MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Trapped Ion Depletion by Anomalous
Diffusion Due to the Dissipative
Trapped Ion Instability

H.K. Wimmel

IPP 6/137   July 1975

IPP 6/137   H.K. Wimmel   Trapped Ion Depletion by Anomalous

Diffusion Due to the Dissipative

Trapped Ion Instability

July 1975   (in English)

Abstract

At high temperatures the KADOMTSEV – POGUTSE diffusion in tokamaks can
become so large as to cause depletion of trapped ions if these are
replaced with free ions by means of collisions rather than being
directly recycled or injected. Modified KADOMTSEV – POGUTSE diffusion
formulas are employed in order to estimate this effect in the cases of
classical and anomalous collisions. The maximum trapped-ion depletion is
estimated from the PENROSE stability condition. For anomalous collisions
a BOHM – type diffusion is derived. Numerical examples are given for JET-
like parameters (JET = Joint European Torus). Depletion is found to
reduce diffusion by factors of up to 10 and more.

1. INTRODUCTION

In 1970 KADOMTSEV and POGUTSE published their well-known formula of the
anomalous diffusion coefficient due to the dissipative trapped-ion
instability (see KADOMTSEV and POGUTSE 1970, 1971). This formula has
been applied to experimental conditions in tokamaks by FURTH (1973),
PFIRSCH et al. (1974), and GREEN et al. (1974). SAISON and WIMMEL (1975)
extended the diffusion formula to plasmas with an ion charge Ze , $Z > 1$,
with the result that the diffusion coefficient drops approximately in
proportion to $1/Z^3$ for not too large Z. KADOMTSEV and POGUTSE (1970,1971)
derive their diffusion formula from dissipative continuity equations for
trapped ions and electrons together with the quasi neutrality condition,
in which certain approximate expressions for the free-particle densities
appear.

In deriving their diffusion formula KADOMTSEV and POGUTSE assumed that
the average trapped-ion and electron densities are not affected by the
anomalous diffusion process. It is this assumption that I want to
replace by a more realistic one. A simple estimate shows that, at high
temperatures, the anomalous diffusion can cause depletion of the trapped
ions if these are only replaced with free ions by means of collisions –
COULOMB or anomalous – rather than being injected directly from outside.
It will turn out that for a small degree of depletion (at moderately
high temperatures) the collisions are of the classical COULOMB type.

For depletions larger than a critical value (at still higher temperatures) an inverse loss-cone instability develops that leads to anomalous collisions. The two cases will be studied separately.

The question exactly what set of basic equations should be used in treating the effect of trapped-ion depletion deserves some special attention for the following reason. An improved set of trapped-fluid equations has been established by the present author (WIMMEL 1975) with corrected expressions for the free-particle densities and the collision terms, and with additional source terms in the continuity equations. In particular, source terms coming from $\nabla \delta_o$ (δ_o = equilibrium fraction of trapped particles) have been taken into account, and the corrected equations show that the anomalous trapped-ion diffusion depends in sign on the total density gradient rather than the trapped-density gradient. This is important near the magnetic axis of the plasma torus. Apart from this effect, the new equations yield diffusion fluxes of similar magnitude to those of the original KADOMTSEV - POGUTSE equations for such values of the aspect ratio as occur in practice. Only in the mathematical limit of very large aspect ratios does the correction factor for the diffusion flux become very small compared with unity. For this reason a corrected version of the KADOMTSEV - POGUTSE equations that includes the $\nabla \delta_o$ correction will be used as the starting point of these calculations.

Section 2 presents the basic trapped-fluid equations and the general expressions for the anomalous trapped-particle diffusion flux. Section 3 gives the formulas for depleted transport with COULOMB collisions. In Section 4 the inverse loss-cone instability leading to anomalous collisions is studied by means of the PENROSE stability condition. Section 5 gives the formulas for depleted transport with anomalous collisions. In Section 6 the limiting temperatures separating the different diffusion regimes are derived. Section 7 presents numerical examples of the resulting anomalous diffusion as a function of temperature. Section 8 completes the study with a summary and discussion.

2. BASIC EQUATIONS AND TRANSPORT

The trapped-fluid equations in two spatial dimensions, as introduced by KADOMTSEV and POGUTSE (1970, 1971) but supplemented by a $\nabla \delta_0$ correction term, read:

$$\frac{\partial n_i}{\partial t} + \underline{v} \cdot \nabla n_i = -\nu_{ieff}(n_i - n_0) + N_p \underline{v} \cdot \nabla \delta_0 \tag{2.1}$$

$$\frac{\partial n_e}{\partial t} + \underline{v} \cdot \nabla n_e = -\nu_{eff}(n_e - n_0) + N_p \underline{v} \cdot \nabla \delta_0 \tag{2.2}$$

$$\phi = (T/2eN_p)(n_i - n_e) \tag{2.3}$$

$$\underline{v} = (cT/2eBN_p)\,\hat{z} \times \nabla(n_i - n_e). \tag{2.4}$$

Here n_i, n_e are the trapped-particle densities, $\underline{\upsilon}$ is the common E x B

drift velocity, $\boldsymbol{\nu}_{iff}$, $\boldsymbol{\nu}_{eff}$ are the effective collision frequencies of the

trapped particles, Φ is the electric potential, $n_0 = \delta_0 N_p$ and N_p are

the trapped and total particle densities in equilibrium; the remaining

notation is standard, except for $T = 2T_e T_i /(T_e + T_i)$. It is assumed that

$\nabla B = \nabla(T/N_p) = 0$, hence $\nabla \cdot \underline{\upsilon} = 0$. A Cartesian slab model, with the

coordinates x, y, will be used throughout.

