{eff}\frac{T}{2T_e}\rho \,. \tag{5.19}$$

The collision terms used instead by HORTON et al. (1974) and LAQUEY et al. (1975) do not agree with eqs. (5.18) and (5.19) because these authors again neglect the δ-effect and, in addition, any changes in the total plasma density that arise from $\underline{E} \times \underline{B}$ drifts.

The local $\delta(r,\theta)$ was used for the above calculations. In order to obtain average results relevant for all θ and for a definite value of r, δ should now be identified with the θ-average $\delta_0(r) \approx \sqrt{r/R}$.

By inserting the above expressions into eqs. (5.1) to (5.3) the following new set of trapped-fluid equation is then obtained:

$$\partial_t(\tilde{n}_i - c_i \rho) + \delta_0 \underline{v} \cdot \nabla N_p + \lambda_i \tilde{n}_i - \mu_i \rho = N.L.T. \tag{5.20}$$

$$\partial_t(\tilde{n}_e + c_e \rho) + \delta_0 \underline{v} \cdot \nabla N_p + \lambda_e \tilde{n}_e + \mu_e \rho = N.L.T. \tag{5.21}$$

$$\phi = -\frac{T}{2eN_p} \frac{\delta_0}{1-\delta_0} \rho \tag{5.22}$$

$$\underline{v} = A_2 \underline{V} \rho \equiv -\frac{cT}{2eBN_p} \cdot \frac{\delta_0}{1-\delta_0} \underline{V} \rho , \tag{5.23}$$

with the abbreviations:

$$c_{i,e} = \frac{1}{1-\delta_0} \frac{T}{2T_{i,e}} \tag{5.24}$$

$$\lambda_{i,e} = \nu_{iff,\,eff} (1-\delta_0) \tag{5.25}$$

$$\mu_{i,e} = \nu_{iff,\,eff} \frac{T}{2T_{i,e}} . \tag{5.26}$$

It is worth noting that the relation between ϕ and ρ [eqs. (5.22), (5.23)] differs from the original KADOMTSEV - POGUTSE one not only in magnitude, but also in sign. Nevertheless, on account of the other ϕ-corrections, occurring in the continuity equations, it will turn out that the linear dispersion equation is modified only to a moderate extent.

6. NEW DISPERSION EQUATION AND NEW CRITICAL MODE PROPERTIES

From the new trapped-fluid equations, eqs. (5.20) to (5.23), the following dispersion equation is easily derived:

$$(-i\omega+\gamma)^2 + \nu_1(-i\omega+\gamma) - i\omega_2\nu_2 + \nu_3^2 = 0 \tag{6.1}$$

or

$$-i\omega+\gamma = \frac{\nu_1}{2}\left\{-1 \pm \left(1 + \frac{4i\omega_2\nu_2 - 4\nu_3^2}{\nu_1^2}\right)^{\frac{1}{2}}\right\} \tag{6.2}$$

with

$$\nu_1 = (\nu_{eff} + \nu_{iff})(1-\delta_0) \tag{6.3}$$

$$\nu_2 = -(\nu_{eff} - \nu_{iff})(1-\delta_0)^2/\delta_0 \tag{6.4}$$

$$\nu_3^2 = \nu_{iff}\nu_{eff}(1-\delta_0)^2 \tag{6.5}$$

$$\omega_2 = -K_y \frac{cT\delta_0^2(\partial N_p/\partial x)}{2eBN_p(1-\delta_0)} = -\frac{\omega_1\delta_0}{1-\delta_0} . \tag{6.6}$$

Asymptotic expressions of eq. (6.2) are, for large K_y (with ν_{iff} neglected):

$$-i\omega+\gamma \approx \pm\left[\tfrac{1}{2}|\omega_1|\nu_{eff}(1-\delta_0)\right]^{\frac{1}{2}}(1 + i\,\mathrm{sign}\,\omega_1) , \tag{6.7}$$

and for small K_y (unstable mode):

$$-i\omega + \gamma = +i\omega_1 + \left[\frac{\omega_1^2}{\nu_{eff}(1-\delta_0)} - \nu_{iff}(1-\delta_0)\right], \tag{6.8}$$

ω_1 being given, as in Section 4, by

$$\omega_1 = K_y \frac{cT\delta_0(\partial N_p/\partial x)}{2eBN_p(1-\delta_0)} . \tag{6.9}$$

It is seen that the dispersion equation is not modified much by the "ϕ-effect" except for $\delta_0 \to 1$; still, the dissipative trapped-particle mode is somewhat <u>more unstable</u> (larger growth rate) than predicted from the earlier theory.

