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Abstract

A multiple-time-scale treatment of the development of a Tokamak discharge in the collisional regime is presented. Self-consistent, resistive, finite- β , MHD equations are used, with inertial and viscous effects included to obtain stabilization of the well-known poloidal spin-up. This stabilization occurs at a plasma state such that the geodesic speed remains subsonic. Only resistive (diffusive) time scales are considered. On the fast, cylindrical, skin time, the field penetrates the plasma, and the poloidal spin-up instability arises and then saturates due to viscosity. On the slower, toroidal, skin time, the plasma would move across the field, and the final density and pressure profiles develop. In addition, toroidal angular momentum would build up to a value depending on the extent of poloidal rotation. Present experiments, however, are unlikely to operate long enough to reach this final state.

Introduction

The nature of resistive plasma losses in toroidal geometry was first studied by Pfirsch and Schlüter [1], who found that toroidal effects lead to an enhancement of classical resistive diffusion. In this initial work, use was made of the resistive MHD equations in the low- β approximation with inertial terms omitted. Self consistent MHD calculations, also neglecting inertia [2], [3], [4], later showed that the Pfirsch-Schlüter result is essentially unchanged by finite- β effects.

Stringer [5] noted that combined inertial and resistive effects tend to make the low- β model unstable; in the absence of viscosity, an initial poloidal rotation would tend to increase until it reached a critical speed, namely, the sound speed times the ratio of the poloidal to the toroidal magnetic field.

Inertial effects were also considered by Zehrfeld and Green [6,7]; they found that rotation tends to enhance diffusion above and beyond the Pfirsch - Schlüter factor.

Hazeltine, Lee, and Rosenbluth [8] extended the works of Zehrfeld and Green and of Stringer; using a low- β model, with inertial effects included, they predicted, in the absence of viscosity, that the rotational spin-up could be stabilized at the critical speed by a weak shock. Going further, the same authors [9] showed that the rotational

spin-up instability also exists in a self-consistent (finite- β) Tokamak model in the limit of negligible inertial effects. In this latter work, however, they did not suggest the mechanism by which the spin-up might be stabilized, nor did they follow the development of the plasma beyond the initial spin-up stage.

A self consistent two fluid model, with resistivity and finite Larmor radius effects included, was considered by Haines [10]. He took inertial effects into account by treating them as small perturbations within the general solution. Haines once again found the diffusion to be increased beyond the Pfirsch - Schlüter factor and also found a tendency for the plasma to rotate.

At about the same time, Greene, Johnson, Weimer and Winsor [11] treated the low- β , one fluid MHD model of a toroidal plasma by means of a systematic expansion, employing an ordering scheme which introduced the resistivity in third order in the inverse aspect ratio, ϵ . Treating the time dependence as a perturbation, they derived a dispersion relation for small disturbances about a stationary state. Again the authors found, among other things, a rotational instability.

Stationary toroidal equilibria without resistivity were investigated by Dobrott and Greene [12] and Zehrfeld and Green [13], the former authors considering a guiding center plasma. Without employing an

expansion, both sets of authors were able to give general conditions on the flow quantities which allow an equilibrium to exist. However, in an MHD model without resistivity, ("ideal MHD"), and with significant fluid motion, four free functions remain undetermined, i.e., these four functions can be prescribed virtually arbitrarily within the framework of the governing equations for ideal MHD flow.

In a previous study of this problem [14], two of the present authors considered stationary, self consistent solutions of the one fluid equations, with significant inertia and finite but small resistivity, employing the standard Tokamak ordering [9]. Viscous effects were neglected in that initial study, but in contrast to Ref. [11], the resistivity was introduced as an independent small parameter. A variety of interesting results were obtained, many of which can be recovered (for vanishing viscosity) from results given at the end of the present paper. Perhaps the most significant result of the early work, however, was simply that all free functions of ideal MHD become uniquely determinable when finite resistivity is introduced in such an analysis; we shall later see that this is also true with additional inclusion of viscosity. In order to achieve a stationary state in full generality, sources for the mass- and angular momentum fluxes had to be introduced; indeed, this early study seemed to offer a generalization of the special steady-state solution, for no sources, given by Grad and Hogan [3]. The general stationary solutions always included important plasma rotation, reducing the achievable β_p significantly below the value indicated in Ref. [3].

Despite the apparently interesting nature of these early results, their usefulness remained in some doubt, primarily because of two considerations: 1) the existence of the poloidal spin-up instability [9] left open the question of how - or even whether - the plasma was to achieve stationarity⁺; 2) the time required to achieve a steady state is significantly longer than the total operating time for virtually all present Tokamak experiments - - hence, a theory that could take that fact into account was clearly called for.

Such a theory must of course, include time dependence, as well as a mechanism for achieving stabilization of the poloidal rotation. One obvious mechanism deserving consideration for the latter purpose is that of viscosity. Recently, several authors have included viscous effects in their study of toroidal discharge [15], [16]. For example, Grimm and Johnson [15] considered the effects of viscosity and heat conduction on the containment of a toroidal plasma. Unfortunately, an error in their treatment of the viscous stresses left open the question of how, or whether, viscous effects were sufficient to stabilize the rotation. However, the results of Ref. [16] strongly indicated that such stabilization was to be expected.

In the present paper we present a model suitable to describe the gross temporal behavior of a Tokamak; we use the time-dependent, one

⁺ Further calculations, extending the results of Ref. [9] to include finite inertial effects, were carried out; these showed the plasma still to be unstable to poloidal rotation in the absence of viscosity.

fluid MHD equations including finite viscosity and resistivity. For our present purposes we wish to exclude pure MHD effects, especially MHD waves and instabilities; to accomplish this we assume certain poloidal symmetries and use flux-surface averages wherever appropriate. Under these conditions we are in effect looking at the plasma in such a way so that we do not "resolve" time variations as fast as those associated with the Alfvén speed; formally, it is as if we were to treat the Alfvén speed as infinite. Our time scales are then the various skin times, i.e., diffusion times, needed by the field to penetrate the toroidal plasma and the plasma to set up appropriate fluxes.

We find, as previously noted by Grad and Hogan [3], that the behavior of the discharge is characterized by two distinct time scales: first, on the fast time scale, i.e., that of the skin time associated with the minor radius, the field penetrates, leaving the plasma practically fixed - - except for the build up of poloidal rotation previously noted - -, whereas on the slower time scale, namely that of the full toroidal skin time, the field stays fixed and the plasma moves across magnetic surfaces. The second time scale is slower by the square of the inverse aspect ratio than the first. In principle, an intermediate time scale is possible; in the following, however, we show that our model leads to constancy of all significant quantities over that time period.

We use Ohm's law in its simplest form⁺ and do not employ an energy equation, although the early phase of a Tokamak discharge is almost certainly governed primarily by energy transport, radiation cooling, etc., mainly through impurities and such. However, within our model, temperature, resistivity and viscosity can be prescribed as arbitrary functions, very nearly constant on magnetic surfaces. Because of the high parallel thermal conductivity this seems to present a plausible physical picture. Since the solution turns out to be remarkably insensitive to the details of the energy profiles we feel that this approach is adequate for our present purposes, and may be even more useful than a classical transport model, as we can, in principle, simulate any temporal history and spatial distribution of temperature on the time scales considered.

We restrict ourselves to the approximation in which magnetic surfaces remain circular cross sections and employ an expansion with respect to the inverse aspect ratio, following Shafranov. [17] As in our earlier work on stationary states, the resistivity is not ordered artificially with respect to the aspect ratio but is treated as an independent small parameter. Viscous effects are treated together with resistivity, the two effects are in fact of the same order for certain Tokamak applications. For the viscosity we use the form given by Braginsky [18] in the limit $\omega\tau \gg 1$; otherwise no simplifications are necessary.