In an earlier paper (WIMMEL 1975, henceforth called (I)) it was shown

that the anomalous trapped-particle diffusion following from eqs. (2.1)

to (2.4) and periodic boundary conditions in y is ambipolar and obeys

the general relation for the flux density:

$$\gamma_x = -\left(cT/2 e B N_p \nu_{eff}\right) < \partial_t \rho \cdot \partial_y \rho > , \qquad (2.5)$$

with $\rho = n_i - n_e$, the pointed brackets denoting a time average. In terms

of critical FOURIER modes the random-phase approximation gave the esti-

mate

$$\gamma_x = \sum_k \frac{A \nu_{eff} k_y \omega_k}{\nu_{eff}^2 + \omega_k^2} < |n_{ik}|^2 > , \qquad (2.6)$$

with Z_k = number of critical modes of common wave vector $\underline{K}$ and

frequency ω_k; n_{ik} = FOURIER transform of $(n_i - \bar{n}_i)$, $\bar{n}_i$ being the

time-averaged n_i ; and A = $cT/2eBN_p$. In (I) the mean square

amplitude of n_{ik} was estimated by the mixing-length hypothesis in the

form of

$$Z_k \, k_x^2 \, \langle |n_{ik}|^2 \rangle \approx |\nabla n_0|^2 , \tag{2.7}$$

with $k_x^2 = k_y^2$. In the present case of trapped ion depletion we make instead the plausible ansatz:

$$Z_k \, k_x^2 \, \langle |n_{ik}|^2 \rangle \approx |\nabla \bar{n}_i|^2 , \tag{2.8}$$

again with $k_x^2 = k_y^2$. This leads to the smaller flux density

$$\gamma_x = \frac{A \nu_{eff} \, \omega_k}{k_y (\nu_{eff}^2 + \omega_k^2)} \, |\nabla \bar{n}_i|^2 . \tag{2.9}$$

Let us turn to determining the critical mode parameters K_y and ω_k. In (I) it was assumed that ω_k obeys the linear dispersion equation $\omega_k = \omega(k_y)$ associated with eqs. (2.1) to (2.4), where ω is the real part of the complex frequency. In the present case one may surmise that the trapped ion depletion, with $\bar{n}_i < n_0$, leads to smaller effective values of $\omega_k(K_y)$ than given by the linear dispersion equation. For simplicity, the linear dispersion equation will nevertheless be used. In this way an upper bound for the flux density γ_x will be obtained.

The dispersion equation is given by (see (I))

$$-i\omega + \gamma = -\nu_s \left\{ 1 \pm \left(1 + \frac{4i\omega_1 \nu_d - \nu_{eff}\,\nu_{iff}}{\nu_s^2} \right)^{\frac{1}{2}} \right\} , \tag{2.10}$$

of which we use the values of the unstable modes. Here $\omega_1 = K_y v_1$, $v_1 = A \delta_0 \partial_x N_p$, $\nu_s = \frac{1}{2}(\nu_{eff} + \nu_{iff})$, $\nu_d = \frac{1}{4}(\nu_{eff} - \nu_{iff})$, ω is the (real) frequency, and γ is the growth rate. The critical ω_K in the case of small-scale turbulence ($|K_y|a > \pi$) and the critical K_y in the case of large-scale turbulence ($|K_y|a \sim \pi$), where a is the small plasma radius, are determined as in (I). In this way, for small-scale turbulence the flux density becomes

$$\gamma_x = - \frac{A^2 \delta_0 |\nabla \bar{n}_i|^2}{2 \sqrt{5} \, \nu_{eff}} \, \partial_x N_p \, , \tag{2.11}$$

with

$$|K_y^{crit}| = \frac{\sqrt{5} \, \nu_{eff}}{|v_1|} . \tag{2.12}$$

For large-scale turbulence one obtains instead

$$\gamma_x = - \left(\frac{a}{\pi |\partial_x N_p|} \right)^{\frac{3}{2}} |\nabla \bar{n}_i|^2 \left(\frac{2A \nu_{eff}}{\delta_0} \right)^{\frac{1}{2}} \partial_x N_p \, , \tag{2.13}$$

with

$$|\omega_{crit}| = \left(\frac{\pi}{2} \frac{\nu_{eff} |v_1|}{a} \right)^{\frac{1}{2}} . \tag{2.14}$$

In order to determine $\bar{n}_i$ in eqs. (2.11), (2.13), the particle balance equation of trapped ions must be used in addition. This balance equation

is obtained by time-averaging and integrating over x eq. (2.1), whence

$$\gamma_x \equiv \langle n_i \, \upsilon_x \rangle = \int_0^x dx' \, \nu_{iff} \, (n_o - \bar{n}_i) \qquad (2.15)$$

Here the term involving $\nabla \delta_o$ disappears owing to $\langle \upsilon_x \rangle = 0$.