Next we consider the <u>critical modes</u> and their properties. As in Section 3, we choose as the critical mode frequency the largest eigenvalue of the collision terms. From standard algebra the largest eigenvalue is found to be: $\nu_{max} \approx \nu_{eff}(1-\delta_0)$, where ν_{iff} has been neglected; hence the critical-mode frequency is

$$\omega_c \approx \nu_{eff}(1-\delta_0) . \tag{6.10}$$

Inserting this into eq. (6.1), the quantities γ and ω_2 are easily determined. In the limit $\nu_3 \to 0$ (i.e. $\nu_{iff} \to 0$):

$$\gamma_c = \left[\left(\frac{\nu_1}{2}\right)^2 + \omega_c^2\right]^{\frac{1}{2}} - \frac{\nu_1}{2} = \frac{\sqrt{5}-1}{2}\nu_{eff}(1-\delta_0) \tag{6.11}$$

$$\omega_{2c} = -\frac{\omega_c}{\nu_2}\left(\nu_1^2 + 4\omega_c^2\right)^{\frac{1}{2}} = +\sqrt{5}\,\delta_0\,\nu_{eff} \tag{6.12}$$

for the unstable modes. In terms of ω_1 :

$$\omega_{1c} \equiv -\omega_{2c} \frac{1-\delta_0}{\delta_0} = -\sqrt{5}\,\nu_{eff}(1-\delta_0). \tag{6.13}$$

This is to be compared with $\omega_{0c} = -\sqrt{5}\,\nu_{eff}$ in Section 3 and $\omega_{1c} = -\sqrt{5}\,\nu_{eff}$ in Section 4, respectively. Again, the critical mode properties are not significantly modified by the " ϕ -effect", except for $\delta_0 \to 1$.

7. NEW ANOMALOUS DIFFUSION FORMULA

The derivation of the diffusion flux density is analogous to Section 3. Now, however

$$\gamma_x \equiv \langle n_i v_x \rangle = -A_2 \langle \tilde{n}_i \, \partial_y \varrho \rangle . \tag{7.1}$$

Inserting

$$\tilde{n}_i \approx \left[1 - \frac{T}{2T_e(1-\delta_0)} - \frac{1}{\nu_{eff}} \frac{\delta_0}{(1-\delta_0)^2} \partial_t \right] \varrho \tag{7.2}$$

which follows from eqs. (5.20) to (5.23), and observing the periodic boundary conditions in y yields

$$\gamma_x = \frac{A_2}{\nu_{eff}} \frac{\delta_0}{(1-\delta_0)^2} \langle \partial_t \varrho \cdot \partial_y \varrho \rangle$$

$$= -\frac{A}{\nu_{eff}} \frac{\delta_0^2}{(1-\delta_0)^3} \langle \partial_t \varrho \cdot \partial_y \varrho \rangle , \tag{7.3}$$

with, again, $A = cT/2eBN_p$. Using once more a <u>mode representation</u> and the <u>random-phase approximation</u> leads to

$$\gamma_x = \frac{A}{\nu_{eff}} \sum_k \omega K_y \frac{\delta_0^2}{(1-\delta_0)^3} \langle |\varrho_k|^2 \rangle . \tag{7.4}$$

From eq. (7.2) one obtains

$$\langle |\varrho_k|^2 \rangle = (1-\delta_0)^2 \nu_{eff}^2 \left\{ \left(1-\delta_0-\frac{T}{2T_e}\right)^2 \nu_{eff}^2 + \frac{\delta_0^2 \omega^2}{(1-\delta_0)^2} \right\}^{-1} \langle |n_{ik}|^2 \rangle . \tag{7.5}$$

