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Energy Principle for 2-d Resistive Instabilities

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Abstract

An energy principle for 2-d resistive instabilities has been found. It leads to a necessary and sufficient condition for stability allowing the use of test functions. One simple consequence is that the current density in a plasma with arbitrary cross section should not increase to the outside. Otherwise the plasma would be unstable against resistive instabilities.

Resistive instabilities are important for the long-time confinement of plasmas. It is difficult to study them because of the character and the higher order of the equations governing a dissipative plasma.

We consider here the most general 2-d class of simultaneously resistive and static equilibria. Around such an equilibrium we consider incompressible perturbations which are also 2-dimensional, conserve symmetry and are otherwise general.

Then an energy principle can be found in the form of a quadratic functional. Its positiveness is a necessary and sufficient condition for stability.

I. Equilibrium

The equations governing the equilibrium are:

$$\underline{j}_o \times \underline{B}_o = \nabla p_o \qquad \underline{v}_o = 0 \tag{1}$$

$$\nabla \times \underline{B}_o = \underline{j}_o \tag{2}$$

$$\nabla \cdot \underline{B}_o = 0 \tag{3}$$

$$\nabla \times \eta_o \underline{j}_o = 0 \tag{4}$$

$$\nabla \cdot \eta_o \underline{j}_o = 0 \tag{5}$$

$\underline{B}_o$ is the magnetic field, p_o the pressure and η_o the resistivity assumed constant on pressure surfaces.

Equations (1), (4) and (5) exclude an exact zero velocity toroidal equilibrium. For small flow velocities the torus might be approximated by the straight case in many situations. The meridional current must vanish because of $\oint \underline{j}_0 \cdot \underline{dl} = 0$ in the plasma section. So using Eqs. (1), (2) and (3) we obtain:

$$\underline{j}_0 = \underline{e}_z J_0(\psi) \tag{6}$$

$$\underline{E}_0 = \underline{ct} = \underline{e}_z \eta_0(\psi) J_0(\psi) \tag{7}$$

$$\underline{B}_0 = \underline{e}_z \times \nabla\psi + (\underline{e}_z \cdot \underline{B}_0)\, \underline{e}_z \tag{8}$$

ψ being the meridional magnetic flux and $\underline{e}_z$ the basis vector along the ignorable coordinate.

$$\nabla^2\psi = J_0(\psi) = -\frac{dP_0}{d\psi} \tag{9}$$

II. Perturbations and Stability Equations

The perturbations around the equilibrium are indexed by 1 and the displacement vector is called $\underline{\xi}$. The perturbed equations of motion lead to:

$$\rho_0 \ddot{\underline{\xi}} + \nabla p_1 - \underline{j}_1 \times \underline{B}_0 - \underline{j}_0 \times \underline{B}_1 = 0 \tag{10}$$

$$\nabla \cdot \underline{\xi} = 0 \tag{11}$$

$$\dot{\underline{A}}_1 + \eta_0 \nabla\times\nabla\times \underline{A}_1 + \eta_1 \underline{j}_0 - \dot{\underline{\xi}} \times \underline{B}_0 = 0 \tag{12}$$

$$\underline{B}_1 = \nabla\times \underline{A}_1 \tag{13}$$

$$\eta_1 = -\underline{\xi} \cdot \nabla\eta_0 \tag{14}$$

If the perturbations do not depend on the ignorable coordinate z, then it follows from eqs. (11) and (13) that

$$\underline{\xi} = \underline{e}_z \times \nabla U \qquad \underline{B}_1 = -\underline{e}_z \times \nabla A = \nabla \times \underline{e}_z A$$

$$\underline{j}_1 = -\underline{e}_z \nabla^2 A \qquad \nabla \times \underline{j}_1 = -\nabla(\nabla^2 A) \times \underline{e}_z$$

where U and A are two scalars independent of z

Substituing these expressions as well as the expression for η_1 from eq. (14) in eqs. (10) and (12) and then taking the curl of eq. (10), we obtain

$$-\nabla \cdot \rho_0 \nabla \ddot{U} - \underline{B}_0 \cdot \nabla(\nabla^2 A) - \nabla \times \underline{j}_0 \cdot \nabla A = 0 \tag{15}$$

$$\dot{A} + \underline{B}_0 \cdot \nabla \dot{U} - \eta_0 \nabla^2 A + y_0 \left(\underline{e}_z \times \nabla \eta_0 \cdot \nabla U \right) = 0 \tag{16}$$

In the derivation of eq. (16) the term $-(\underline{e}_z \cdot \underline{B}_0) \nabla \dot{U}$ has been omitted because the vector potential $\underline{A}_1$ can always be redefined to eliminate any gradient.

If $\nabla^2 A$ is taken from eq. (16) ($\eta_0 \neq 0$) and inserted in eq. (15), we obtain the following system of equations in matrix operatorial form:

$$\begin{pmatrix} -\nabla \cdot \rho_0 \nabla & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{U} \\ \ddot{A} \end{pmatrix} + \begin{pmatrix} \dfrac{-(\underline{B}_0 \cdot \nabla)^2}{\eta_0} & \dfrac{-(\underline{B}_0 \cdot \nabla)}{\eta_0} \\ \dfrac{\underline{B}_0 \cdot \nabla}{\eta_0} & \dfrac{1}{\eta_0} \end{pmatrix} \begin{pmatrix} \dot{U} \\ \dot{A} \end{pmatrix} + \tag{17}$$

$$+ \begin{pmatrix} -\gamma_0 \underline{B}_0 \cdot \nabla \, \underline{e}_z \times \frac{\nabla \eta_0}{\eta_0} \cdot \nabla & -\nabla \times \underline{j}_0 \cdot \nabla \\ \gamma_0 \, \underline{e}_z \times \frac{\nabla \eta_0}{\eta_0} \cdot \nabla & -\nabla^2 \end{pmatrix} \begin{pmatrix} u \\ A \end{pmatrix} = 0$$

or $$N \ddot{Y} + M \dot{Y} + Q Y = 0 \qquad Y = \begin{pmatrix} u \\ A \end{pmatrix}$$

and $\nabla \times \underline{j}_0 = \gamma_0 \, \underline{e}_z \times \frac{\nabla \eta_0}{\eta_0}$ because of eq. (4)