The effects of trapped-ion depletion will be estimated in the sections to follow by means of a zero-dimensional analysis. This means that I shall put $\partial_x \longrightarrow 1/a$, where a is the minor plasma radius. Therefore eq. (2.15) will be used in the form

$$|\gamma_x| = a \, \nu_{iff} \, (n_o - \bar{n}_i), \qquad (2.16)$$

while eqs. (2.11) and (2.13) will be used in the forms

$$|\gamma_x| = \frac{A^2 \, \bar{n}_i^2 \, n_o}{2 \sqrt{5} \, \nu_{eff} \, a^3} \qquad (2.17)$$

and

$$|\gamma_x| = \pi^{-3/2} \, \bar{n}_i^2 \left(\frac{2 \, \nu_{eff} \, A}{n_o} \right)^{\frac{1}{2}} . \qquad (2.18)$$

In plasma regimes where anomalous collisions are effective γ_{eff} and γ_{iff} will be replaced by anomalous values. In addition, for the numerical examples to be considered, a characteristic LAWSON quantity

$$n\tau = a \, N_p^2 / |\gamma_x| \qquad (2.19)$$

will be plotted as a function of temperature (and other plasma parameters) rather than the flux density γ_x itself. This will make an assessment of the numerical results in terms of experimental performance easier.

3. DEPLETED TRANSPORT WITH COULOMB COLLISIONS

In the case of small-scale turbulence the diffusion is determined by eqs. (2.16) and (2.17). Hence, putting $\overline{n}_i = \delta_i N_p$, γ_x and $n\tau$ are determined by

$$|\gamma_x| = a \nu_{\text{iff}} N_p (\delta_o - \delta_i) = \frac{A^2 \delta_o N_p^3}{2\sqrt{5} \, a^3 \nu_{\text{eff}}} \delta_i^2 \tag{3.1}$$

and

$$n\tau = \frac{N_p}{\nu_{\text{iff}} (\delta_o - \delta_i)} = \frac{2\sqrt{5} \, a^4 \nu_{\text{eff}}}{A^2 N_p^2 \delta_o \, \delta_i^2} \tag{3.2}$$

with δ_i satisfying the quadratic equation

$$\delta_i^2 + K_1 (\delta_i - \delta_o) = 0, \tag{3.3}$$

where

$$K_1 = \frac{2\sqrt{5} \, a^4 \nu_{\text{eff}} \, \nu_{\text{iff}}}{A^2 N_p^2 \delta_o} . \tag{3.4}$$

For large values of K_1 the solution of eq. (3.3) approaches the equilibrium value, $\delta_i \to \delta_0$.

In the case of large-scale turbulence the diffusion is determined by eqs. (2.16) and (2.18). Hence

$$|\gamma_x| = a \nu_{iff} N_p (\delta_0 - \delta_i) = \left(\frac{N_p}{\pi}\right)^{\frac{3}{2}} \left(\frac{2A \nu_{eff}}{\delta_0}\right)^{\frac{1}{2}} \delta_i^2 \qquad (3.5)$$

and

$$n\tau = \frac{N_p}{\nu_{iff}(\delta_0 - \delta_i)} = \left(\frac{\delta_0 N_p}{2A \nu_{eff}}\right)^{\frac{1}{2}} \frac{\pi^{\frac{3}{2}} a}{\delta_i^2} \qquad (3.6)$$

with δ_i determined by

$$\delta_i^2 + K_2 (\delta_i - \delta_0) = 0, \qquad (3.7)$$

where

$$K_2 = \pi^{3/2} a \nu_{iff} \left(\frac{\delta_0}{2AN_p \nu_{eff}}\right)^{\frac{1}{2}}. \qquad (3.8)$$

Again, for large values of K_2 the equilibrium solution of δ is approached, $\delta_i \to \delta_0$.

The results of this section hold in the absence of anomalous collisions, i.e. for trapped-ion densities greater than the critical density given in the next section, eq. (4.14).

4. INVERSE LOSS-CONE INSTABILITY

Only for trapped-ion depletions that are not too large do the collisions of ions and electrons remain classical. For larger depletions an inverse loss-cone instability develops that will provide for fluctuating electric fields and anomalous collisions. This instability will be referred to as "inverse loss-cone" because here the "loss-cones" actually remain filled with circulating particles, while the complementary volume in velocity space becomes partially depleted of trapped particles. In the following, the critical depletion for the onset of the instability will be estimated from the PENROSE criterion (PENROSE 1960, McCUNE 1965). A rough approximation for the ion distribution function will be employed and only perturbations with $\underline{K} \parallel \underline{B}_o$ will be considered because for other directions of $\underline{K}$ the PENROSE criterion does not apply in the presence of $\underline{B}_o \neq 0$.

The PENROSE criterion yields instability if and only if

$$I =: P \int\limits_{-\infty}^{+\infty} \frac{du \, F'(u)}{u - u_o} > 0 \; , \qquad (4.1)$$

with

$$F(u) = \sum_{i,e} \omega_{p\nu}^2 \int d^2\underline{v}_\perp \, f_\nu (\underline{v}_\perp, u) \qquad (4.2)$$

and $F'(u_o) = 0$. Because of symmetry we have $u_o = 0$. Putting $F = F_i + F_e$, $I = I_i + I_e$, a Maxwell distribution of electrons (no depletion) yields

$$F_e(u) = \omega_{pe}^2 \left(\frac{m_e}{2\pi T_e}\right)^{\frac{1}{2}} e^{-m_e u^2/2T_e} \tag{4.3}$$

and

$$I_e = -\ 4\pi N_p e^2 / T_e \ . \tag{4.4}$$

For the ions we make the ansatz of a step function in velocity space, with overpopulation in the loss-cone region (circulating ions) and underpopulation for the trapped ions, viz.