⁺On the time scales of interest in this problem, the normal Hall term and electron pressure gradient term tend to cancel. Also, the $\partial \underline{j} / \partial t$ term and the pseudodyadic terms are negligible. See Ref. [19].

1. Basic Equations.

We treat the equations in conservation form,

$$\frac{\partial}{\partial t} (\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \underline{v}) + \nabla p = \frac{1}{\mu_0} (\nabla \times \underline{B}) \times \underline{B} + \underline{M} \quad 1.1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = Q \quad 1.2)$$

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) - \nabla \times (\eta \underline{J}) \quad 1.3)$$

where we have used Ohm's law in the form

$$\eta \underline{J} = \underline{E} + \underline{v} \times \underline{B} \quad 1.4)$$

Q is a mass source, \underline{M} contains the momentum source as well as the viscous forces. For convenient use in the curvilinear coordinates to be introduced below it is useful to write the viscous force density in invariant form [19]: $\underline{M} = \underline{M}_s + \underline{F}_v$;

$$\underline{F}_v = 3 \nabla \cdot (\underline{b} \underline{b} D) - \nabla D; \quad \underline{b} \equiv \frac{\underline{B}}{|\underline{B}|} \quad 1.5)$$

$$D = \mu \left[\underline{b} \underline{b} : \nabla \underline{v} - \frac{1}{3} \nabla \cdot \underline{v} \right] \quad 1.6)$$

Furthermore, we let

$$p = c^2 \rho; \quad c^2 = c^2(r, t) \quad 1.7)$$

so that local changes of state are treated isothermally. This seems justified as we are considering only slow changes. We assume axisymmetry, and use magnetic flux surfaces as coordinate surfaces. In general, these surfaces change in time, and this effect will be self consistently included in the analysis. Let us, however, first consider a general orthogonal r, θ, ϕ system having the topology of Fig. 1

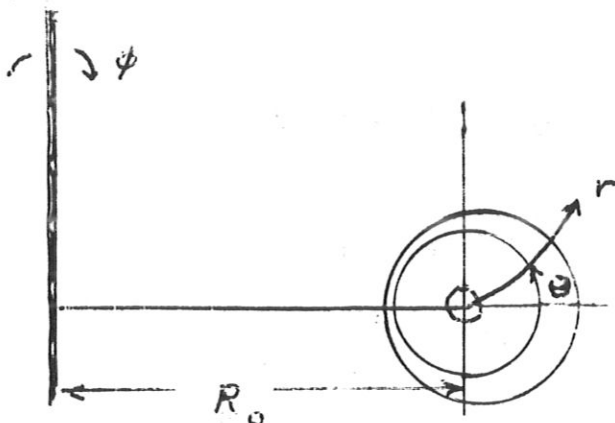


Fig. 1

We then have the line element

$$ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\phi^2 d\phi^2 \quad (1.8)$$

For the moment we leave the scale factors, h_r, h_θ, h_ϕ undetermined; the form that they take in a torus, using flux coordinates within our expansion scheme to $O(\epsilon)$, will be described later in this Section.

Using $\nabla \cdot \underline{B} = 0$, taking advantage of the axisymmetry, $\partial/\partial\phi = 0$, and letting \hat{e}_ϕ be the unit vector in the ϕ - direction, we may write

for the magnetic field:

$$\underline{B} = \frac{\hat{e}_\phi}{h_\phi} \times \nabla \psi + \frac{\hat{e}_\phi}{h_\phi} \Lambda \quad 1.9)$$

or

$$\underline{B} = \left\{ -\frac{1}{h_\theta} \frac{\partial \psi}{\partial \theta}, \frac{1}{h_r} \frac{\partial \psi}{\partial r}, \Lambda \right\} / h_\phi \quad 1.10)$$

and, taking the curl, the current density becomes:

$$\mu_0 \underline{J} = \left\{ \frac{1}{h_\theta} \frac{\partial \Lambda}{\partial \theta}, \frac{1}{h_r} \frac{\partial \Lambda}{\partial r}, \Delta^* \psi \right\} / h_\phi \quad 1.11)$$

with

$$\Delta^* \psi \equiv h_\phi^2 \nabla \cdot \left(\frac{\nabla \psi}{h_\phi^2} \right) \quad 1.12)$$

To rewrite the momentum equations, we use the identity:

$$\nabla \cdot (\rho \underline{v} \underline{v}) = \underline{v} \nabla \cdot (\rho \underline{v}) + \rho \left[\nabla \frac{v^2}{2} - \underline{v} \times \underline{\Omega} \right]; \quad 1.13)$$

$$\underline{\Omega} \equiv \nabla \times \underline{v}$$

Using the shorthand notation

$$T_{xi} \equiv \rho \left[\frac{v_r^2}{h_r} \frac{\partial}{\partial x_i} h_r + \frac{v_\theta^2}{h_\theta} \frac{\partial}{\partial x_i} h_\theta + \frac{v_\phi^2}{h_\phi} \frac{\partial}{\partial x_i} h_\phi \right]; x_i = r, \theta \quad 1.14)$$

we have the 3 momentum conservation laws:

$$\begin{aligned} h_r \frac{\partial}{\partial t} (\rho v_r) + \nabla \cdot (\rho \underline{v} h_r v_r) + \frac{\partial}{\partial r} (c^2 \rho) - T_r \\ = -\frac{1}{\mu_0} \frac{1}{h_\phi^2} \left(\Lambda \frac{\partial \Lambda}{\partial r} + \Delta^* \psi \frac{\partial \psi}{\partial r} \right) + h_r M_r \end{aligned} \quad 1.15)$$

$$\begin{aligned} h_0 \frac{\partial}{\partial t} (\rho v_\theta) + \nabla \cdot (\rho \underline{v} h_0 v_\theta) + \frac{\partial}{\partial \theta} (c^2 \rho) &= T_\theta \\ &= -\frac{1}{\mu_0} \frac{1}{h_\phi^2} \left(\Lambda \frac{\partial \Lambda}{\partial \theta} + \Delta^* \psi \frac{\partial \psi}{\partial \theta} \right) + h_\theta M_\theta \end{aligned} \quad 1.16)$$

$$\begin{aligned} h_\phi \frac{\partial}{\partial t} (\rho v_\phi) + \nabla \cdot (\rho \underline{v} h_\phi v_\phi) &= -\frac{1}{\mu_0} \frac{1}{h_r h_\theta h_\phi} \left(\frac{\partial}{\partial r} \left(\Lambda \frac{\partial \psi}{\partial \theta} \right) - \right. \\ &\quad \left. - \frac{\partial}{\partial \theta} \left(\Lambda \frac{\partial \psi}{\partial r} \right) \right) + h_\phi M_\psi \end{aligned} \quad 1.17)$$

the continuity equation

$$\frac{\hat{D} \rho}{\hat{D} t} + \rho \nabla \cdot \underline{v} = 0 ; \quad \frac{\hat{D}}{\hat{D} t} \equiv \frac{\partial}{\partial t} + \underline{v} \cdot \nabla , \quad 1.18)$$

and the three components of Faraday's law:

$$-\frac{\partial}{\partial t} \frac{\partial \psi}{\partial \theta} = -\frac{1}{\mu_0} \frac{\partial}{\partial \theta} (\eta \Delta^* \psi) + \frac{\partial}{\partial \theta} (\underline{v} \cdot \nabla \psi) \quad 1.19)$$

$$\frac{\partial}{\partial t} \frac{\partial \psi}{\partial r} = \frac{1}{\mu_0} \frac{\partial}{\partial r} (\eta \Delta^* \psi) - \frac{\partial}{\partial r} (\underline{v} \cdot \nabla \psi) \quad 1.20)$$

$$\begin{aligned} \frac{\partial \Lambda}{\partial t} &= \frac{1}{\mu_0} h_\phi^2 \nabla \cdot \left(\eta \frac{\nabla \Lambda}{h_\phi^2} \right) - \frac{h_\phi}{h_r h_\theta} \left[\frac{\partial}{\partial r} \left(\frac{v_\phi}{h_\phi} \frac{\partial \psi}{\partial \theta} \right) - \right. \\ &\quad \left. - \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{h_\phi} \frac{\partial \psi}{\partial r} \right) \right] - h_\phi^2 \nabla \cdot \left(\frac{\Lambda \underline{v}}{h_\phi^2} \right) \end{aligned} \quad 1.21)$$

Eqs 1.19) and 1.20) may be integrated immediately:

$$\frac{\hat{D} \psi}{\hat{D} t} = \frac{\eta}{\mu_0} \Delta^* \psi - C(t) \quad 1.22)$$

where $C(t)$ is directly related to the applied electric field.