This, together with the critical-mode hypothesis and the mixing-length hypothesis for isotropic, strong turbulence (see Section 3) now leads to

$$\gamma_x = A \nu_{eff} \frac{\omega \delta_0^2}{K_y (1-\delta_0)} \left\{ \left(1-\delta_0-\frac{T}{2T_e}\right)^2 \nu_{eff}^2 + \frac{\delta_0^2 \omega^2}{(1-\delta_0)^2} \right\}^{-1} |\nabla n_0|^2 . \tag{7.6}$$

It can be shown that the amplitudes determined by the mixing-length hypothesis satisfy the validity condition for linearizing $\delta_{i,e}(\phi)$ for typical machine parameters in a wide range of δ_0, including $\delta_0 \to 0$. The validity condition itself has the form $2e|\phi| < 3\delta_0^2 T_{i,e}$. Inserting the critical-mode parameters leads to the final expression for anomalous diffusion:

$$\gamma_x = -\frac{1}{2\sqrt{5}} \frac{v_0^2 \delta_0}{\nu_{eff}} \partial_x N_p \cdot K_1\left(\delta_0, \frac{T}{T_e}\right), \tag{7.7}$$

with

$$K_1 = \frac{2\delta_0^2}{1-\delta_0}\left\{\left(1-\delta_0-\frac{T}{2T_e}\right)^2+\delta_0^2\right\}^{-1}. \tag{7.8}$$

The correction factor K_1 is defined in such a way that eq. (7.7) agrees with eq. (4.6) for $K_1 = 1$. It is seen that $K_1 \to 0$ for $\delta_0 \to 0$. However, in a fat torus the relevant values of δ_0 are between, say, 0.3 and 0.7, if the corresponding partial volumes are considered. In this range K_1 falls between 2 and 6 for $T_i \lesssim T_e$, as is seen from Table I.

In conclusion of this Section, even though the electrostatic corrections considered drastically change the trapped-fluid equations (they are in fact not "corrections", but an essential part of the physics involved), the final expressions for the dispersion equation and the diffusion flux are not altered in order of magnitude in the range of physically relevant values of δ_0.

8. SUMMARY

The diffusion coefficient induced by the dissipative trapped-ion instability has been rederived by a method that is able to determine to what density gradient (∇N_p, ∇n_0, or whatever) the diffusion flux is proportional (a result that cannot be obtained from the method used by KADOMTSEV and POGUTSE, 1970), and that can be applied to more general, modified trapped-fluid equations than the ones introduced by KADOMTSEV and POGUTSE.

The validity of the original KADOMTSEV - POGUTSE trapped-fluid equations has been investigated. Terms accounting for effects omitted from the original equations have been added, and, thus, a new set of trapped-fluid equations has been established. The additional effects accounted for refer to $\nabla \delta_0$, and to $E_\parallel$ of the instability.

From the new equations the anomalous diffusion has been derived, and it is found that the new result exhibits a different dependence on δ_0 as compared to the KADOMTSEV - POGUTSE formula, but does not deviate in order of magnitude in the range of relevant values of δ_0. This order-of-magnitude agreement of the two diffusion coefficients is not a completely trivial one, in view of the fact that the two sets of trapped-fluid equations differ drastically from one-another.

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$\left(\frac{r}{R}\right)^2$	$\frac{R}{r}$	δ_0	K_1 $(T_i = T_e)$	K_1 $(T_i = \frac{1}{2} T_e)$
0.01	10.0	0.316	2.18	1.32
0.02	7.07	0.376	2.90	2.01
0.04	5.00	0.447	3.56	2.92
0.06	4.08	0.495	3.96	3.54
0.08	3.54	0.532	4.26	4.02
0.10	3.16	0.562	4.51	4.41
0.12	2.89	0.589	4.76	4.78
0.14	2.67	0.612	4.98	5.11
0.16	2.50	0.632	5.21	5.42
0.18	2.36	0.651	5.44	5.73
0.20	2.24	0.669	5.67	6.04

Table I. Numerical values of the correction factor K_1 for the anomalous diffusion flux density [see eqs. (7.7), (7.8)] .