$$f_i \, d^2\underline{\xi}_\perp \, d\eta = \frac{k}{\pi^{3/2}} e^{-(\xi_\perp^2 + \eta^2)} \left[H(|\eta| \, tg\, \varphi - \xi_\perp) \right.$$

$$\left. + \Delta \cdot H(\xi_\perp - |\eta| \, tg\, \varphi) \right] d^2\underline{\xi}_\perp \, d\eta \ . \tag{4.5}$$

Here H is the HEAVISIDE step function, and

$$\xi_\perp^2 = \frac{m_i}{2T_i} v_\perp^2 \ , \quad \eta^2 = \frac{m_i}{2T_i} u^2 \ , \tag{4.6}$$

$$\cos\varphi = \delta_o \ , \quad \sin\varphi \geqq 0 \ , \tag{4.7}$$

$$k = (1 - \delta_i)/(1 - \delta_o) \ , \tag{4.8}$$

$$\Delta \cdot k = \delta_i / \delta_o \ , \tag{4.9}$$

hence f_i satisfies the relations

$$\int_{total} f_i \, d^2\underline{\xi}_\perp \, d\eta = 1 \ , \quad \int_{trapped} f_i \, d^2\underline{\xi}_\perp \, d\eta = \delta_i \ . \tag{4.10}$$

It follows that

$$F_i(\eta)\,d\eta = \frac{\omega_{pi}^2\, K}{\sqrt{\pi}} \left\{ e^{-\eta^2} - (1-\Delta)\, e^{-\eta^2/\delta_o^2} \right\} d\eta \qquad (4.11)$$

and

$$I_i = -\, 4\pi N_p\, e^2\, S/T_i \quad , \qquad (4.12)$$

with

$$S = (\delta_i - \delta_o + \delta_i\, \delta_o)/\delta_o^2 . \qquad (4.13)$$

Evaluating the PENROSE condition $I > 0$ yields instability for depletions δ_i $< \delta_c$, with

$$\delta_c = \frac{\delta_o}{1+\delta_o}\left(1 - \delta_o\, \frac{T_i}{T_e}\right). \qquad (4.14)$$

5. DEPLETED TRANSPORT WITH ANOMALOUS COLLISIONS

For overcritical depletion of trapped ions, i.e. for $\delta_i < \delta_c$ [see eq. (4.14)] , anomalous collisions of ions and electrons will be produced by the inverse loss-cone instability. The anomalous effective collision frequencies of trapped ions and electrons will be called ν_{ifan} and ν_{efan} , respectively. A rough estimate of them is gained by assuming BROWNIAN motion in velocity space with $K\vartheta_{th} \gtrsim \omega$ and $K\lambda_b \sim 1$, viz.

$$\nu_{ifan} \sim \frac{8\pi^2}{\delta_o^2}\, \alpha\omega_{pi} \left(T_e/T_i\right)^{3/2} , \qquad (5.1)$$

$$\nu_{efan} \sim \frac{8\pi^2}{\delta_0^2} \propto \omega_{pe} \; , \tag{5.2}$$

where $\alpha = E^2/8\pi N_p T_e$. Putting, for example, $\alpha \sim 10^{-2}$ would lead to $\nu_{ifan} \sim \omega_{pi} \gg \nu_{iff}$. This shows that α will easily adjust itself in such a way that the trapped-ion density is prevented from dropping below the critical density as determined by eq. (4.14). Hence one is justified in putting $\delta_i \approx \delta_c$ and determining the anomalous collision frequency ν_{ifan} from the trapped-ion particle balance. In addition, it is seen that

$$\nu_{ifan}/\nu_{efan} = (m_e/m_i)^{\frac{1}{2}} (T_e/T_i)^{\frac{3}{2}}. \tag{5.3}$$

The following procedure is now followed in order to compute the depleted diffusion in the regime of anomalous collisions. In eqs. (2.16) to (2.18) the quantities ν_{iff} and ν_{eff} are replaced by ν_{ifan} and ν_{efan}. Then ν_{efan} is eliminated by means of eq. (5.3). After that, eq. (2.16) together with eq. (2.17), or (2.18), provides a system of 2 equations for the unknowns $\nu_{\star}$ and ν_{ifan} , while $\overline{n}_i$ is fixed by $\overline{n}_i = n_c = \delta_c N_p$. One notices that this approximate procedure is largely independent of the specific instability responsible for the anomalous collisions, as long as the relations $K\nu_{th} \gtrsim \omega$, $K\lambda_D \sim 1$ are satisfied; with the exception that the critical depletion δ_c was determined on the basis of a specific ion distribution function roughly describing trapped-ion depletion.

