Ohmic Losses of a Relativistic Electron Ring Moving Close to Conducting Walls of Finite Thickness

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Abstract

A relativistic electron ring moving along a conducting wall is acted on by a retarding force due to the ohmic losses of the image currents in the wall. This force is calculated for an electron ring moving coaxially along and close to a cylinder of finite thickness. The dependence of the force on the thickness and conductivity of the cylinder and ring velocity is discussed.

I. Introduction

In the electron ring accelerator [1] it may be favourable to use conducting structures near the electron rings, e.g. to suppress the negative mass instability or to improve the focussing of the rings by means of image effects. In the case of an electron ring moving parallel to such a conducting structure, there are image currents flowing in the conductor. The finite conductivity of the structure leads to ohmic losses, which give rise to a retarding force on the electron ring, thus modifying the dynamics of the ring motion.

This paper considers the retarding force for an infinitely thin electron ring of major radius R moving coaxially inside or outside a hollow cylinder of thickness d and conductivity σ (fig.1). If attention is confined to the case where the distance a of the ring to the conductor is sufficiently small $a/R \ll 1$, linear treatment of the problem will be a sufficient approximation. The problem is thus reduced to calculating the drag force on an infinitely thin electron beam moving transversely to the beam direction along a conducting plane sheet of thickness d. The case of an infinitely thick sheet ($d = \infty$) has been solved by Voskresensky et al. [2] . But because the results for sheets of finite thickness are very much different from the $d = \infty$ case, it seems worthwhile to discuss the more general situation.

An electron beam with a velocity v_z in the z-direction is moving in the x-direction with a velocity v_x at a distance a from a plane conducting sheet located at $y: 0 > y > -d$, as shown in fig.2. v_x, v_z are the components of the velocity in the laboratory system. The beam velocity v_z^o in the rest frame of the beam is given by $v_z^o = v_z \gamma_x$, $\gamma_x = (1-v_x^2)^{-1/2}$. The force F_x on a beam electron - we call F_x the drag force - is composed of an electric and a magnetic component:

$$F_x = e E_x - e v_z B_y = F_e + F_M$$

Because E_x is produced by the beam charge only and B_y by the beam current only, the contributions of the beam charge and the beam current to the drag force are additive and can be calculated separately.

II. The ring current case

To begin with the case of a neutral current beam, the starting point is Maxwell's equations

$$\text{(1)} \quad \begin{aligned} &\text{curl}\ \underline{E} = -\dot{\underline{B}}, && \text{div}\ \underline{E} = 4\pi\rho, \\ &\text{curl}\ \underline{B} = 4\pi \underline{j} + \dot{\underline{E}}, && \text{div}\ \underline{B} = 0. \end{aligned}$$

(velocity of light c = 1 and $\dot{} \equiv \frac{\partial}{\partial t}$), which have to be solved for the following charge and current densities ρ, j in the three different regions $I: y > 0$, $II: 0 > y > -d$, $III: -d > y$

$$\text{(2)} \quad \begin{aligned} &I: && \rho = 0, && \underline{j} = \{0, 0, J\,\delta(x - v_x t)\,\delta(y-a)\}, \\ &II: && \rho = 0, && \underline{j} = \sigma \underline{E}, \\ &III: && \rho = 0, && \underline{j} = 0, \end{aligned}$$

where J is the beam current and σ the conductivity of the metal sheet.

Because of $\frac{\partial}{\partial z} \equiv 0$ and the special form of eqs. (2) the only non vanishing components of the electric and magnetic fields are: E_z, B_x, B_y. Eliminating E_z and B_x, one gets the following equations for B_y in the three different regions:

$$\text{(3)} \quad \begin{aligned} &I: && \Delta B_I - \ddot{B}_I = -\frac{4\pi J}{v_x}\,\delta(x - v_x t)\,\delta(y-a), \\ &II: && \Delta B_{II} - \ddot{B}_{II} - 4\pi\sigma \dot{B}_{II} = 0, \\ &III: && \Delta B_{III} - \ddot{B}_{III} = 0. \end{aligned}$$

(The index y is replaced by the region label.)

As a consequence of Maxwell's equations the tangential components of the electric and magnetic fields have to be continuous at the surface of the conducting sheet, which leads to the following boundary conditions for $B_y(x, y, t)$:

$$(4)\qquad \begin{aligned} &B_I(x,y=0,t) = B_{II}(x,y=0,t), \quad \frac{\partial}{\partial y} B_I(x,y=0,t) = \frac{\partial}{\partial y} B_{II}(x,y=0,t), \\ &B_{II}(x,y=-d,t) = B_{III}(x,y=-d,t), \quad \frac{\partial}{\partial y} B_{II}(x,y=-d,t) = \frac{\partial}{\partial y} B_{III}(x,y=-d,t), \end{aligned}$$

and for $|y| \to \infty$ it is required that the fields vanish:

$$(5)\qquad B_I(x,y,t) = 0 \quad for\ y \to \infty, \quad B_{III}(x,y,t) = 0 \quad for\ y \to -\infty .$$

If the beam velocity v_x is kept constant it can be assumed that after a suffiently long time the fields become stationary in the sense that all quantities only depend on $x-v_x t$ and y. Introducing the Fourier transform $B(\omega, y)$ defined by

$$(6)\qquad B(x-v_x t, y) = \int_{-\infty}^{+\infty} B(\omega,y)\, e^{i\frac{\omega}{v_x}(x-v_x t)}\, d\omega, \quad with\ B^*(\omega,y) = B(-\omega,y)$$

where $B^*(\omega, y)$ is the complex conjugate of $B(\omega, y)$, and inserting (6) in (3), one obtains

$$(7)\qquad \begin{aligned} &I: \left(\frac{\partial^2}{\partial y^2} - \frac{\omega^2}{v_x^2 \gamma_x^2}\right) B_I(\omega,y) = i\omega \frac{2J}{v_x^2}\, \delta(y-a) \\ &II: \left(\frac{\partial^2}{\partial y^2} - \frac{\omega^2}{v_x^2 \gamma_x^2} + 4\pi\sigma\omega i\right) B_{II}(\omega,y) = 0 \\ &III: \left(\frac{\partial^2}{\partial y^2} - \frac{\omega^2}{v_x^2 \gamma_x^2}\right) B_{III}(\omega,y) = 0 \end{aligned}$$

where $\gamma_x = (1-v_x^2)^{-1/2}$ is the relativistic factor of the x-motion.

To calculate $B(x-v_x t, y)$ one needs the Fourier transform $B(\omega, y)$ for positive ω only because of the reality condition $B^*(\omega, y) = B(-\omega, y)$. Therefore, in the following it will be assumed throughout that $\omega > 0$. The general solution of eq.(7) satisfying the boundary conditions (7) is given by

(8)
$$\mathrm{I}: \quad B_I(\omega,y) = B_I(\omega)\, e^{-\frac{\omega}{v_x\gamma_x} y} - i\frac{\gamma_x J}{v_x} e^{-\frac{\omega}{v_x\gamma_x}|y-a|},$$
$$\mathrm{II}: \quad B_{II}(\omega,y) = B^+_{II}(\omega)\, e^{\frac{s}{v_x\gamma_x} y} + B^-_{II}(\omega)\, e^{-\frac{s}{v_x\gamma_x} y},$$
$$\mathrm{III}: \quad B_{III}(\omega,y) = B_{III}(\omega)\, e^{\frac{\omega}{v_x\gamma_x} y},$$

with $s = (\omega^2 - 4\pi\sigma\gamma_x^2 v_x^2 \omega i)^{1/2}$. The coefficients $B_I(\omega)$, $B^+_{II}(\omega)$, $B^-_{II}(\omega)$ and $B_{III}(\omega)$ can be determined by the four boundary conditions (4). Finally one obtains for the complete solution

(9)
$$\mathrm{I}: \quad B_I(\omega,y) = -i\frac{\gamma_x J}{v_x} e^{-\frac{\omega}{v_x\gamma_x}(a+y)} \frac{(\omega^2-s^2)\left(1-e^{-\frac{s}{v_x\gamma_x}2d}\right)}{(\omega+s)^2-(\omega-s)^2 e^{-\frac{s\,2d}{v_x\gamma_x}}} - i\frac{\gamma_x J}{v_x} e^{-\frac{\omega}{v_x\gamma_x}|a-y|},$$
$$\mathrm{II}: \quad B_{II}(\omega,y) = -i\frac{\gamma_x J}{v_x} e^{-\frac{\omega}{v_x\gamma_x}a}\, 2\omega\, \frac{(\omega+s)\, e^{\frac{sy}{v_x\gamma_x}} - (\omega-s)\, e^{-\frac{s}{v_x\gamma_x}(y+2d)}}{(\omega+s)^2 - (\omega-s)^2 e^{-\frac{s}{v_x\gamma_x}2d}},$$
$$\mathrm{III}: \quad B_{III}(\omega,y) = -i\frac{\gamma_x J}{v_x} e^{-\frac{\omega}{v_x\gamma_x}a} \frac{4\omega s\, e^{\frac{\omega}{v_x\gamma_x}(y+d)}\, e^{-\frac{s}{v_x\gamma_x}d}}{(\omega+s)^2 - (\omega-s)^2 e^{-\frac{s}{v_x\gamma_x}2d}}.$$

The force on each electron $F_M = -e v_z B_y(x-v_x t=0, y=a)$ can now be calculated. Inserting eqs. (9) in eq. (6) yields for the force F_M in the x-direction.