In fact, if we solve 1.22) with the boundary condition

$$\psi = 0 \quad \text{at} \quad r = 0 \quad 1.23)$$

then

$$C(t) = E_{\phi} h_{\phi} \equiv \frac{U_0}{2\pi R_0} \quad 1.24)$$

as can best be seen by considering the stationary state and comparing with Ohm's law. C is then time-independent so long as the applied voltage is held constant.

So far we have considered a general orthogonal coordinate system. We now introduce flux coordinates so that ψ depends on r only i.e., it is independent of θ . But then, during the field penetration phase, the coordinate system becomes time dependent too, because $r = r(t)$. This does not affect the spatial derivatives, but the time derivatives above are Eulerian, i.e., they are to be taken at a fixed point in space where both r and θ change and the unit vectors \hat{e}_r and \hat{e}_{θ} also rotate. This is easily taken into account.

Let $\partial r / \partial t$ and $\partial \theta / \partial t$ be the rate of change of r and θ at a fixed point. We then have to substitute⁺

$$\frac{\partial f}{\partial t} \Rightarrow \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial t} \quad 1.25)$$

In vector equations the rotation of \hat{e}_r and \hat{e}_{θ} generally also has to be taken care of; however, to the order to which we carry our expansion scheme we will never actually need to consider this effect. Details are available in Ref. [19].

⁺ $\partial r / \partial t$ and $\partial \theta / \partial t$ as defined above are opposite to the "velocity" of the coordinate system.

In the next Sections we apply the standard Tokamak expansion with respect to the inverse aspect ratio

$$\epsilon \equiv \frac{r}{R_0} \quad (1.26)$$

and restrict ourselves to the order in which ψ has circular cross sections. The lines $r = \text{const.}$ in a cartesian $x - y$ frame are then circles with a "shifted" origin: [17]

$$(x + \xi(r))^2 + y^2 = r^2, \quad \xi = O(\epsilon) \quad (1.27)$$

and except for an $O(\epsilon)$ correction, θ is the usual polar angle.

Keeping x and y fixed, we therefore have

$$\frac{\partial r}{\partial t} = \frac{x}{r} \frac{\partial \xi}{\partial t} + \dots = \frac{\partial \xi}{\partial t} \cos \theta + O(\epsilon^2) \quad (1.28)$$

Now

$$\sin \theta = y/r + O(\epsilon) \quad (1.29)$$

so

$$\frac{\partial \theta}{\partial t} = -\frac{1}{r} \frac{\partial \xi}{\partial t} \sin \theta + O(\epsilon^2) \quad (1.30)$$

Our substitution rule to account for the moving coordinate system then simply reads:

$$\frac{\partial f}{\partial t} \Rightarrow \frac{\partial f}{\partial t} + \frac{\partial \xi}{\partial t} \left(\frac{\partial f}{\partial r} \cos \theta - \frac{\partial f}{\partial \theta} \frac{\sin \theta}{r} \right) + O(\epsilon^2) \quad (1.31)$$

Our scale factors now are

$$h_r = 1 + \epsilon x \cos \theta + O(\epsilon^2) \quad (1.32)$$

$$h_\theta = r (1 + \epsilon y \cos \theta) + O(\epsilon^2) \quad 1.33)$$

$$h_\phi = 1 + \epsilon \cos \theta + O(\epsilon^2) \quad 1.34)$$

with

$$\epsilon x = -\xi'(r) \quad 1.35)$$

$$\epsilon y = -\int_0^r \frac{d\tilde{r}}{\tilde{r}} \xi'(\tilde{r}) \quad 1.36)$$

where of course x and y in 1.32) and 1.33) have nothing to do with the cartesian frame temporarily introduced in Equ. 1.27) above. From now on, those two letters will retain the definitions given in 1.35) and 1.36).

At the end of this Section we give the general Ansatz describing our expansion scheme, which we will justify by showing that it is internally consistent and sufficient to determine the solution. The θ - dependence is listed explicitly, all quantities below are then functions of r and t only with the exception of Λ_{-1} which is a constant; Λ_{-1}/h_ψ is just the externally applied toroidal field. The numerical index gives the ϵ - order, perturbations caused by finite resistivity are noted by an index η :

$$\begin{aligned}
V_r &= u_{\eta 1} \cos \theta + u_{\eta 2} + O(\eta \epsilon^3) \\
V_\theta &= v_1(r) + v_2 \cos \theta + v_{\eta 1} \sin \theta + O(\epsilon^3) + O(\eta \epsilon^2) \\
V_\psi &= w(r) + w_1 \cos \theta + w_{\eta 1} \sin \theta + O(\epsilon^4) + O(\eta \epsilon) \\
\epsilon^2 \rho &= \epsilon^2 [\rho(r) + R_1 \cos \theta + R_{\eta 1} \sin \theta] + O(\epsilon^2) + O(\eta \epsilon) \\
\psi &= \psi(r, t) \\
\Lambda &= \Lambda_{-1} + \Lambda_1(r) + \Lambda_2 \cos \theta + \Lambda_{\eta 1} \sin \theta + O(\epsilon^3) + O(\eta \epsilon^2)
\end{aligned}
\tag{1.37}$$

This arrangement is identical with the usual Tokamak ordering, except that we have shifted down the order of the main magnetic field, and correspondingly that of all other quantities. This has the advantage that the quantities of main physical interest are then of order unity. The velocities are ordered so as to give equal order contributions to $\underline{v} \times \underline{B}$. The exceptional behavior of v_r will become clearer in the following. Recall that, by definition, $\frac{\partial \psi}{\partial \theta} = 0$ in the flux coordinate system. We are led to the above scheme by our earlier work on stationary states. [14]

We now assume that time dependence is caused by resistive effects only, thereby ruling out all fast MHD motion - together with MHD instabilities. We treat η as a small parameter independent of ϵ and restrict ourselves to first order effects in η only. Our zeroth order is then found by putting $\frac{\partial}{\partial t} = \eta = u_c = Q = M = 0$, leading to steady ideal MHD flow including inertial effects when w, w_1, v_1 and v_2 are not zero.