For small-scale turbulence the results are as follows:

$$|\gamma_x| = a \nu_{ifan} N_p \left(\delta_o - \delta_c\right)$$

$$= \left(\frac{m_e}{5 m_i}\right)^{\frac{1}{4}} \left(\frac{T_e}{T_i}\right)^{\frac{3}{4}} \left(\frac{\delta_o(\delta_o - \delta_c)}{2}\right)^{\frac{1}{2}} \cdot \frac{\delta_c N_p c T}{2 a e B} \tag{5.4}$$

and

$$n \tau = \frac{N_p}{\nu_{ifan}\left(\delta_o - \delta_c\right)}$$

$$= \left(\frac{5 m_i}{m_e}\right)^{\frac{1}{4}} \left(\frac{T_i}{T_e}\right)^{\frac{3}{4}} \left(\frac{2}{\delta_o(\delta_o - \delta_c)}\right)^{\frac{1}{2}} \frac{2 e B N_p a^2}{c T \delta_c} , \tag{5.5}$$

with

$$\nu_{ifan} = \left(\frac{m_e}{5 m_i}\right)^{\frac{1}{4}} \left(\frac{T_e}{T_i}\right)^{\frac{3}{4}} \left(\frac{\delta_o}{2(\delta_o - \delta_c)}\right)^{\frac{1}{2}} \frac{\delta_c c T}{2 a^2 e B} , \tag{5.6}$$

and δ_c given by eq. (4.14). As is seen from eqs. (5.4), (5.5), this is BOHM-type diffusion.

For large-scale turbulence one has instead

$$|\gamma_x| = a \nu_{ifan} N_p \left(\delta_o - \delta_c\right)$$

$$= \frac{1}{\pi^3} \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \left(\frac{T_i}{T_e}\right)^{\frac{3}{2}} \frac{\delta_c^4}{(\delta_o - \delta_c)\delta_o} \frac{c T N_p}{a e B} \tag{5.7}$$

and

$$n\tau = \frac{N_p}{\nu_{ifan}\left(\delta_o - \delta_c\right)}$$

$$= \pi^3 \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \left(\frac{T_e}{T_i}\right)^{\frac{3}{2}} \frac{\delta_o\left(\delta_o - \delta_c\right)}{\delta_c^4} \frac{a^2 e B N_p}{cT} , \quad (5.8)$$

with a different value of ν_{ifan}, viz.

$$\nu_{ifan} = \frac{1}{\pi^3} \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \left(\frac{T_i}{T_e}\right)^{\frac{3}{2}} \frac{\delta_c^4}{\delta_o\left(\delta_o - \delta_c\right)^2} \frac{cT}{a^2 e B} . \quad (5.9)$$

This again is BOHM-type diffusion.

6. LIMITING TEMPERATURES

In Sections 3 and 5 four different diffusion formulas have been given for the cases of small-scale and large-scale turbulence, with COULOMB and anomalous collisions, respectively. In order to apply these formulas, their ranges of validity in parameter space must be known. For this purpose the limiting temperatures of the four ranges are given in the following. Which of these are applicable for a specific plasma will be determined for the numerical examples considered in the next section. Because the limiting temperatures follow by elementary calculations from the formulas of Sections 3, 4, and 5, the results are simply listed.

For small-scale turbulence the limiting temperature between the ranges of COULOMB collisions and anomalous collisions is found from the

condition $\delta_i = \delta_c$ to be

$$T_{e1} = \frac{1}{\delta_0} \left(a Z_{eff} \right)^{4/5} \left[10^{-22} \frac{e B N_p}{c \delta_c} \left(1 + \frac{T_e}{T_i} \right) \right]^{\frac{2}{5}}$$

$$\left(\frac{T_e}{T_i} \right)^{3/10} \left[2 \sqrt{5} \left(\delta_0 - \delta_c \right) \right]^{1/5} \left(\frac{m_e}{m_i} \right)^{1/10} \tag{6.1}$$

where the following expressions for y_{eff} and y_{iff} have been used:

$$\nu_{eff} = 10^{-22} N_p Z_{eff} / T_e^{3/2} \delta_0^2 , \tag{6.2}$$

$$\nu_{iff} = 10^{-22} \left(N_p Z_{eff}^3 / T_i^{3/2} \delta_0^2 \right) \left(m_e / m_i \right)^{1/2} . \tag{6.3}$$

All formulas are in c.g.s. units. For large-scale turbulence one has

instead

$$T_{e2} = Z_{eff}^2 \left(\frac{\pi T_e}{T_i} \right)^{6/5} \left[\frac{a \left(\delta_0 - \delta_c \right)}{\delta_c^2} \right]^{4/5}$$

$$\left[10^{-22} \frac{e B N_p}{2 c \delta_0} \cdot \frac{m_e}{m_i} \left(1 + \frac{T_e}{T_i} \right) \right]^{2/5} \tag{6.4}$$

as the limit between the ranges of COULOMB and anomalous collisions.

In the range of COULOMB collisions the limiting temperature between the ranges of small-scale and large-scale turbulence is obtained, from the condition $\left| K_y^{crit} \right| = \pi / a$, as:

$$T_{e3} = \left[\frac{\sqrt{5}}{\pi} 10^{-22} \frac{Z_{eff}\, e B\, N_p\, a^2}{c\, \delta_o^3} \left(1 + \frac{T_e}{T_i}\right) \right]^{2/5}. \qquad (6.5)$$

On the contrary, in the range of anomalous collisions a limiting temperature T_{e4} between the ranges of small-scale and large-scale turbulence does not exist. Independent of T_e , the turbulence is small-scale or large-scale for

$$f =: \frac{\pi}{\delta_c} \left[2 \delta_o (\delta_o - \delta_c) \right]^{\frac{1}{2}} \left(\frac{m_e}{5 m_i}\right)^{\frac{1}{4}} \left(\frac{T_e}{T_i}\right)^{\frac{3}{4}} \lessgtr 1, \qquad (6.6)$$

respectively.