(10)
$$F_M = -e\frac{v_z\gamma_x J}{v_x}\, 2\, \mathrm{Im}\int_0^\infty d\omega\, e^{-\frac{2\omega a}{v_x\gamma_x}} \frac{(\omega^2-s^2)\left(1-e^{-\frac{s}{v_x\gamma_x}2d}\right)}{(\omega+s)^2-(\omega-s)^2 e^{-\frac{s}{v_x\gamma_x}2d}}$$

or substituting $\frac{2\omega a}{v_x\gamma_x} \to \omega$ and introducing the dimensionless quantity $p = 4\pi\sigma v_x\gamma_x a$

$$(11)\qquad F_M = -eJv_x\gamma_x^2\, 2p\, \mathrm{Re}\int_0^\infty d\omega\, e^{-\omega}\, \frac{\omega\left(1-e^{-s\frac{d}{a}}\right)}{(\omega+s)^2-(\omega-s)^2 e^{-s\frac{d}{a}}}$$

$$s = (\omega^2 - 2ip\omega)^{1/2}$$

In terms of ring data the current J is given by $J = \frac{eN_e}{2\pi R} v_z$, where N_e is the number of electrons in the ring, R the major ring radius and v_z the transverse velocity of the electrons. In accordance with the plane approach the velocity v_φ of the ring electron is called the transverse velocity and denoted by the index z. Now, in order to discuss the dependence of the force on the longitudinal velocity v_x one has to know the transverse velocity v_z as a function of v_x. If one assumes an electric acceleration in the x-direction, the transverse momentum is a constant of motion. From this it follows that $v_z = v_z^0/\gamma_x$, where v_z^0 is the transverse velocity at the beginning of acceleration and $\gamma_x = (1-v_x^2)^{-1/2}$ is the longitudinal relativistic factor. Inserting this in eq.(11) one gets for the drag force

$$(12)\qquad F_M = -\frac{m_e r_0 N_e v_z^{0\,2}}{2\pi R a}\, g\left(p, \frac{d}{a}\right)$$

with

$$(13)\qquad g\left(p,\frac{d}{a}\right) = p\, \mathrm{Re}\int_0^\infty d\omega\, e^{-\omega}\, \frac{1}{\omega - ip + (\omega^2-2ip\omega)^{1/2}\, \mathrm{ctgh}\left[(\omega^2-2ip\omega)^{1/2}\frac{d}{2a}\right]}$$

(r_0 = classical electron radius, m_e electron mass and $p = 4\pi\sigma a v_x \gamma_x$.

For the two limiting cases $\frac{d}{a} = \infty$ and $\frac{d}{a}$ small the integral (13) can be evaluated analytically. For $\frac{d}{a} = \infty$ we obtain [2]

$$(14)\qquad g\left(p, \frac{d}{a}=\infty\right) = -\frac{1}{p} + \frac{\pi}{2}\left(J_1(p)\sin p + N_1(p)\cos p\right),$$

where $J_1(p)$ and $N_1(p)$ are the Bessel and Neumann functions resp. For $(1+p^2)^{1/4}\frac{d}{a} \ll 1$ one obtains the very simple result

$$(15)\qquad g\left(p, \frac{d}{a}\right) = \frac{\frac{pd}{2a}}{1+\left(\frac{pd}{2a}\right)^2}.$$

The asymptotic behaviour $p \gg 1$ of $g(p, \frac{d}{a})$ is found to be

$$g(p \gg 1, \frac{d}{a}) \approx \frac{1}{2}\left(\frac{\pi}{d}\right)^{3/2} \tag{16}$$

The function $g(p, \frac{d}{a})$ is plotted as a function of p for different values of d/a in fig.3. For sufficiently small d/a the maximum value of $g(p, \frac{d}{a})$ is doubled relative to the $d = \infty$ case, and the maximum is shifted to higher values of p. The maxima of $g(p, \frac{d}{a})$ are reached for $d = \infty$ at $2\pi\sigma a v_x \gamma_x = 1$ and for small d at $2\pi\sigma d v_x \gamma_x = 1$.

<u>III. The ring charge case</u>

The contribution to the retarding force due to the beam charge can be calculated in an anologous way. The equations for the charge and current densities in the three regions (fig.2) are

$$\begin{aligned} &I: && \rho = 4\pi Q\,\delta(x - v_x t)\,\delta(y-a), && \underline{j} = \{Qv_x, 0, 0\}, \\ &II: && \rho = 0, && \underline{j} = \sigma \underline{E}, \\ &III: && \rho = 0, && \underline{j} = 0, \end{aligned} \tag{17}$$

where Q is the charge line density.

Now the only non vanishing components of the electric and magnetic fields are B_z, E_x, E_y. To find the retarding force, we need the component E_x of the electric field. Inserting eqs. (17) in eqs. (1) and eliminating B_z and E_y yields for the equations (the index x is dropped and replaced by the region label)

$$\begin{aligned} &I: && \Delta E_I - \ddot{E}_I = -4\pi \frac{Q}{\gamma_x^2 v_x}\,\dot{\delta}(x - v_x t)\,\delta(y-a), \\ &II: && \Delta E_{II} - \ddot{E}_{II} - 4\pi\sigma \dot{E}_{II} = 0, \\ &III: && \Delta E_{III} - \ddot{E}_{III} = 0 . \end{aligned} \tag{18}$$

The boundary conditions at the boundaries of the regions I, II and III are the same as in the first case. The tangential components of the electric and magnetic fields have to be continuous. These conditions can be formulated as conditions for $E_x(x,y,t)$ and $\frac{\partial}{\partial y} E_x(x,y,t)$. They are (again dropping the index x)

$$E_I(x,0,t) = E_{II}(x,0,t)\,, \quad E_{II}(x,-d,t) = E_{III}(x,-d,t)\,,$$

(19) and
$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}-4\pi\sigma\frac{\partial}{\partial t}\right)\frac{\partial}{\partial y}E_I(x,0,t) = \left(\frac{\partial}{\partial t}+4\pi\sigma\right)\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)\frac{\partial}{\partial y}E_{II}(x,0,t)$$

$$\frac{\partial}{\partial y}E_{II}(x,-d,t) = \frac{\partial}{\partial y}E_{III}(x, y=-d, t)$$

As in the former case the vanishing of the fields for $|y| \to \infty$ will be required. The equations (18) are identical with the equations (3) if one replaces J by $-\frac{Q}{v_x\gamma_x^2}$. Introducing the Fourier transform (6) of E_x the equations (8) are the solution of the equations (18) if J is replaced by $-\frac{Q}{v_x\gamma_x^2}$. Using the boundary conditions (19), one gets the Fourier transform $E_x(\omega, y)$ in the three regions $(\omega > 0)$

$$\text{I}:\; E_I(\omega,y) = -i\frac{Q}{v_x\gamma_x}e^{-\frac{\omega}{v_x\gamma_x}(a+y)}\frac{(s+\omega+4\pi i\sigma)(s-\omega-4\pi i\sigma)\left(1-e^{-\frac{s}{v_x\gamma_x}2d}\right)}{(s+\omega+4\pi i\sigma)^2-(s-\omega-4\pi i\sigma)^2\,e^{-\frac{s}{v_x\gamma_x}2d}} - i\frac{Q}{v_x\gamma_x}e^{-\frac{\omega}{v_x\gamma_x}|a-y|}\,,$$