2. Ideal MHD Flow.

The integrated part of Faraday's law, Equ. 1.22) yields now

$$v_r \psi' = 0 \quad 2.1)$$

justifying our Ansatz for v_r . Here and in the following a prime denotes d/dr . With the definition

$$F \equiv \psi' \quad 2.2)$$

we consider the lowest ϵ orders of our equations of motion. The r - component of momentum, Equ. 1.15) then gives to $O(1)$ and $O(\epsilon)$ respectively:

$$(c^2 \rho)' = -\frac{1}{\mu_0} \left[\Lambda_{-1} \Lambda_1' + \frac{F}{r} (rF)' \right] \quad 2.3)$$

$$-\frac{1}{R_0} \rho w^2 + (c^2 R_1)' = -\frac{1}{\mu_0} \left\{ \Lambda_{-1} \Lambda_2' - 2\epsilon \left[\Lambda_{-1} \Lambda_1' + \frac{F}{r} (rF)' \right] \right\} \\ - \frac{F}{\mu_0} [\Delta^* \psi]_1 \quad 2.4)$$

where $[\Delta^* \psi]_1$ is the coefficient of $\cos \theta$ in the expansion of $\Delta^* \psi$:

$$[\Delta^* \psi]_1 = -2\epsilon x \frac{1}{r} (rF)' + F [\epsilon(\gamma - x - 1)]' \quad 2.5)$$

The θ - component of momentum gives to $O(\epsilon)$:

$$\epsilon \rho w^2 - c^2 R_1 = \frac{1}{\mu_0} \Lambda_{-1} \Lambda_2 \quad 2.6)$$

while the ϕ - component, to $O(\epsilon^2)$, gives

$$-\int \rho (v_1 w_1 + v_2 w + \epsilon (x+2) v_1 w) - R_1 v_1 w = -\frac{i}{\mathcal{M}_0} F \Lambda_2 \quad (2.7)$$

The continuity Equ. 1.18) and the ϕ - component of Faraday's law yield the following $O(\epsilon)$ and $O(\epsilon^2)$ results:

$$v_2 + \left[\frac{R_1}{\rho} + \epsilon (x+1) \right] v_1 = 0, \quad (2.8)$$

and

$$\Lambda_{-1} (v_2 + \epsilon (x+1) v_1) + F (\epsilon w - w_1) = 0, \quad (2.9)$$

respectively. Higher order terms than the ones just considered would introduce new terms in the expansion, but we now have exactly enough information to express all MHD perturbations in terms of the free functions ρ, w, v_1, F . We first write them in terms of R_1/ρ : Equ. 2.8) above then gives:

$$v_2 = - \left[\frac{R_1}{\rho} + \epsilon (x+1) \right] v_1 \quad (2.10)$$

If we then introduce the "geodesic speed"

$$Y \equiv \frac{\Lambda_{-1}}{F} v_1 \quad (2.11)$$

we can rewrite Equ. 2.9) in the form

$$w_1 = \epsilon w - Y \left(\frac{R_1}{\rho} + 2\epsilon \right) \quad (2.12)$$

Equ. 2.6) determines Λ_2 , so Equ. 2.7) may be solved for w_1 :

$$Y w_1 = \epsilon w (w - Y) - c^2 \frac{R_1}{\rho}$$

If we now eliminate w_1 between the last two expressions, we find

$$\frac{R_1}{\rho} = \epsilon \frac{w^2 + 2Y(Y-w)}{c^2 - Y^2} \quad 2.14)$$

We still have to consider the two Eqs. 2.3) and 2.4) describing the r - component of momentum to $O(1)$ and $O(\epsilon)$. The first determines Λ_1 whereas the second will give us the Shafranov shift, ξ , and therefore determine our scale factors. By using 2.6) to eliminate Λ_2 we can rewrite 2.4) in the form

$$2\epsilon \left[\rho c^2 \left(1 + \frac{w^2}{2c^2} \right) \right]' = - \frac{F}{\mu_0} [\Delta^* \psi]_1 \quad 2.15)$$

where we have also used 2.3). Finally, inserting $[\Delta^* \psi]_1$ from 2.5) and using $\epsilon y' = \frac{\epsilon x}{r}$, $\epsilon x = -\xi'$, we find Shafranov's equation for the shift modified by inertia:

$$\frac{1}{\mu_0 r} (r F^2 \epsilon x)' = 2\epsilon \left[\rho c^2 \left(1 + \frac{w^2}{2c^2} \right) \right]' - \frac{F^2}{\mu_0 R_c} \quad 2.16)$$

The remaining components of Faraday's Law integrate in the ideal MHD limit to yield simply $\Phi' = v_1 \Lambda_1 - w F^2$ to $O(1)$.[†] Higher order corrections to Φ need not concern us here. The solution still contains four free functions which we will determine in the following sections.

Inertial effects are seen to influence the shift of the magnetic surfaces as well as to introduce θ -dependent contributions to density and velocity. The magnetic surfaces are therefore no longer pressure

[†] since $\frac{\partial}{\partial t} = u_0 = 0$ in this Section, \underline{E} (ideal) = $-\nabla \phi$ and $E_r = \phi'(r)$

to lowest order

surfaces but the solution is still symmetric with respect to the equatorial plane. As already indicated by our Ansatz, resistive effects will destroy this symmetry. We furthermore see that the geodesic speed is limited by the sound speed, i.e., there is a sound barrier. The next Section deals with resistive effects together with time dependence.

3. Introducing Resistivity.

As indicated in the formal Ansatz of section 1, finite resistivity will introduce new perturbations with different θ - symmetry. We furthermore anticipate different time scales and introduce a multiple time scale formalism by writing

$$t \rightarrow t + \epsilon t + \epsilon^2 t + \dots \rightarrow t, t_1, t_2 \quad 3.1)$$

We introduce the η -perturbations of our general Ansatz and consider first the fast time scale, $\partial/\partial t = O(\eta)$. In the sequel we restrict ourselves to first order terms in η only. Then, to $O(\eta)$, the ϕ - component of momentum and continuity yield respectively

$$\frac{\partial}{\partial t} (\rho w) = 0, \quad \frac{\partial}{\partial t} \rho = 0 \quad 3.2)$$

so ρ and w do not change on the fast time scale. The $O(\eta)$ part of the flux equation 1.22) reads:

$$\frac{\partial \psi}{\partial t} = \frac{\eta}{\mu_c} \frac{1}{r} (r \psi')' - \frac{U_c}{2\pi R_c} \quad 3.3)$$

and we have the very important result that ψ , and therefore F , depends solely on the external voltage U_0 and on the behavior of the resistivity η . It is only through η , μ , and c^2 that the energy transport, which we do not consider in any detail here, couples to the rest of the equations. This is further reason for our feeling that it is useful here simply to simulate all detailed energy behavior by arbitrarily prescribing η , μ and c^2 as

functions of r and t .

As ρ and w do not change on the fast time scale, the r, t dependence of Λ_1 is completely determined by relation 2.3), and also from relation 2.14) the shift is known as a function of r and t , once the solution of 3.3) for $F(r, t)$ is known. In other words, the toroidal field and also the coordinate system follow from the solution of the simple diffusion equation 3.3) for ψ or $F(r, t)$, which is independent of the rest of the equations. If the voltage U_0 is kept constant, the solution reaches a quasi equilibrium provided the temperature does so; the details of this state depend strongly on energy losses and on the initial conditions.

The rest of this Section is devoted to finding an equation for the one remaining free function of Section 2 which changes on the fast time scale, viz. v_1 . Now the flux equation just considered has an $O(\eta \epsilon)$ part where the effect of the changing scale factor appears as given in Equ. 1.31).

$$F \frac{\partial \xi}{\partial t} = \frac{\eta}{\mu_0} [\Delta^* \psi]_1 - F u_{\eta 1} \quad 3.4)$$

Returning to Equ. 2.15), we find immediately

$$u_{\eta 1} = -2 \frac{r}{R_0} \frac{\eta}{F^2} \left[\rho c^2 \left(1 + \frac{w^2}{2c^2} \right) \right]' - \frac{\partial \xi}{\partial t} \quad 3.5)$$

where the r.h.s. contains only known quantities. It is this part of the radial velocity that leads to Pfirsch - Schlüter diffusion.

We now treat the ϕ - component of Faraday's law, Equ. 1.21).