In addition, two more characteristic temperatures will be given that determine whether certain necessary conditions for the validity of the diffusion formulas are satisfied. Neglecting any effects of LANDAU damping, one necessary validity condition requires that the critical wavelength be greater than an ion gyroradius. For smaller critical wavelengths the drift approximation breaks down. This is satisfied for

$$T_e > T_{e5} = \left(\frac{10\, m_i\, T_i}{T_e}\right)^{\frac{1}{4}} \left[10^{-22} \frac{a\, N_p\, Z_{eff}}{\delta_o^3} \left(1 + \frac{T_e}{T_i}\right) \right]^{\frac{1}{2}}. \qquad (6.7)$$

This limiting temperature has been evaluated only for the case of small-scale turbulence with COULOMB collisions. This suffices for the

numerical examples considered in the next section. Another necessary

validity condition requires that the frequency of the critical modes be

smaller than the transiting frequency of the untrapped ions with respect

to the minor circumference of the torus. One reason for this requirement

is to justify the unequal treatment of trapped and untrapped particles

in the basic eqs. (2.1) to (2.4). The condition is satisfied for

$$T_e > T_{e6} = \left(\frac{m_i T_e}{3 T_i} \right)^{\frac{1}{4}} \left[10^{-22} \frac{R q N_p Z_{eff}}{\delta_o^3} \right]^{\frac{1}{2}} , \qquad (6.8)$$

where R is the major torus radius, and $q = 2\pi/\iota$ is the usual safety

factor. Again, this limiting temperature has been evaluated only for

small-scale turbulence with COULOMB collisions because this is sufficient

for the numerical examples to follow.

7. NUMERICAL EXAMPLES

In order to illustrate the practical consequences of the above

formulas, I shall present two numerical examples. The parameters used

are as follows: $N_p = 10^{13}$ and 10^{14} cm^{-3}, B = 3 x 10^4 G, $T_i = T_e$,

$Z_{eff} = 1$, a = 100 cm, R = 300cm, q = 2.5, $\delta_o = 0.5$. The functions

plotted are $n\tau(T_e)$; see Figs. 1 and 2.

In order to construct the graphs of Figs. 1 and 2, one must first

determine which of the four formulas of eqs. (3.2), (3.6), (5.5), (5.8)

apply in what temperature range. Hence the critical temperatures T_{e1}, T_{e2}, T_{e3}, and the quantity f must be evaluated. For the case $N_p = 10^{13}$ cm^{-3} one obtaines $T_{e1} = 12.0$ keV, $T_{e2} = 6.65$ keV, $T_{e3} = 12.4$ keV, f = 0.929 < 1. It follows that for $T_e < T_{e1}$ small-scale turbulence with COULOMB collisions obtains, while for $T_e > T_{e1}$ small-scale turbulence with anomalous collisions obtains. The critical temperatures T_{e2} and T_{e3} do not have physical significance here because the conditions for their validity are not satisfied in the pertinent temperature ranges. For the case $N_p = 10^{14}$ cm^{-3} the discussion is analogous, the critical values being $T_{e1} = 30.1$ keV, $T_{e2} = 16.7$ keV, $T_{e3} = 31.1$ keV, f = 0.929.

Figures 1 and 2 each show three diffusion regimes, viz. undepleted KADOMTSEV-POGUTSE diffusion at low temperatures, with COULOMB collisions effective; then depleted KADOMTSEV-POGUTSE diffusion at higher temperatures, still with COULOMB collisions in effect; and, finally, depleted BOHM-like diffusion, with anomalous collisions. The extrapolated curves (broken lines) show, first, that for $T_e \approx T_{e1}$ the effect of depletion reduces the particle losses by a factor of approximately 10 in the examples considered, and, second, that without anomalous collisions, $n\tau(T_e)$ would show a minimum value near $T_e = T_{e1}$ and increase again for larger temperatures, $T_e > T_{e1}$. In addition to T_{e1}, the critical temperatures T_{e5} and T_{e6} have been indicated, showing that the corresponding necessary validity conditions of the theory are satisfied in the temperature range of interest.

8. SUMMARY AND DISCUSSION

It has been shown that at high temperatures the anomalous diffusion
due to the dissipative trapped-ion instability causes depletion of
trapped ions if these are not directly replaced by injection from out-
side. This depletion leads to lower diffusion than that according to
the formula of KADOMTSEV and POGUTSE. At high depletion an inverse
loss-cone instability develops and induces anomalous collisions. In
this regime a BOHM-like diffusion law is obtained. The numerical
examples presented in Section 7 show that under experimental conditions
of current interest the depletion effect is potentially important
because it leads to reductions of the diffusion by a factor of 10
and more.

In keeping with the original theory by KADOMTSEV and POGUTSE only the
case of ions with $Z = 1$ and no impurities present has been considered.
As is known from previous work (SAISON and WIMMEL, 1975; DOBROWOLNY,
1974), a multiple ion charge Ze, with $Z > 1$, or the presence of
impurities with $Z > 1$ may reduce the anomalous diffusion considerably.
Under such conditions, therefore, the depletion of trapped ions usually
will not be a large effect. This justifies the restriction to $Z = 1$ in
the present study. As has been mentioned elsewhere (WIMMEL, 1975),
BOHM diffusion due to the collisionless trapped-ion instability
may be superimposed on the regimes of large-scale turbulence, depending

on whether the former can be stabilized or not. In the numerical
examples considered above this problem does not arise because only
small-scale turbulence occurs there.