(20) $$\text{II}:\; E_{II}(\omega,y) = -i\frac{Q}{v_x\gamma_x}e^{-\frac{\omega}{v_x\gamma_x}a}\,e^{\frac{s}{v_x\gamma_x}y}\,2s\,\frac{(s+\omega+4\pi i\sigma)-(s-\omega-4\pi i\sigma)\,e^{-\frac{s}{v_x\gamma_x}2(y+d)}}{(s+\omega+4\pi i\sigma)^2-(s-\omega-4\pi i\sigma)^2\,e^{-\frac{s}{v_x\gamma_x}2d}}\,,$$

$$\text{III}:\; E_{III}(\omega,y) = -i\frac{Q}{v_x\gamma_x}e^{-\frac{\omega}{v_x\gamma_x}a}\,4\omega\,\frac{(\omega+4\pi i\sigma)\,e^{\frac{\omega}{v_x\gamma_x}(y+d)}\,e^{-\frac{s}{v_x\gamma_x}d}}{(s+\omega+4\pi i\sigma)^2-(s-\omega-4\pi i\sigma)^2\,e^{-\frac{s}{v_x\gamma_x}2d}}\,.$$

with $s = (\omega^2 - 4\pi i\sigma v_x^2\gamma_x^2\omega)^{1/2}$. Inserting eqs. (20) in eq. (6) and substituting $\frac{2\omega a}{v_x\gamma_x} \to \omega$ the force on each ring electron yields for

(21) $$F_e = \frac{eQ}{a}\,\mathrm{Im}\int_0^\infty d\omega\, e^{-\omega}\,\frac{\left(s+\omega+2i\frac{\lambda}{v_x\gamma_x}\right)\left(s-\omega-2i\frac{\lambda}{v_x\gamma_x}\right)\left(1-e^{-s\frac{d}{a}}\right)}{\left(s+\omega+2i\frac{\lambda}{v_x\gamma_x}\right)^2-\left(s-\omega-2i\frac{\lambda}{v_x\gamma_x}\right)^2 e^{-s\frac{d}{a}}}$$

with $s = (\omega^2 - 2i\lambda v_x\gamma_x \omega)^{1/2}$ and $\lambda = 4\pi\sigma a$. Expressing the line density Q in forms of ring data, $Q_e = \frac{e N_e}{2\pi R}$ (N_e number of ring electrons, R major ring radius) the drag force can be written as

$$F_e = - \frac{m_0 r_0 N_e}{2\pi R a} h(\lambda, \tfrac{d}{a}, v_x\gamma_x) \tag{22}$$

with

$$h(\lambda, \tfrac{d}{a}, v_x\gamma_x) = \mathrm{Im} \int_0^\infty d\omega\, e^{-\omega} \frac{\lambda \left[i\omega \left(v_x\gamma_x + \frac{2}{v_x\gamma_x}\right) - \frac{2\lambda}{v_x\gamma_x} \right]}{\omega^2 - i\omega\left(v_x\gamma_x - \frac{2}{v_x\gamma_x}\right)\lambda - \frac{2\lambda^2}{v_x^2\gamma_x^2} + \left(\omega + i\frac{2\lambda}{v_x\gamma_x}\right) s \,\mathrm{ctgh}\left(\frac{s}{2}\frac{d}{a}\right)} \tag{23}$$

In general the function $h(\lambda, \frac{d}{a}, v_x\gamma_x)$ depends on the three variables $\lambda = 4\pi\sigma a$, $\frac{d}{a}$ and $v_x\gamma_x$. But there are regions in which $h(\lambda, \frac{d}{a}, v_x\gamma_x)$ will be approximately a function of only one variable.

Beginning with an infinite thick sheet ($d = \infty$) one finds on the assumption that $d = \infty$, $\lambda \gg \frac{1}{v_x\gamma_x}$, $\lambda \gg v_x\gamma_x$

$$h(\lambda, \tfrac{d}{a} = \infty, v_x\gamma_x) = \int_0^\infty d\omega\, e^{-\omega} \frac{2\sqrt{\kappa\omega}}{\omega - 2\sqrt{\kappa\omega} + 2\kappa}, \quad \kappa = \frac{\lambda}{(v_x\gamma_x)^3} \tag{24a}$$

that $d = \infty$, $\lambda \ll \frac{1}{v_x\gamma_x}$

$$h(\lambda, \tfrac{d}{a} = \infty, v_x\gamma_x) = -\frac{\lambda}{v_x\gamma_x}\left[\cos\frac{\lambda}{v_x\gamma_x}\, \mathrm{Ci}\left(\frac{\lambda}{v_x\gamma_x}\right) + \sin\frac{\lambda}{v_x\gamma_x}\, \mathrm{Si}\left(\frac{\lambda}{v_x\gamma_x}\right)\right] \tag{24b}$$

and that $d = \infty$, $\lambda \ll v_x\gamma_x$

$$h(\lambda, \tfrac{d}{a} = \infty, v_x\gamma_x) = \frac{1}{v_x\gamma_x\lambda} - \frac{\pi}{2}\left[\sin(v_x\gamma_x\lambda) J_1(v_x\gamma_x\lambda) - \cos(v_x\gamma_x\lambda) N_1(\lambda v_x\gamma_x)\right] \tag{24c}$$

where $Ci(\alpha) = -\int_\alpha^\infty \frac{\cos t}{t}\,dt$, $Si(\alpha) = \int_0^\alpha \frac{\sin t}{t}\,dt$ are the sine and cosine integrals [3]. For the $d=\infty$ case $h(\lambda, \frac{d}{a}, v_x\gamma_x)$ is plotted as a function of $v_x\gamma_x$ for different values of λ in fig.4. As can be seen, eq. (24a) describes the curves near the maximum for $\lambda = 10^2$ to $\lambda = 10^8$; the curves depend on $\kappa = \lambda/(v_x\gamma_x)^3$ and have their maximum at $\kappa \approx 0.4$. For very small λ ($\lambda \leq 10^{-2}$) curves have two maxima, at $v_x\gamma_x = \lambda$ and at $v_x\gamma_x = 2/\lambda$.

The curves near the maxima are given by eqs. (24b) and (24c).

For thin sheets there are also simple expressions for the function $h(\lambda, \frac{d}{a}, v_x\gamma_x)$. Assuming $[1+(2\lambda v_x\gamma_x)^2]^{1/4}\frac{d}{a} \ll 1$ one obtains for $|v_x\gamma_x + \frac{2}{v_x\gamma_x}| \ll \frac{2\lambda}{v_x^2\gamma_x^2}$

(25a) $$h(\lambda, \frac{d}{a}, v_x\gamma_x) = \frac{\frac{2a v_x\gamma_x}{d\lambda}}{1+\left(\frac{2a v_x\gamma_x}{d\lambda}\right)^2}$$

and for
$|v_x\gamma_x + \frac{2}{v_x\gamma_x}| \gg \frac{2\lambda}{v_x^2\gamma_x^2}$, $\frac{4a}{d v_x^2\gamma_x^2} \ll 1$, $\frac{2a}{d\lambda v} \cong 1$

(25b) $$h(\lambda, \frac{d}{a}, v_x\gamma_x) = \frac{\frac{2a}{v_x\gamma_x d\lambda}}{1+\left(\frac{2a}{v_x\gamma_x d\lambda}\right)^2}$$

and

$|v_x\gamma_x + \frac{2}{v_x\gamma_x}| \gg \frac{2\lambda}{v_x^2\gamma_x^2}$, $\frac{2a}{d\lambda v_x\gamma_x} \ll 1$

(25c) $$h(\lambda, \frac{d}{a}, v_x\gamma_x) = \pi\, e^{-\frac{4a}{d v_x^2\gamma_x^2}}\,\frac{4a}{d v_x^2\gamma_x^2}.$$

The asymptotic behaviour $v_x\gamma_x \gg 1$ of $h(\lambda, \frac{d}{a}, v_x\gamma_x)$ is found to be

(26) $$h(\lambda, \frac{d}{a}, v_x\gamma_x) = \frac{1}{2}\left(\frac{\pi}{v_x\gamma_x\lambda}\right)^{1/2}$$

In fig.5a-5g the function $h(\lambda, \frac{d}{a}, v_x\gamma_x)$ is plotted as a function of $v_x\gamma_x$ for different values of $\frac{d}{a}$ and λ. The equations (24a-c) and (25a-c) give for most cases a good approximation of the curves near the maxima. To give an example, in fig.5b the curves near the maxima at $v_x\gamma_x = 2\left(\frac{a}{d}\right)^{1/2}$ are approximated by (25c), where $h(\lambda, \frac{d}{a}, v_x\gamma_x)$ is independent of $\lambda = 4\pi\sigma a$ and near the maxima at $v_x\gamma_x = \frac{d}{a}\frac{\lambda}{2}$ by eq. (25b). To present an overall picture of the position of the maxima of h in fig.6 they are plotted as functions of $v_x\gamma_x$, λ and $\frac{d}{a}$.