The $O(\eta)$ contribution is

$$\Lambda_{-1} \left[(r u_{\eta 1})' + v_{\eta 1} \right] - w_{\eta 1} r = 0 \quad 3.6)$$

To $O(\eta \epsilon)$ new terms would appear; however, the θ - average of the corresponding equation is simply

$$\left\langle \frac{h_r h_\theta}{h_\phi} \frac{\partial \Lambda_1}{\partial t} \right\rangle_{\eta \epsilon} = \frac{1}{\mu_0} (\eta r \Lambda_1')' - \left\langle \frac{h_\theta}{h_\phi} \Lambda v_r \right\rangle_{\eta \epsilon} \quad 3.7)$$

and this yields, to the required order,

$$r \frac{\partial \Lambda_1}{\partial t} = \left[\frac{\eta}{\mu_0} r \Lambda_1' - r \Lambda_{-1} (u_{\eta 2} + \frac{\epsilon}{2} (y-1) u_{\eta 1}) \right]' \quad 3.8)$$

As $\partial \Lambda_1 / \partial t$ is known from 2.3) with 3.3), this result simply serves to determine the second contribution to the radial velocity which corresponds to the so-called classical diffusion:

$$u_{\eta 2} = \frac{\eta}{\mu_0} \frac{\Lambda_1'}{\Lambda_{-1}} + \frac{1}{2} \epsilon (1-y) u_{\eta 1} - \frac{1}{\Lambda_{-1}} \frac{1}{r} \int_0^r \tilde{r} \frac{\partial \Lambda_1}{\partial t} d\tilde{r} \quad 3.9)$$

We now turn to the θ - component of momentum Equ. 1.16). Viscous forces enter here; we use the results given in appendix A to obtain their lowest order contributions. To $O(\eta)$ we find:

$$c^2 R_\eta = - \frac{1}{\mu_0} \Lambda_{-1} \Lambda_{\eta 1} - k Y \quad 3.10)$$

with

$$k \equiv \frac{F \mu}{r \Lambda_{-1}} \left[\frac{2}{3} \frac{R_1}{\rho} + \epsilon \right] \quad 3.11)$$

If we consider the same equation to $O(\eta\epsilon)$ and average with respect to Θ we obtain the time derivative; conveniently, the influence of the moving coordinates is still one order higher and can be neglected.

Thus

$$r\rho \frac{\partial v_r}{\partial t} + \epsilon \left(\rho w w_\eta + \frac{1}{2} R_\eta w^2 \right) = \epsilon \left(\frac{1}{\mu_0} \Lambda_{-1} \Lambda_{\eta 1} + \frac{3}{2} k Y \right) \quad (3.12)$$

Now the r - component of momentum to $O(\eta)$ just produces the r -derivative of Equ. 3.10) above, so nothing new comes from it. The fact that this is also true for the viscous term is confirmed in Appendix A. Contributions not yet considered are those of the ϕ - component of momentum and of continuity to $O(\eta\epsilon)$; these formally bring in the time scale t_1 as well as $\frac{\partial \xi}{\partial t}$:

$$\begin{aligned} & \frac{\partial}{\partial t_1} (\rho w) + \frac{\partial \xi}{\partial t} (\rho w)' \cos \Theta + \frac{\partial}{\partial t} (R_1 w + \rho w_1) \cos \Theta \\ & + \frac{1}{r} \frac{\partial}{\partial r} (r \rho u_{\eta 1} w) \cos \Theta + \frac{1}{r} \frac{\partial}{\partial \Theta} \sin \Theta (R_\eta v_{1w} + \rho (v_{\eta 1} w + v_1 w_\eta)) \\ & = \frac{1}{r \mu_0} \frac{\partial}{\partial \Theta} \sin \Theta (\Lambda_{\eta 1} F) + \frac{3}{r} \frac{F}{\Lambda_{-1}} k Y \cos \Theta \end{aligned}$$

We therefore conclude

$$\frac{\partial}{\partial t_1} (\rho w) = 0 \quad (3.13)$$

and

$$\begin{aligned} & r \left(w \frac{\partial}{\partial t} R_1 + \rho \frac{\partial}{\partial t} w_1 + \frac{\partial \xi}{\partial t} (\rho w)' \right) + (r \rho u_{\eta 1} w)' + \\ & + (R_\eta v_1 + \rho v_{\eta 1}) w + \rho v_1 w_\eta = \frac{1}{\mu_0} F \Lambda_{\eta 1} + \frac{3F}{\Lambda_{-1}} k Y \end{aligned} \quad (3.14)$$

Similarly we find from continuity

$$\frac{\partial \rho}{\partial t_1} = 0 \quad 3.15)$$

and

$$r \left(\frac{\partial R_1}{\partial t} + \frac{\partial \xi}{\partial t} \rho' \right) + (r \rho u_{\eta_1})' + R_{\eta} v_1 + \rho v_{\eta_1} = 0 \quad 3.16)$$

We may now use Equ. 3.6) to eliminate $(r u_{\eta_1})' + v_{\eta_1}$
and Equ. 3.10) to eliminate $\frac{1}{\mu_0} \Lambda_{-1} \Lambda_{\eta_1}$ from the last two results.

Thus

$$r \left[w \frac{\partial R_1}{\partial t} + \rho \frac{\partial w_1}{\partial t} + (\rho w)' \frac{\partial \xi}{\partial t} \right] + r u_{\eta_1} (\rho w)' + \frac{F}{\Lambda_{-1}} \rho w w_{\eta} \\ + R_{\eta} v_1 w + \rho v_1 w_{\eta} = - \frac{F}{\Lambda_{-1}} c^2 R_{\eta} + 2 \frac{F}{\Lambda_{-1}} h Y$$

$$r \left[\frac{\partial R_1}{\partial t} + \rho' \frac{\partial \xi}{\partial t} \right] + r u_{\eta_1} \rho' + \frac{F}{\Lambda_{-1}} \rho w_{\eta} + R_{\eta} v_1 = 0$$

To separate time derivatives, we subtract w times the last equation
from the first:

$$r \left[\rho \left(\frac{\partial w_1}{\partial t} + \frac{\partial \xi}{\partial t} w' + u_{\eta_1} w' \right) \right] + \rho v_1 w_{\eta} = - \frac{F}{\Lambda_{-1}} (c^2 R_{\eta} - 2 h Y)$$

We now introduce the shorthand notation:

$$\frac{DR_1}{Dt} \equiv \frac{\partial R_1}{\partial t} + \left(\frac{\partial \xi}{\partial t} + u_{\eta_1} \right) \rho' \quad 3.17)$$

as well as

$$\frac{Dw_1}{Dt} \equiv \frac{\partial w_1}{\partial t} + \left(\frac{\partial \xi}{\partial t} + u_{\eta_1} \right) w' - 2 \frac{F}{\Lambda_{-1}} \frac{h Y}{\rho} \quad 3.18)$$

If we further use 3.10) to eliminate $\Lambda_{\eta 1}$ from Equ. 3.12), we have finally the following system of three equations containing time derivatives on the fast time scale:

$$r \frac{\partial v_1}{\partial t} + \epsilon \left[w w_\eta + \frac{R}{\rho} \eta (c^2 + \frac{w^2}{2}) - \frac{1}{2} \frac{k Y}{\rho} \right] = 0 \quad 3.19)$$

$$r \frac{D w_1}{D t} + v_1 w_\eta + \frac{F}{\Lambda_{-1}} c^2 \frac{R \eta}{\rho} = 0 \quad 3.20)$$

$$\frac{r}{\rho} \frac{D R_1}{D t} + \frac{F}{\Lambda_{-1}} w_\eta + \frac{R \eta}{\rho} v_1 = 0 \quad 3.21)$$

But in the previous section, we found that R_1 and w_1 are determined in terms of the functions w and Y , so their time derivatives are not free in view of 3.19). We therefore use the last two equations above to determine the two remaining η -perturbations R_η and w_η :

$$(c^2 - Y^2) \frac{R_\eta}{\rho} = \frac{\Lambda_{-1}}{F} r \left(\frac{Y}{\rho} \frac{D R_1}{D t} - \frac{D w_1}{D t} \right) \quad 3.22)$$

$$(c^2 - Y^2) w_\eta = \frac{\Lambda_{-1}}{F} r \left(Y \frac{D w_1}{D t} - \frac{c^2}{\rho} \frac{D R_1}{D t} \right) \quad 3.23)$$