Equation (17) is the stability equation. Let us now investigate the matrix operators of this equation.

III. Boundary Conditions and Symmetry Properties

The simplest set of boundary conditions is to let u and A be univalued and u and A go to zero at infinity in the section of the plasma. We prove now that all the matrix operators N, M, Q are symmetric, and that N and M are positive definite.

For N and M we have

$$-\int f \, \nabla \cdot \rho_0 \nabla g \, d\tau = \int \rho_0 \nabla g \cdot \nabla f \, d\tau$$

the integral over the divergence term vanishes because of the boundary conditions.

$$-\int f_1 \frac{(\underline{B}_0\cdot\nabla)^2}{\eta_0} g_1 \, d\tau - \int f_1 \frac{\underline{B}_0}{\eta_0}\cdot\nabla g_2 \, d\tau + \int f_2 \frac{\underline{B}_0}{\eta_0}\cdot\nabla g_1 \, d\tau + \int \frac{f_2 g_2}{\eta_0} d\tau =$$

$$\int \frac{(\underline{B}_0\cdot\nabla f_1)(\underline{B}_0\cdot\nabla g_1)}{\eta_0} d\tau + \int g_2 \frac{\underline{B}_0}{\eta}\cdot\nabla f_1 \, d\tau - \int g_1 \frac{\underline{B}_0}{\eta_0}\cdot\nabla f_2 \, d\tau + \int \frac{f_2 g_2}{\eta_0} d\tau$$

$\nabla\cdot\underline{B}_0 = 0$ and $\eta_0 = \eta_0(\Psi)$ and the fact that $\underline{f}$ and $\underline{g}$ are univalued make the symmetry of M evident.

The positiveness of (Y, NY) and (Y, MY) is also evident. Similarly it is possible to prove that the operator Q is symmetric. The parentheses mean the integral over the volume of the salar product $\underline{Y}\cdot\underline{N}Y$ and $\underline{Y}\cdot\underline{M}Y$

These properties allow us to use a theorem [1] on the stability of dissipative systems which states that the necessary and sufficient condition for exponential stability is

$$(Y, QY) \geqq 0 \tag{18}$$

where the scalar product is defined by the integration in L^2 space.

Viscosity would not alter this criterion but would modify the operator M without affecting its symmetry and positiveness and hence would only change the growth rates of instabilities.

IV. Explicit Criterion and Application

Using eq. (8) and $\eta_0' = \frac{d\eta_0}{d\psi}$ we can write (Y, QY) in the following form:

$$(Y, QY) = \int d\tau \, y_0 \frac{\eta_0'}{\eta_0} \left(\underline{e}_z \times \nabla\psi \cdot \nabla u\right)^2 + 2\int d\tau \, y_0 \frac{\eta_0'}{\eta_0} A \left(\underline{e}_z \times \nabla\psi \cdot \nabla u\right) + \int d\tau \, |\nabla A|^2 \tag{19}$$

Let us now distinguish two cases:

a) $A = 0$

then $$(Y, QY) = \int d\tau \, y_0 \frac{\eta_0'}{\eta_0} \left(\underline{e}_z \times \nabla\psi \cdot \nabla u\right)^2$$

$y_0 = -\frac{dP_0}{d\psi}$ is positive in a simply connected equilibrium.

If in any finite region in ψ, $\eta_0' < 0$ then it is possible to localize $(\underline{e}_z \times \nabla\psi \cdot \nabla u)^2$ inside this region and the system would be unstable.

It is easy to show that this instability is absent in the ideal MHD case.

To stabilize, the current density should decrease to the outside. This instability might be responsible for the anomalous skin effect in a

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b) Let $A \neq 0$ and $\eta_0' > 0$. We then call $(\underline{e}_z \times \nabla\psi . \nabla u) = V$ and minimize with respect to V, which trivially leads to

$$V = \bar{A} - A \qquad\qquad \bar{A} = \frac{\oint \frac{dl}{B} A}{\oint \frac{dl}{B}}$$

$$(Y, QY) = \int d\tau \, |\nabla A|^2 - \int d\tau \, y_0 \frac{\eta_0'}{\eta_0} \left(A^2 - \bar{A}^2 \right)$$

The Euler equation corresponding to the minimization with respect to A leads to

$$\nabla^2 A + (A - \bar{A}) \, y_0 \frac{\eta_0'}{\eta_0} = \lambda A$$

λ is the Lagrange factor. If $\lambda > 0$ it is unstable. This is an eigenvalue problem whose solution depends on $y_0 \frac{\eta_0'}{\eta_0}$ which in general is not trivial

In [2] the problem of neighbouring equilibria for the 2-d. Vlassov equation was reduced to an eigenvalue problem of the same type.

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References

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