IV. Application to the Electron Ring Accelerator

Characteristic data of an electron ring deviece for the acceleration of heavy ions are: number of electrons $N_e = 5 \cdot 10^{12}$, major ring radius R = 2.5 cm, distance of the ring from the conducting wall a = 1 cm, the transverse velocity $v_z^0 = c$ and the longitudinal velocity v_x: $0 < v_x\gamma_x < 1$. If the conductivity of the applied wall material is varied between copper ($\sigma = 5.2 \cdot 10^{17} sec^{-1}$) and graphite ($\sigma = 7 \cdot 10^{12} sec^{-1}$) the parameter λ will be in the range of $10^3 < \lambda < 2 \cdot 10^8$. If furthermore, the thickness d of the conducting cylinder is assumed to be $d/a \gg 10^{-3}$ the drag force F_e due to the longitudinal current of the ring electrons is very small and can be neglected. In the above described parameter region a good approximation of F_e is obtained by taking eq. (24a) in the limit $\kappa = \frac{\lambda}{(v_x\gamma_x)^3} \gg 1$

(27) $$F_e = - \frac{m_e r_0 N_e}{2\pi R a} \frac{1}{4} \frac{(v_x\gamma_x)^{3/2}}{(\sigma a)^{1/2}} ,$$

which for the given ring data, the conductivity of copper and $v_x\gamma_x = 1$ yields F_e = 0.05 volt/cm.

The main contribution to the drag force is given by F_M due to the transverse motion of the ring electrons. For a thick cylinder ($d = \infty$) it follows from eqs. (14) and (12) (see fig.3) that F_M reaches its maximum for copper at $v_x\gamma_x = 10^{-8}$ and, inserting the above given ring data, the maximum force is $F_M = 11 \frac{kev}{cm}$.

For the more interesting case of a thin cylinder ($d/a < 1$) the retarding force is given by eqs. (12) and (15). The force depends only on $v_x\gamma_x$ and the product σd. This suggest the introduction of the surface resistivity $\rho = (\sigma d)^{-1}$. Then F_M can be written as

$$(28) \qquad F_M = - \frac{m_e r_0 N_e}{2\pi R a} \frac{\frac{2\rho[\Omega]}{377 v_x \gamma_x}}{1 + \left(\frac{2\rho[\Omega]}{377 v_x \gamma_x}\right)^2} .$$

With the ring data the maximum force is $F_M = 22 \frac{keV}{cm}$ at a velocity $v_x \gamma_x = \frac{\rho[\Omega]}{188.5}$ (v_x in units of the velocity of light). These results for the thin cylinder are in good agreement with the numerical calculation of Herrmann [4], who has treated the thin wall case in cylindrical geometry.

It is thus shown that the drag force F_M on an intense relativistic electron ring due to image currents in conducting walls can reach high values. The characteristic behaviour of the retarding force is, that it first increases with the velocity v_x, then reaches a maximum and finally decreases for higher v_x. It should be pointed out that above the maximum $\left(\frac{\partial F_M}{\partial v_x} < 0\right)$ the contribution F_M to the accelerating forces leads to an unstable situation for the longitudinal motion.

Acknowledgements:

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References

[1] C. Andelfinger et al., 9th Int.Conf.on High Energy Accelerators, May 5-7, 1974, Stanford, Cal./USA

[2] G.V. Voskresensky et al., Symposium on Collective Methods of Acceleration, Dubna (1972)

[3] W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and Theorem for the Special Functions of Mathematical Physics

[4] W. Herrmann, IPP Report 0/25 (1974)

Figure Captions

fig.1 schematic of configuration

fig.2 the linear beam configuration

fig.3 the retarding force $F_M = - \frac{m_e r_0 N_e v_z^{o2}}{2\pi R a} \, g(p, \frac{d}{a})$ acting on the ring current, is plotted as function of $p = 4\pi\sigma a v_x \gamma_x$ and d/a .

fig.4 for an infinitely thick cylinder ($d=\infty$) the retarding force $F_e = - \frac{m_e r_0 N_e}{2\pi R a} \, h(\lambda, \frac{d}{a}, v_x\gamma_x)$ acting on the ring charge, is plotted as function of $\lambda = 4\pi\sigma a$ and $v_x\gamma_x$.

fig.5a-5g the retarding force $F_e = - \frac{m_e r_0 N_e}{2\pi R a} \, h(\lambda, \frac{d}{a}, v_x\gamma_x)$ acting on the ring charge, is plotted as function of $\lambda = 4\pi\sigma a$, $v_x\gamma_x$ and d/a .

fig.6 the position of the maxima of F_e is shown as function of $v_x\gamma_x$, d/a for different values of $\lambda = 4\pi\sigma a$.