Before we use these results in Equ. 3.19), we rewrite it in the form of an equation for the "geodesic speed" Y . From the definition of Y , and the flux equation 3.3), we can write

$$\frac{\partial v_1}{\partial t} = \frac{F}{\Lambda_{-1}} \left[\frac{\partial Y}{\partial t} + \frac{1}{M_0 F} \left(\frac{\eta}{r} (r F)' \right)' \right] \quad 3.24)$$

Finally, introducing

$$q \equiv \frac{\epsilon \Lambda_{-1}}{F} \quad 3.25)$$

we find from Eqs. 3.19), 3.22) and 3.23):

$$\begin{aligned} \frac{\partial Y}{\partial t} + \frac{1}{\mu_0} \frac{Y}{F} \left[\frac{\eta}{r} (rF)' \right]' + \\ + \frac{\eta}{c^2 - Y^2} \frac{\Lambda_{-1}}{F} \left\{ \frac{1}{\rho} \frac{DR_1}{Dt} \left[Y \left(c^2 + \frac{w^2}{2} \right) - c^2 w \right] - \frac{Dw_1}{Dt} \left[c^2 + \frac{w^2}{2} - Yw \right] \right\} \\ = \frac{\epsilon}{2} \frac{\Lambda_{-1}}{F} \frac{\eta Y}{r \rho} \end{aligned} \quad 3.26)$$

Finally we list the time derivatives of R_1 and w_1 , since they follow from the results of Section 2 remembering that ρ and w do not change at the time scale considered here:

$$(c^2 - Y^2) \frac{1}{\rho} \frac{\partial R_1}{\partial t} = 2 \frac{\partial Y}{\partial t} \left[Y \left(\frac{R_1}{\rho} + 2\epsilon \right) - \epsilon w \right] \quad 3.27)$$

and

$$(c^2 - Y^2) \frac{\partial w_1}{\partial t} = -2 \frac{\partial Y}{\partial t} \left[c^2 \frac{R_1}{\rho} + \epsilon \left(c^2 - \frac{w^2}{2} \right) \right] \quad 3.28)$$

Equ. 3.26) requires not only that we substitute in it 3.27) and 3.28) for portions of $\frac{DR_1}{Dt}$ and $\frac{Dw_1}{Dt}$, but it also requires knowledge of $\frac{\partial Y}{\partial t}$, according to the definitions 3.17) and 3.18).

$\frac{\partial F}{\partial t}$, however, is known from $\frac{\partial F}{\partial t}$, because of the Shafranov relation 2.14), and, from 3.3), $\frac{\partial F}{\partial t} = \left(\frac{\eta}{\mu_0 r} (rF)' \right)'$, as we already have noted in deriving 3.24) and 3.26). In this way we see that our problem can be reduced to a single but rather complicated looking equation for Y .

In summary of this Section we find that the fast time scale determines F and Y , the two other "free" functions of MHD remaining

undetermined. We know, however, that the latter quantities also do not change on the time scale t_1 . We will see later on that they are determined only on the time scale t_2 and that this time scale will lead ultimately to a unique equilibrium. The next Section, however, discusses the behavior of Y on the fast time scale and the rotational instability described by it. [9], [19]

4. The Rotation.

The final result of the last Section describes the detailed behavior of Y or v_i . In particular our derivation is valid during the penetration phase of the poloidal field ($\dot{B} \neq 0$), which may give new and interesting results, since in Ref. [9] and related works the electrostatic approximation was used. Here, however, we indicate only how such a complicated equation lends itself to analysis without actually having to integrate it, for we are mainly interested in the time-asymptotic behavior of the solution.

The last three equations of the previous section may be combined formally to yield

$$\frac{\partial Y}{\partial t} = f(Y, F) \quad 4.1)$$

where we only list the time dependent parameters, since the others are simply constants on the time scale considered. We first look for stationary solutions $\frac{\partial Y}{\partial t} = 0$. The last two equations of the previous Section then tell us immediately that

$$\frac{\partial w_i}{\partial t} = \frac{\partial R_i}{\partial t} = 0 \quad ; \quad 4.2)$$

thus $f(Y, F)$ is relatively easily given explicitly and its zeros can be determined. As the function cannot change sign between its zeros, the solution there will either be purely growing or decaying. To determine what is actually happening in such a region, it is sufficient

to analyze the neighbourhood of a zero itself; in other words, a stability analysis of the stationary solutions is all that is required. Care is necessary at possible double zeros, but they probably do not occur here. At simple zeros, f must, of course, change its sign, further facilitating the analysis.

Here we only sketch the procedure for the special case $w = 0$ and with complete field penetration already having taken place, i.e.,

$$\left[\frac{\eta}{\eta_0 r} (rF)' \right]' = 0, \quad \frac{\partial \xi}{\partial t} = 0 \quad (4.3)$$

so

$$\frac{D w_1}{D t} = - \frac{2F}{\Lambda_{-1}} \frac{k \gamma}{\rho r} \quad (4.4)$$

$$\frac{D R_1}{D t} = u_{\eta 1} \rho' \quad (4.5)$$

Stationary solutions are therefore found by solving

$$\frac{\eta}{i - \gamma^2/c^2} \frac{\Lambda_{-1}}{F} \left\{ u_{\eta 1} \frac{\rho'}{\rho} + \frac{2F}{\Lambda_{-1}} \frac{k}{r \rho} \right\} \gamma = \frac{\epsilon}{2} \frac{\Lambda_{-1}}{F} \frac{k \gamma}{r \rho} \quad (4.6)$$

or

$$\gamma \left\{ r g u_{\eta 1} \rho' + \frac{\epsilon}{2} k \left(3 + \frac{\gamma^2}{c^2} \right) \right\} = 0 \quad (4.7)$$

Clearly $\gamma = 0$ is a stationary solution, stable or not. In fact it would be the only one if $k = 0$ (no viscosity). For convenience in the following we repeat

$$k = \frac{F \mu}{r \Lambda_{-1}} \left[\frac{2}{3} \frac{R_1}{\rho} + \epsilon \right] \quad (4.8)$$

and

$$\frac{R_1}{f} = 2\epsilon \frac{Y^2/c^2}{1 - Y^2/c^2} \quad ; \quad (w = c) \quad 4.9)$$

so

$$h_k = \frac{F\mu}{r\Lambda_{-1}} \epsilon \frac{1 + \frac{1}{3}(Y^2/c^2)}{1 - Y^2/c^2} \quad 4.10)$$

The stationary solutions different from $Y = 0$ are therefore found from

$$r^2 q u_{\eta_1} f' + \frac{\epsilon^2}{2} \frac{F}{\Lambda_{-1}} \frac{\mu}{3r} \frac{(3 + Y^2/c^2)^2}{1 - Y^2/c^2} = 0 \quad 4.11)$$

or

$$\frac{1}{6} \frac{(3 + Y^2/c^2)^2}{1 - Y^2/c^2} = - \frac{\Lambda_{-1}}{4\epsilon^2 F} r^2 q u_{\eta_1} f' \quad 4.12)$$