The above analysis of the depletion effect is zero-dimensional.
Generalizing to a 1-dimensional analysis will change the particle
balance equation, eq. (2.16), into a differential equation, with
the flux γ_x expressed by density gradients. A solution in closed
analytical form is then impossible. Rather a numerical transport
code must be used. Also the untrapped particles must then be
described in a selfconsistent way, and the use of the necessary
boundary conditions will require that external particle sources,
charge exchange, and additional diffusion mechanisms be included
in the code. It may even be necessary to use spatially averaged
gradients in the computation in order to represent the nonlocal
character of the turbulent diffusion. It appears that such a
program would have its main interest in the application to specific
machine studies. In order to obtain a general result in the way of
an order-of-magnitude estimate, the present zero-dimensional treat-
ment clearly is more appropriate.

REFERENCES

McCUNE J.E. (1965) Phys. Rev. Lett. 15, 398 and 684.

DOBROWOLNY M. (1974) Nucl. Fusion 14, 599.

FURTH H.P. (1973) Controlled Fusion and Plasma Physics (Proc. 6th
Eur. Conf., Moscow) Vol.2. p. 51 Joint Institute of Nuclear
Research.

GREEN B.J., MARTY D., PFIRSCH D., SAISON R., and WIMMEL H.K. (1974)
Plasma Physics and Controlled Nuclear Fusion Research
(Proc. 5th Conf., Tokyo) paper IAEA-CN-33/A 14-1.
International Atomic Energy Agency.

KADOMTSEV B.B. and POGUTSE O.P. (1970) Reviews of Plasma Physics
(LEONTOVICH M.A., Editor) Vol.5 p. 249. Consultants
Bureau, New York.

KADOMTSEV B.B. and POGUTSE O.P. (1971) Nuclear Fusion 11, 67.

PENROSE O. (1960) Phys. Fluids 3, 2.

PFIRSCH D., SAISON R., WIMMEL H.K. (1974) JET Workshop (Culham)
Paper B. 1. Culham Laboratory.

SAISON R. and WIMMEL H.K. (1975) Waves and Instabilities in Plasmas
(2nd Internat. Congress, Innsbruck) Paper P 3. University
of Innsbruck.

WIMMEL H.K. (1975) Laboratory Report IPP 6/133 (March 1975). Max-Planck-
Institut für Plasmaphysik, Garching. To be published.

FIGURE CAPTIONS

Fig. 1   The function $n\tau\left(T_e\right)$ for JET-like parameters with
$N_p = 10^{13} \; cm^{-3}$.