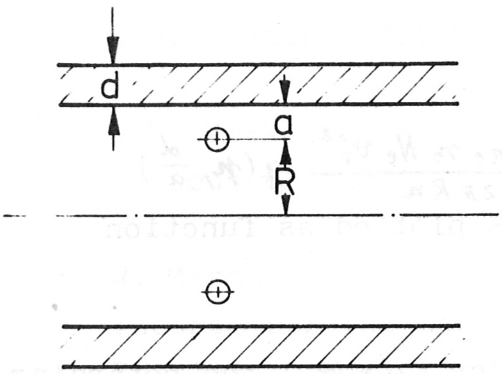


fig. 1

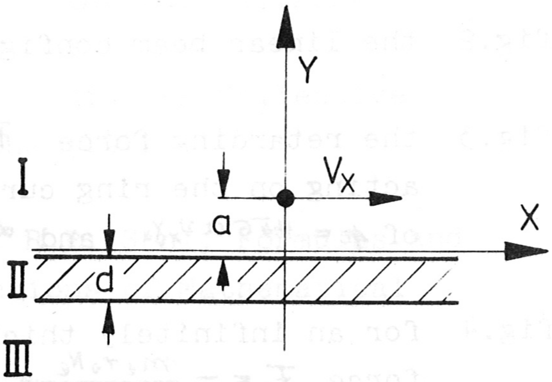


fig. 2

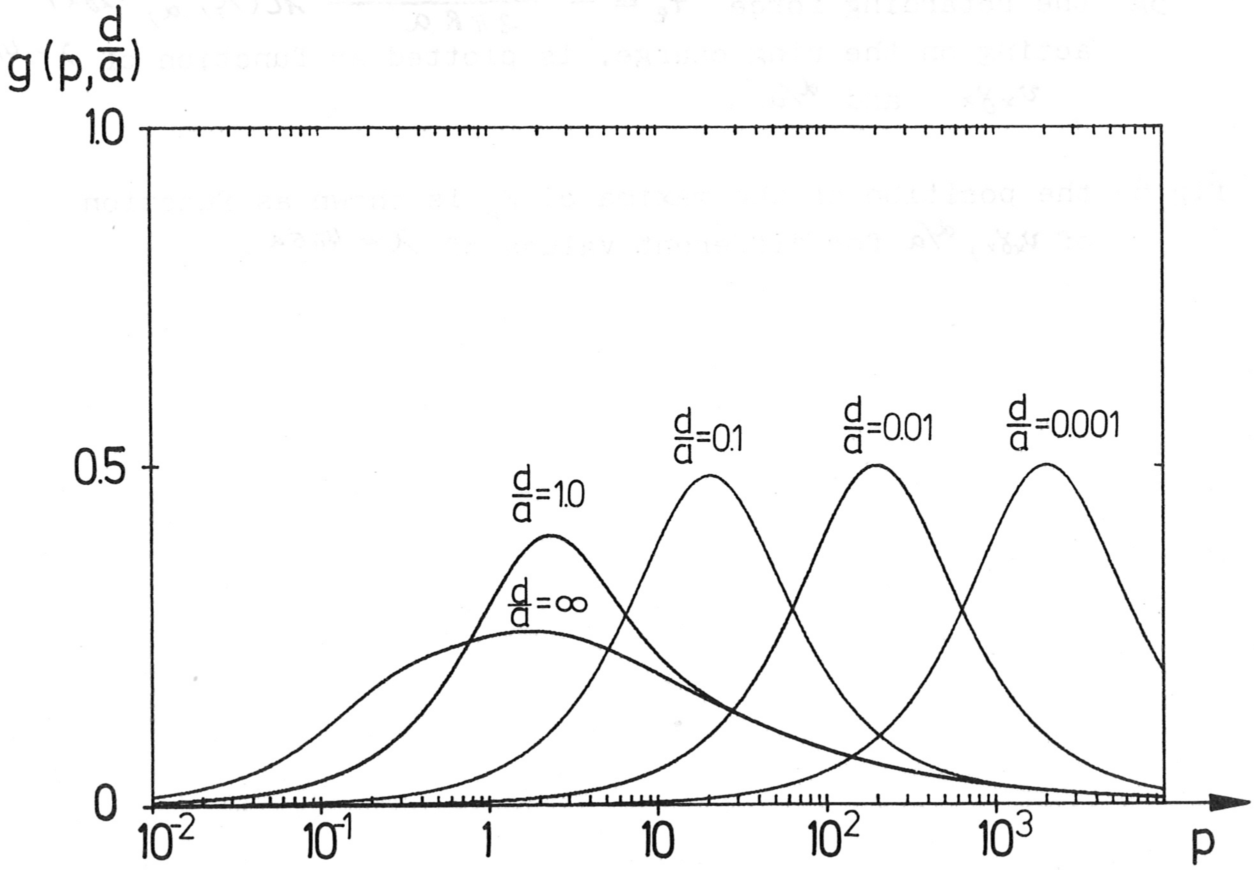


fig. 3

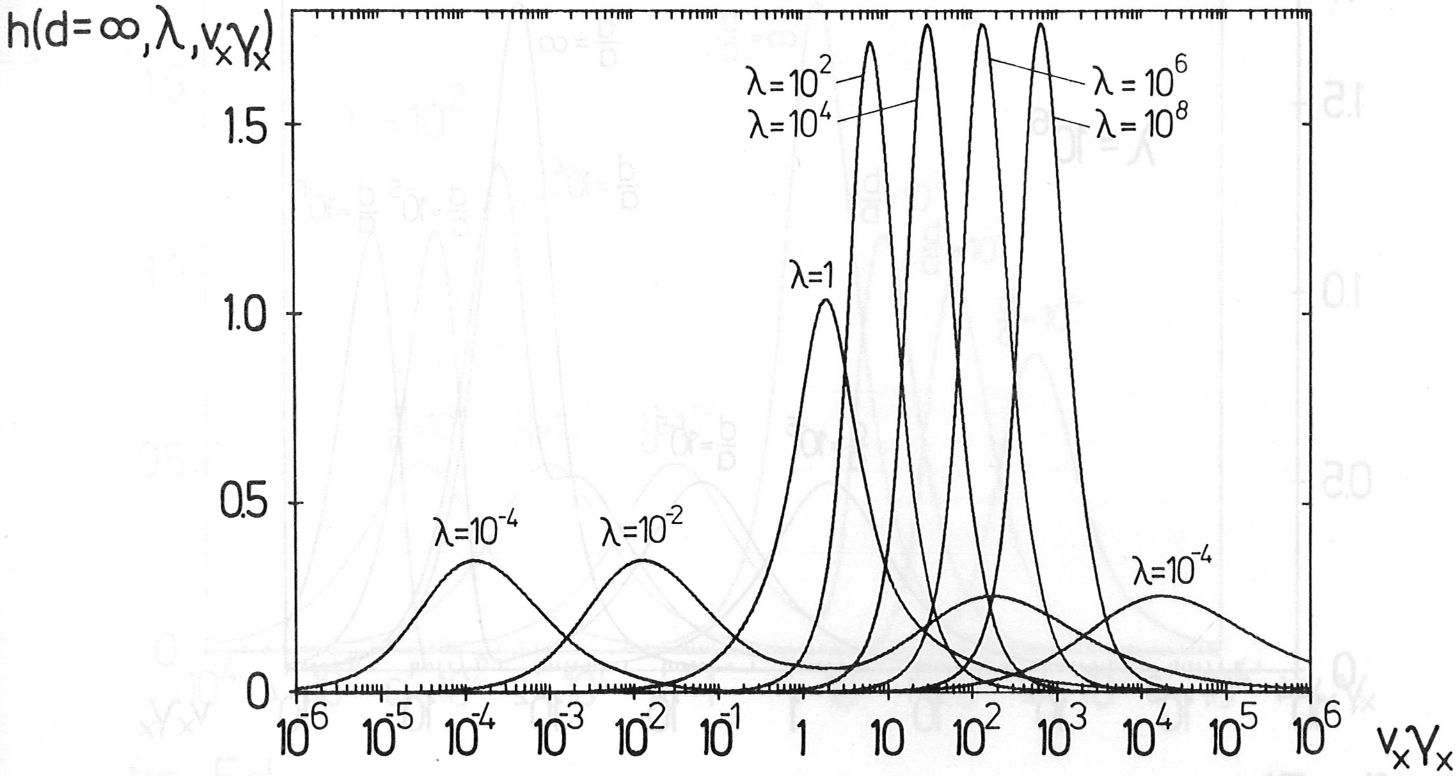

h(d=∞,λ,vxγx)
1.5
1.0
0.5
0
λ=10^2
λ=10^4
λ=10^6
λ=10^8
λ=1
λ=10^-4
λ=10^-2
λ=10^-4
10^-6 10^-5 10^-4 10^-3 10^-2 10^-1 1 10 10^2 10^3 10^4 10^5 10^6
vxγx

fig. 4

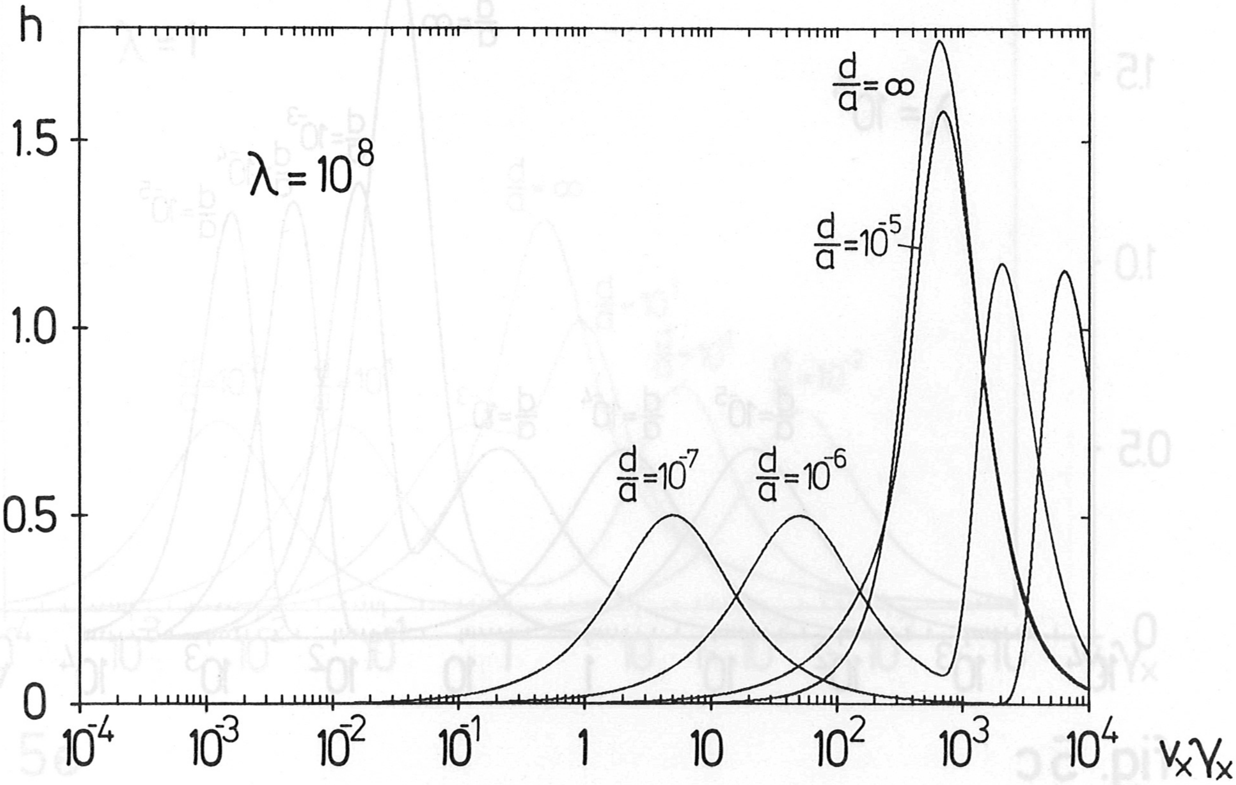

h
1.5
1.0
0.5
0
λ = 10^8
d/a = ∞
d/a = 10^-5
d/a = 10^-7
d/a = 10^-6
10^-4 10^-3 10^-2 10^-1 1 10 10^2 10^3 10^4
vxγx