Now the left-hand side equals $3/2$ for $Y = 0$ and increases monotonically with Y^2/c^2 . We recall that $u_{\eta_1} \sim -(\epsilon^2 \rho)'$ so the r.h.s. is always positive provided the pressure- and density-gradients are in the same direction. If this term is $< 3/2$, no real solution $Y \neq 0$ exists and $Y = 0$ remains the only stationary value. Otherwise there is only one solution for Y^2/c^2 , and, furthermore, it is always on the subsonic branch, $Y^2 < c^2$. All 3 possible roots are simple, so $\frac{\partial Y}{\partial t}$ changes sign there. In order to study the stability it therefore suffices to concentrate on $Y = 0$. We may now proceed by retaining only linear terms, taking care to properly

treat the time derivatives of w_1 and R_1 . In this case $\frac{\partial Y}{\partial t} \sim Y$, so

$$\frac{\partial R_1}{\partial t} \approx 0 \quad 4.13)$$

and

$$\frac{\partial w_1}{\partial t} \approx -2\epsilon \frac{\partial Y}{\partial t} \quad 4.14)$$

Then Equ. 3.26) reads approximately

$$\frac{\partial Y}{\partial t} + \frac{q}{f} \frac{\Lambda_{-1}}{F} \left\{ \frac{1}{f} u_{\eta_1} f' Y + 2\epsilon \frac{\partial Y}{\partial t} + \frac{2F}{\Lambda_{-1}} \frac{kY}{rp} \right\} = \frac{\epsilon \Lambda_{-1} kY}{2f rp} \quad 4.15)$$

or, regrouping

$$(1 + 2q^2) \frac{\partial Y}{\partial t} + \frac{\Lambda_{-1}}{F} \left\{ \frac{q}{f} u_{\eta_1} \frac{f'}{f} + \frac{3}{2} \epsilon \frac{k}{rp} \right\} Y = 0 \quad 4.16)$$

It is interesting to note here the presence of Pfirsch-Schlüter factor, $1 + 2q^2$.

Thus, using only the leading term of k from Equ. 4.10), we find that $Y = 0$ is unstable provided

$$r^2 \frac{q}{f} u_{\eta_1} f' + \frac{3}{2} \frac{4F}{\Lambda_{-1}} \epsilon^2 < 0 \quad 4.17)$$

which, incidentally, is also the condition that the other two roots are real; further, since the latter roots are also simple, $\frac{\partial Y}{\partial t}$ changes sign there, so they themselves are then stable. In other words, when $Y = 0$ is stable the other two roots do not exist, whereas if $Y = 0$ is unstable the other roots do exist, and are stable at $Y^2 < c^2$.

For the special case discussed here we therefore find that viscosity always suffices to stabilize the rotation at either $Y = 0$ or at some

finite value with $Y^2/c^2 < 1$. From the analysis it also follows that it is the inertial contribution to the viscous force, i.e., that part proportional to Y^n , $n \geq 1$, which effects the stabilization no matter how small μ is, provided $\mu \neq 0$ of course.

This example is intended to demonstrate that our equation 3.26) is amenable to analysis even in its implicit form if care is taken. Now for the fast time scale, it is probably sufficient to treat the case $w = 0$, as this is a natural initial condition, but we shall see below that on the slow time scale w will change, especially if we introduce sources and $Y \neq 0$. Particularly if the source yields plasma with the local velocity, w grows and saturates at a value different from zero, so the stability analysis for Y is necessary for all the w values found under way. Furthermore the effect of changes in the poloidal field are interesting in themselves. [19]

5. The Long Time Behavior

Up to now we have determined two of the four free functions of Section 2): F is governed by the external applied electric field, and reaches an asymptotic final state at the end of the fast time scale if $U_0 = \text{const.}$ We have also studied the behavior of v_y (or Y) for the special case $w = 0$ and full field penetration as an illustration of a method for obtaining information about the poloidal spin-up. Asymptotically, Y is either 0, or if that state is unstable, Y saturates at a finite value $Y^2 < c^2$ depending on viscosity. Furthermore, ρ and w keep their initial values during the fast time scale and in addition were shown not to depend on the intermediate time scale t_1 .

We have to go to time scale t_2 in order to find the behavior of ρ and w . We multiply the ϕ - component of momentum and continuity, Eqs. 1.17) and 1.18), by $h_r h_\theta h_\phi$, and, taking the average over θ , find the $O(\eta \epsilon^2)$ contributions. We assume the short time behavior to have reached its time asymptotic state, so $\frac{\partial}{\partial t} = 0$. Since ρ and w do not depend on the intermediate time scale t_1 , we then have formally

$$\left\langle h_r h_\theta h_\phi^2 \frac{\partial}{\partial t} (\rho v_\phi) \right\rangle_{\eta \epsilon^2} = \frac{\partial}{\partial t_2} (\rho w) + \left\langle \frac{\partial}{\partial t_1} (R_1 w + \rho w_1) \cos \theta \right\rangle_{5.1)} + \frac{\partial}{\partial t_1} \left\langle h_r h_\theta h_\phi^2 (\rho w) \right\rangle_{\eta \epsilon^2}$$

so the derivatives with respect to t_1 average out. The same is true for the continuity equation, and both results hold to $O(\eta \epsilon^2)$. U_0 is allowed to vary on the t_1 time scale.

We therefore find

$$\frac{\partial}{\partial t_2} (\rho w) + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[\rho u_{\eta 2} w + \frac{1}{2} u_{\eta 1} (\rho w_1 + R_1 w + \epsilon (2+y) \rho w) \right] \right\} \quad 5.2)$$

$$= P_\phi + \langle h_r h_e h_\varphi^2 F_{v\phi} \rangle_{\eta e 2}$$

and

$$\frac{\partial}{\partial t_1} \rho + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[\rho u_{\eta 2} + \frac{1}{2} u_{\eta 1} (R_1 + \epsilon (y+1) \rho) \right] \right\} = Q \quad 5.3)$$

where P_ϕ is the (average) source of ϕ - directed momentum. Now if $u_0 = \text{const.}$ none of the quantities in 5.2) and 5.3) with the possible exception of R_1 and w_1 , depend on the intermediate time scale t_1^+ . Therefore we must conclude that neither R_1 nor w_1 (and therefore neither v_1 nor Y) can depend on t_1 , since all three quantities are interrelated through the ideal MHD relations. Consequently the time scale t_1 , though formally necessary, in no way affects our final results, to the order studied, unless u_0 is allowed to vary with time.

Actually, viscosity averages out exactly in the toroidal angular momentum equation 5.2 see Appendix A, Equ. A.16).

From the above we see that we may take the asymptotic state of the fast time scale as initial conditions for our equations evolving on the slow time scale. If our source yields plasma with the local speed

$$P_\phi = Q w \quad 5.4)$$

⁺ For the dependence on R_1 of the viscous term in 5.2) and 3.14) see Equ. 4.8) or Appendix A. However, as pointed out in the paragraph below 5.3), the flux average of $F_{v\phi}$ as defined in 5.2) actually vanishes exactly, i.e., to all orders in the expansion.

we see that the state $w = 0$, although being a solution if $Y = 0$ is stable, is unstable by itself, i.e., any initial perturbation would grow, since the source simply delivers angular momentum at the local value. w will therefore tend to a limit $\neq 0$ with this special source arrangement. The density however always changes, until $t_2 \rightarrow \infty$, and the time-asymptotic state is reached. This final state is unique - it does not depend on the initial conditions or on the state reached at the end of the fast time scale.

We briefly consider here, as an illustrative example, the final state for the special, source - free case:

$$P_\phi = 0, \quad Q = 0 \quad 5.5)$$

Letting $\frac{\partial}{\partial t_2} = 0$ and subtracting w times the continuity equation from the momentum equation, we find

$$w_1 + \epsilon w = 0 \quad 5.6)$$

We may rewrite this result using Equ. 2.12):

$$w_1 + \epsilon w = 2\epsilon w - Y \left(\frac{R_1}{\rho} + 2\epsilon \right) = 0$$

and using Equ. 2.14) for R_1 :

$$2c^2(w - Y) - Yw^2 = 0 \quad 5.7)$$

We are mainly interested in the solution under the assumption $Y/c \ll 1$,

so

$$w = Y \left(1 + \frac{1}{2} \frac{Y^2}{c^2} + \dots \right) \quad 5.8)$$

Thus, apart from an inertial correction due to the rotation, $w = Y$, asymptotically. The flow in that case becomes parallel to the magnetic field and the r -component of the η - independent (ideal MHD) electric field $\vec{\Phi}'(r)$ (see below 2.16) vanishes asymptotically on the slow time scale. Thus it is clear, even with no sources, that w would tend to grow from an initial value of zero unless $Y = 0$ and remains so.