Fig. 2   The function $n\tau\left(T_e\right)$ for JET-like parameters with
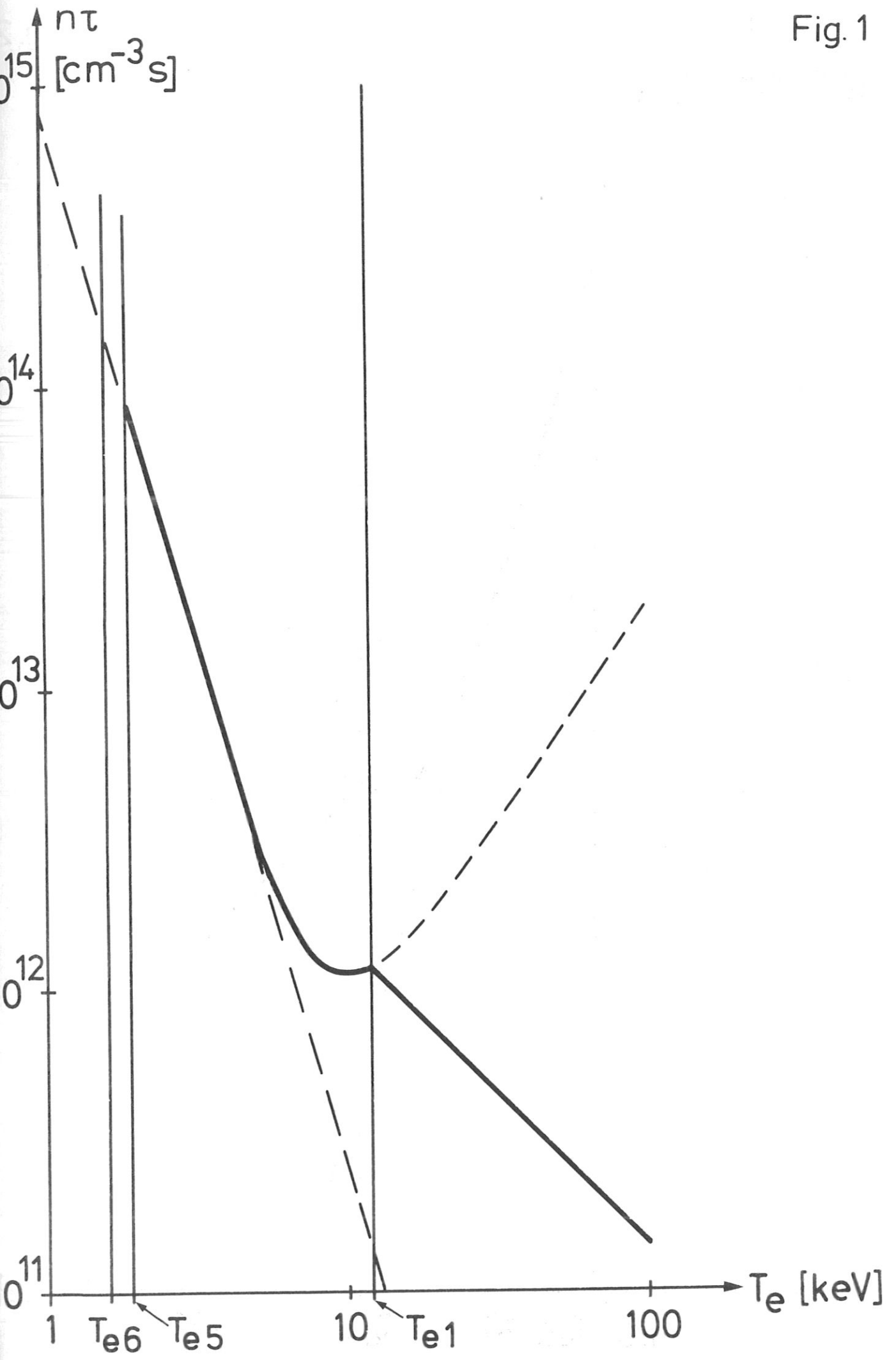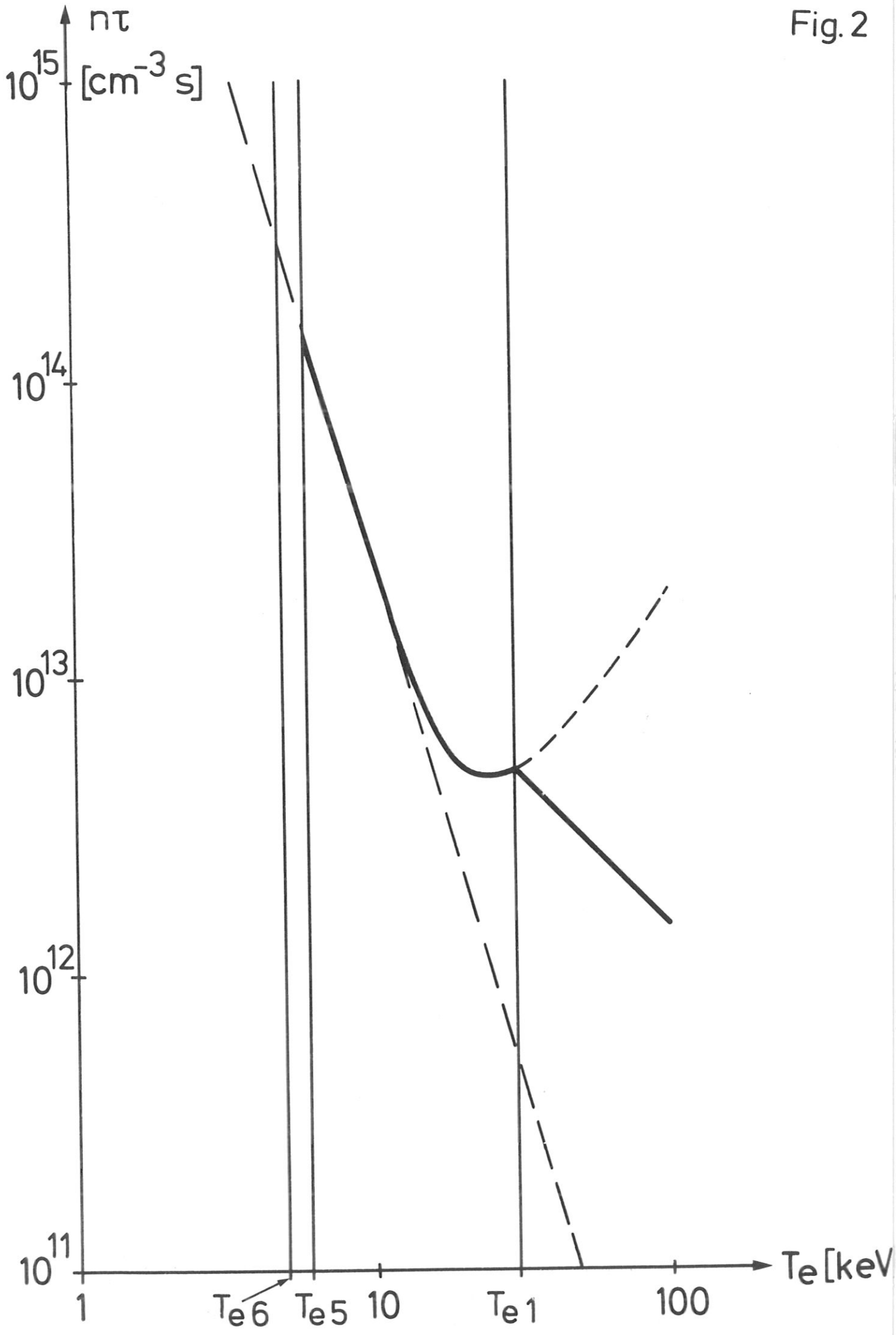$N_p = 10^{14} \; cm^{-3}$.

Fig. 1

Fig. 2