fig. 5a

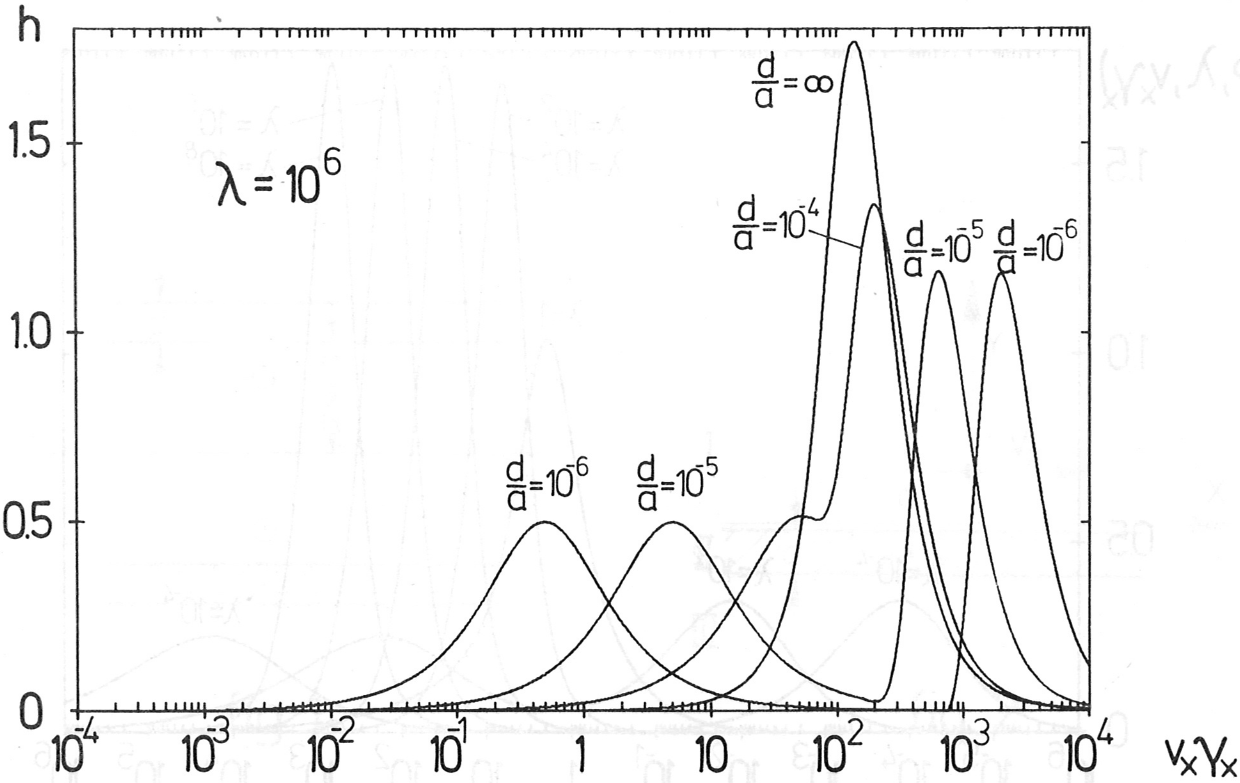

h
λ = 10^6
d/a = ∞
d/a = 10^-4
d/a = 10^-5
d/a = 10^-6
d/a = 10^-6
d/a = 10^-5
1.5
1.0
0.5
0
10^-4
10^-3
10^-2
10^-1
1
10
10^2
10^3
10^4
vₓγₓ

fig. 5b

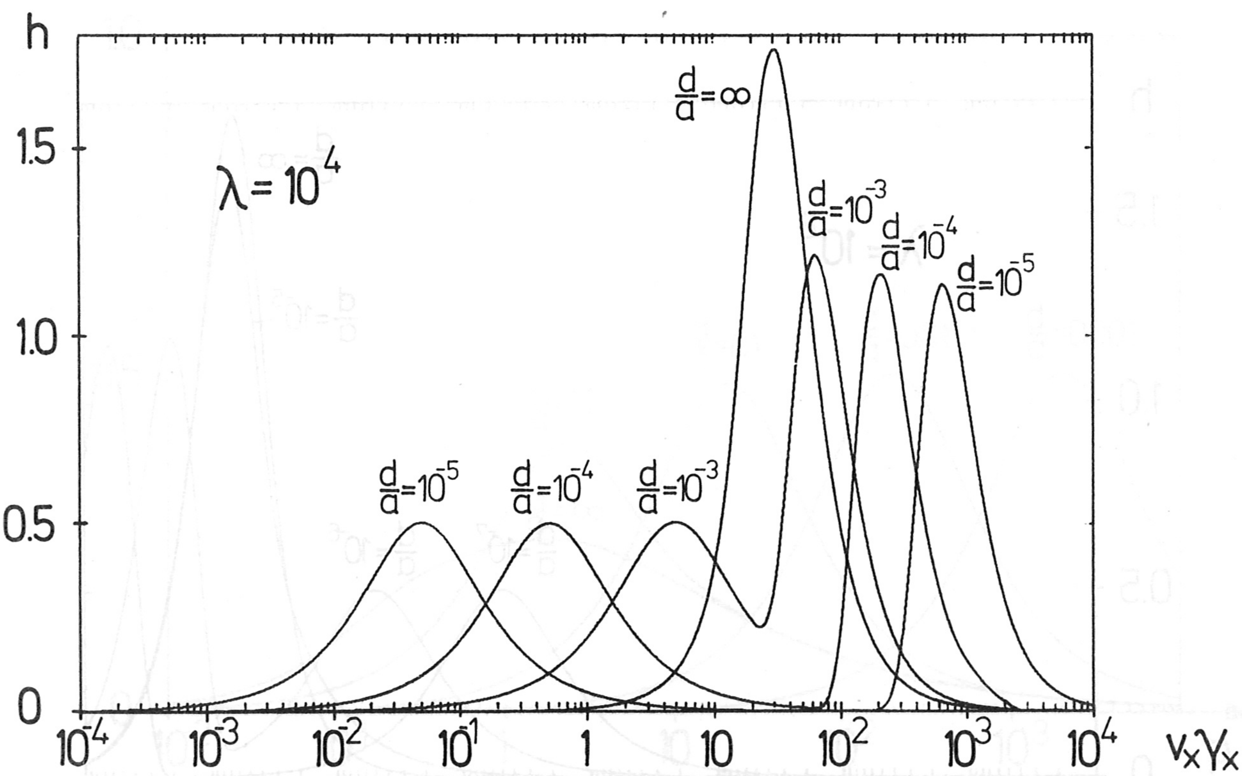

h
λ = 10^4
d/a = ∞
d/a = 10^-3
d/a = 10^-4
d/a = 10^-5
d/a = 10^-5
d/a = 10^-4
d/a = 10^-3
1.5
1.0
0.5
0
10^-4
10^-3
10^-2
10^-1
1
10
10^2
10^3
10^4
vₓγₓ