The final density profile now follows from continuity. The only solution regular at the origin is

$$\rho u_{\eta 2} + \frac{1}{2} u_{\eta 1} (R_1 + \epsilon(1+Y)\rho) = 0 \quad 5.9)$$

which, on rewriting $u_{\eta 1}$ and $u_{\eta 2}$ according to Eqs. 3.5) and 3.9), reads

$$\frac{\eta}{\mathcal{M}_0} \frac{\Lambda_1'}{\Lambda_{-1}} - 2\epsilon \frac{\eta}{F^2} \left[\rho c^2 \left(1 + \frac{w^2}{2c^2} \right) \right]' \left[\epsilon \rho + \frac{R_1}{2} \right] = 0 \quad 5.10)$$

We finally use the ideal MHD relation 2.3) to eliminate Λ_1 , Equ. 3.14) to eliminate R_1 , and replace w everywhere according to Equ. 5.8), keeping all quantities correct to first order in Y^2/c^2 :

$$(c^2 \rho)' + 2q^2 \left[c^2 \rho \left(1 + \frac{w^2}{2c^2} \right) \right]' \left[1 + \frac{Y^2}{2c^2} \right] + \frac{1}{\mathcal{M}_0} \frac{F}{r} (rF)' = 0 \quad 5.11)$$

The final pressure profile therefore depends on whether or not the plasma rotates.

The last term in 5.11) is simply the $\vec{J} \times \vec{\beta}$ force due to the toroidal current, and if we define a local poloidal β value simply

as the ratio of the pressure gradient to that part of the $\underline{J} \times \underline{B}$ force needed to balance it, we find, for a nonrotating equilibrium, [3]

$$\beta_p = \frac{1}{1 + 2q^2} \quad 5.12)$$

For a rotating equilibrium, the relation becomes more complicated; but, qualitatively, it is easy to see that β_p would be reduced by inertial effects.

Appendix A: The Viscous Force.

With reference to Eqs. 1.5) and 1.6) [19] and taking into account $\vec{B}_r = 0$ and $\partial/\partial\phi = 0$, the components of the viscous force entering the terms \underline{M} in our equations of motion are:

$$F_{vr} = -\frac{1}{h_r} \frac{\partial D}{\partial r} - 3D \left(\frac{b_\theta^2}{h_r h_\theta} \frac{\partial h_\theta}{\partial r} + \frac{b_\phi^2}{h_r h_\phi} \frac{\partial h_\phi}{\partial r} \right) \quad A.1)$$

$$\begin{aligned} \bar{F}_{v\theta} = & -\frac{1}{h_\theta} \frac{\partial D}{\partial \theta} + \frac{3}{h_r h_\theta h_\phi} \frac{\partial}{\partial \theta} (h_r h_\phi b_\theta^2 D) \\ & - 3D \left(\frac{b_\phi^2}{h_\phi h_\theta} \frac{\partial h_\phi}{\partial \theta} \right) \end{aligned} \quad A.2)$$

$$F_{v\phi} = \frac{3}{h_r h_\theta h_\phi^2} \frac{\partial}{\partial \theta} (h_r h_\phi^2 b_\theta b_\phi D) \quad A.3)$$

with

$$\underline{b} \equiv \underline{B}/B$$

D contains velocities, but we only need to consider the ideal contributions independent of η , as the coefficient μ is itself considered $O(\eta)$.

$$\begin{aligned} D = \mu \left[\frac{b_\theta^2}{h_\theta} \frac{\partial v_\theta}{\partial \theta} + \frac{b_\theta b_\phi}{h_\theta} \frac{\partial v_\phi}{\partial \theta} + \frac{b_\phi}{h_\theta h_\phi} \frac{\partial h_\phi}{\partial \theta} (v_\theta b_\phi - v_\phi b_\theta) - \right. \\ \left. - \frac{1}{3} \frac{1}{h_r h_\theta h_\phi} \frac{\partial}{\partial \theta} (h_r h_\phi v_\theta) \right] \end{aligned} \quad A.4)$$

Now

$$b_\phi = 1 + O(\epsilon^2), \quad b_r b_\theta = \frac{F}{\Lambda_{-1}} + O(\epsilon^2)$$

Sorting out orders, we find D has no θ -free part, so

$$F_{v\theta} = O(D), \quad F_{v\phi} = O(\epsilon D)$$

to lowest order. Also D is $O(\mu \epsilon^3)$. We want to order μ such that viscosity enters the equations of motion at the earliest possible time (maximal information ordering). But the θ -component of momentum, Equ. 1.16), has an $O(\eta)$ part, so D itself should be $O(\eta)$, i.e.

$$\mu = O(\eta/\epsilon^2) \quad \text{A.5)}$$

In our expansion, we need the lowest orders in ϵ of F_v directly, but the next order only averaged over θ . In all cases, D itself is required to lowest order only:

$$D \doteq -\mu \frac{F}{\Lambda_{-1}} \sin \theta \left\{ \frac{1}{r} (w_1 + \epsilon(Y-w)) - \frac{1}{3r} \left[\epsilon(x+1)Y + \frac{\Lambda_{-1}}{F} v_2 \right] \right\} \quad \text{A.6)}$$

From our ideal MHD results, we may eliminate v_2 by Equ. 2.6)

$$v_2 = - \left(\frac{R_1}{\rho} + \epsilon(x+1) \right) v_1$$

or

$$\frac{1}{3} \left[\epsilon(x+1)Y + \frac{\Lambda_{-1}}{F} v_2 \right] = -\frac{1}{3} \frac{R_1}{\rho} Y \quad \text{A.7)}$$

and using Equ. 2.12), viz.,

$$w_1 - \epsilon w = -Y \left(\frac{R_1}{\rho} + 2\epsilon \right)$$

we find

$$D \doteq \frac{\mu F}{r \Lambda_{-1}} \left[\frac{2}{3} \frac{R_1}{\rho} + \epsilon \right] Y \sin \theta \quad \text{A.8)}$$

Let us introduce the abbreviation

$$\begin{aligned} k &\equiv \frac{\mu F}{r \Lambda_{-1}} \left[\frac{2}{3} \frac{R_1}{\rho} + \epsilon \right] \\ &= O(\mu \epsilon^2) = O(\eta) \end{aligned} \quad \text{A.9)}$$

We may now collect the results needed in Section 3:

$$D = k Y \sin \theta + \dots \quad \text{A.11)}$$

$$h_r F_{vr} = -(k Y)' \sin \theta + \dots \quad \text{A.12)}$$

$$h_\theta F_{v\theta} = -k Y \cos \theta + \dots \quad \text{A.13)}$$

$$h_\phi F_{v\phi} = \frac{3}{r} \frac{F}{\Lambda_{-1}} k Y \cos \theta + \dots \quad \text{A.14)}$$

And finally the two needed mean values (see, for example, 3.12)

$$\frac{1}{r} \langle h_\theta F_{v\theta} \rangle = \frac{3}{2} \epsilon k Y + \dots \quad \text{A.15)}$$

and

$$\langle h_r h_\theta h_\phi^2 F_{v\phi} \rangle = 0 \quad \text{A.16)}$$

where this last result is exact -- cf. Equ. A.3)

We see that k becomes a known function of the "free functions" of ideal MHD.

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