fig. 5c

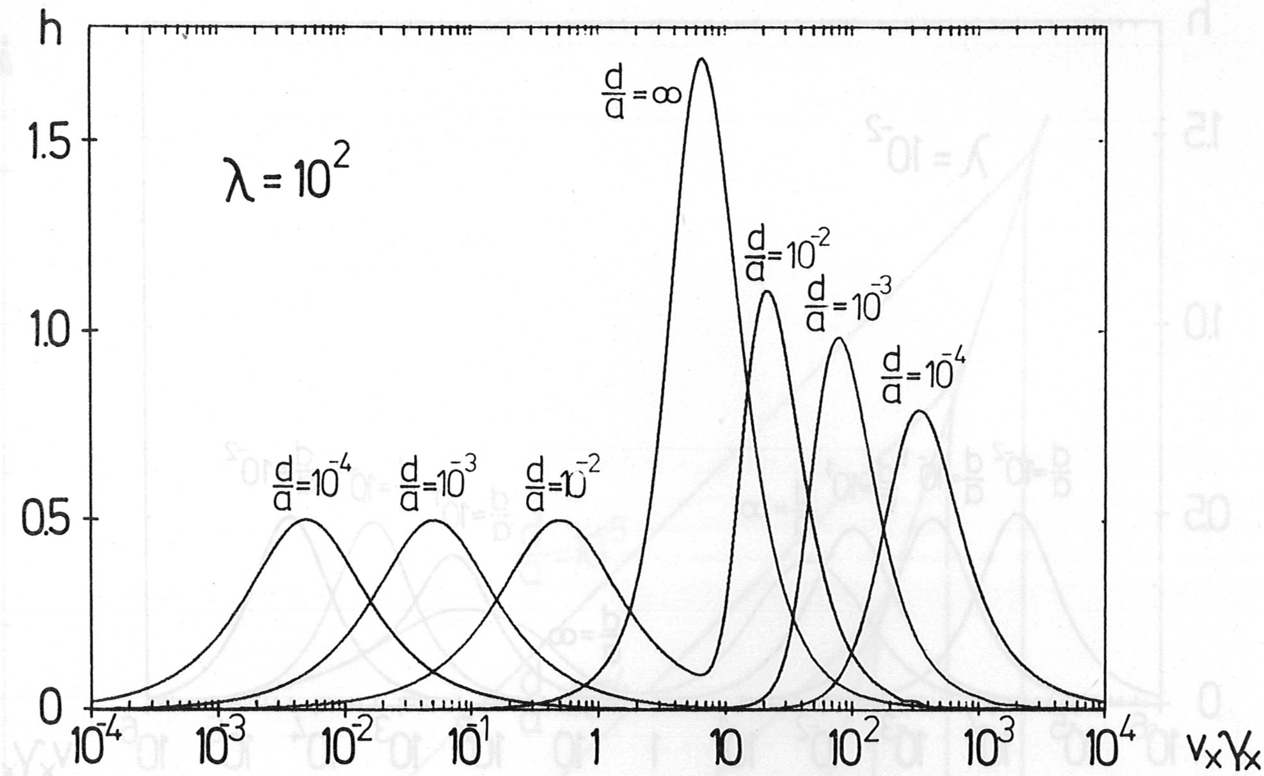

h
λ = 10²
d/a = ∞
d/a = 10⁻²
d/a = 10⁻³
d/a = 10⁻⁴
d/a = 10⁻⁴
d/a = 10⁻³
d/a = 10⁻²
vₓγₓ

fig. 5d

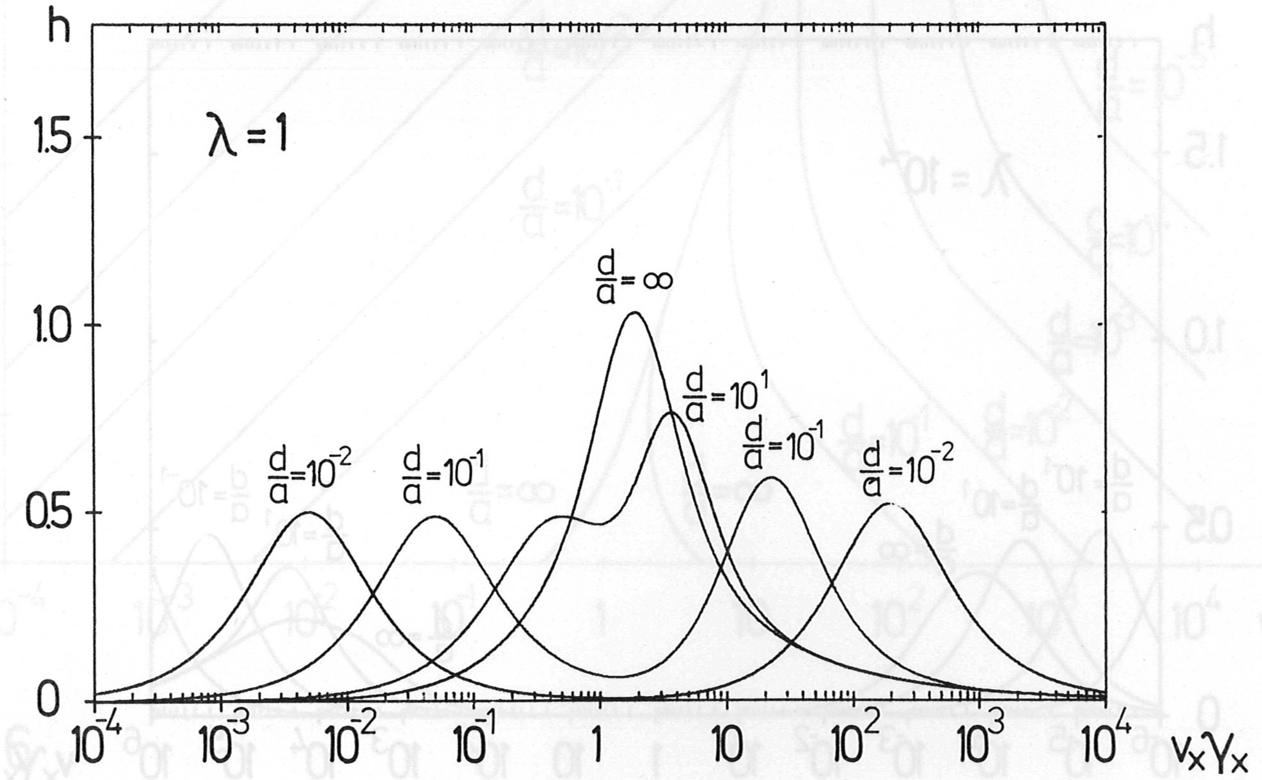

h
λ = 1
d/a = ∞
d/a = 10¹
d/a = 10⁻¹
d/a = 10⁻²
d/a = 10⁻²
d/a = 10⁻¹
vₓγₓ

fig. 5e

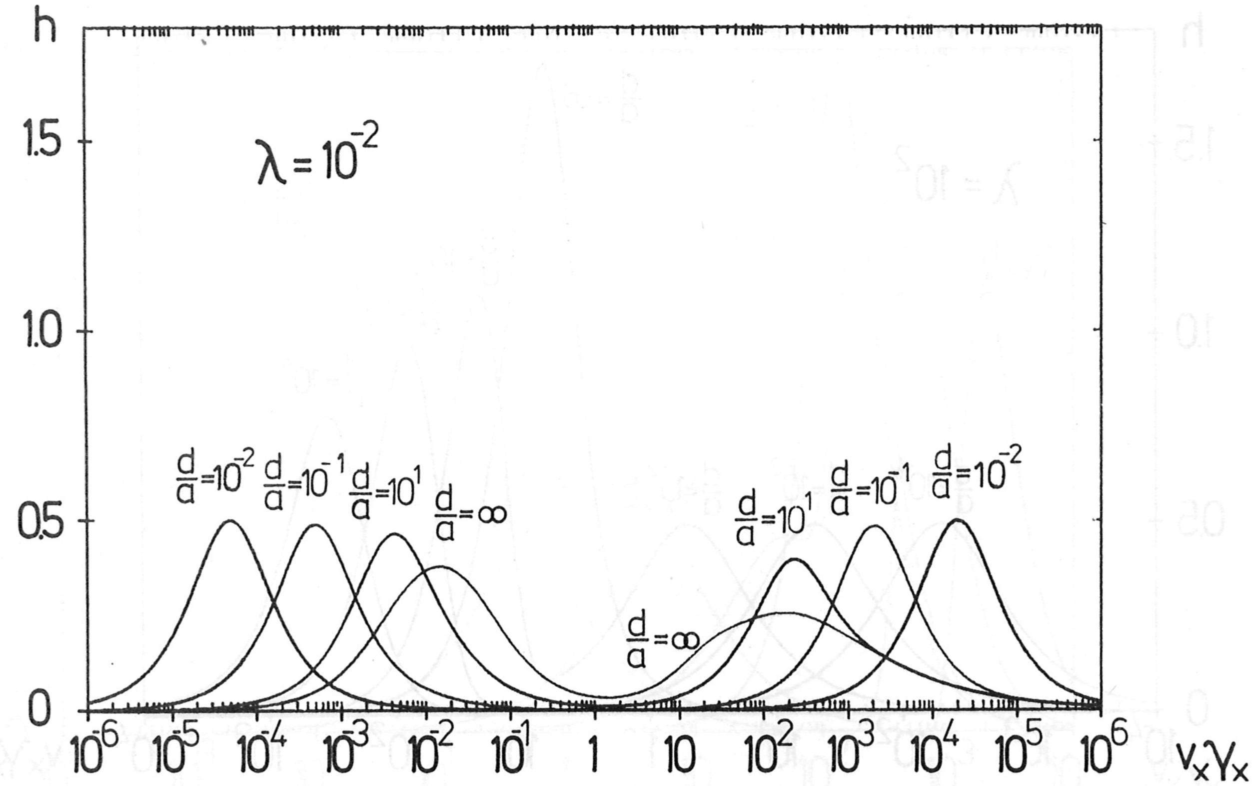

h
1.5
1.0
0.5
0
λ = 10⁻²
d/a = 10⁻²
d/a = 10⁻¹
d/a = 10¹
d/a = ∞
d/a = ∞
d/a = 10¹
d/a = 10⁻¹
d/a = 10⁻²
10⁻⁶ 10⁻⁵ 10⁻⁴ 10⁻³ 10⁻² 10⁻¹ 1 10 10² 10³ 10⁴ 10⁵ 10⁶
vₓγₓ

fig. 5f

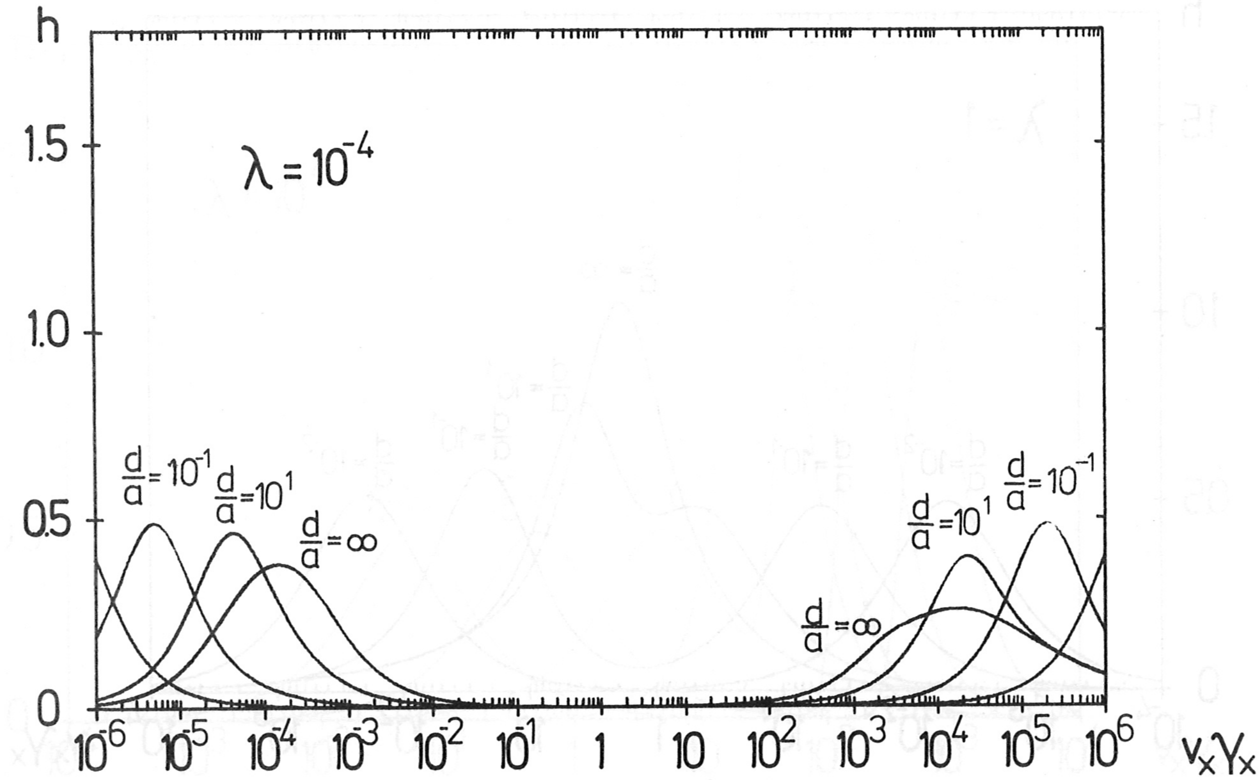
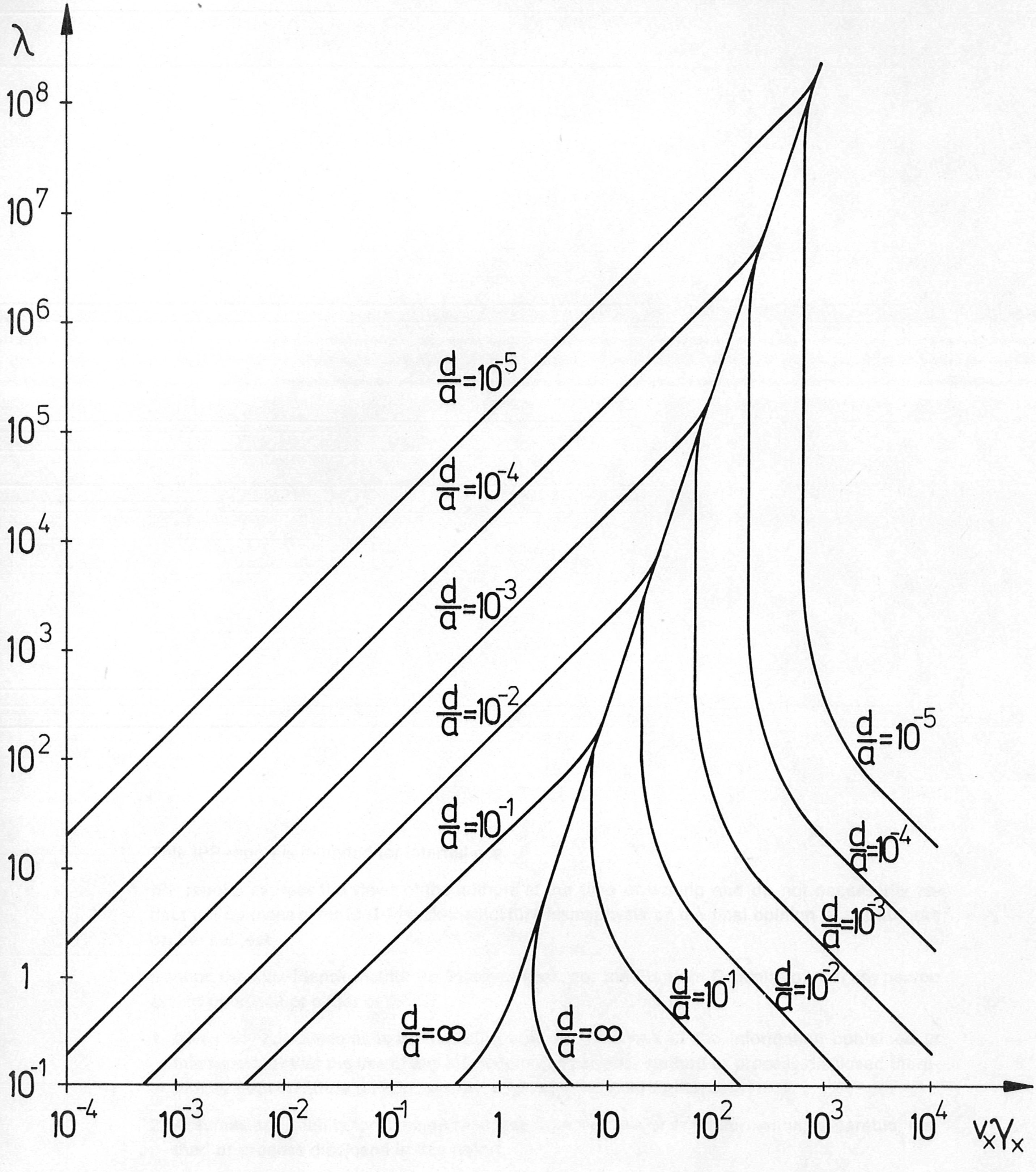

h
1.5
1.0
0.5
0
λ = 10⁻⁴
d/a = 10⁻¹
d/a = 10¹
d/a = ∞
d/a = ∞
d/a = 10¹
d/a = 10⁻¹
10⁻⁶ 10⁻⁵ 10⁻⁴ 10⁻³ 10⁻² 10⁻¹ 1 10 10² 10³ 10⁴ 10⁵ 10⁶
vₓγₓ

fig. 5g



λ
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10
1
10^{-1}
$\frac{d}{a}=10^{-5}$
$\frac{d}{a}=10^{-4}$
$\frac{d}{a}=10^{-3}$
$\frac{d}{a}=10^{-2}$
$\frac{d}{a}=10^{-1}$
$\frac{d}{a}=\infty$
$\frac{d}{a}=\infty$
$\frac{d}{a}=10^{-1}$
$\frac{d}{a}=10^{-2}$
$\frac{d}{a}=10^{-3}$
$\frac{d}{a}=10^{-4}$
$\frac{d}{a}=10^{-5}$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10
10^2
10^3
10^4
$v_x \gamma_x$

fig. 6