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ON THE STABILITY OF HIGH - β ,
HELICALLY SYMMETRIC HYDROMAGNETIC EQUILIBRIA
WITH DECREASING PRESSURE PROFILE

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Abstract

The influence of the geometrical parameters of helically symmetric ideal magnetohydrodynamic equilibria on the plasma β is investigated. For the calculations, it is assumed that the electric current density vanishes on the magnetic axis. Furthermore, only equilibria which satisfy stability criteria in the neighbourhood of the magnetic axis are considered. The plasma β , here defined as β on the magnetic axis, is estimated assuming that the plasma radius is much smaller than the helical wavelength and that the pressure vanishes on the separatrix. Thus, β depends on the geometrical parameters of the equilibrium and on the steepness of the pressure profile, which is limited by the stability criteria. Configurations having elliptical cross-sections with their major axis along the binormal to the magnetic axis and being slightly D-shaped appear to be favourable for stability and high β values.

I. Introduction

It is the purpose of this report to investigate the influence which the geometrical parameters of helically symmetric ideal MHD equilibria have on the plasma β , as estimated under several restricting assumptions specified below.

The investigation of helically symmetric equilibria is of practical interest for configurations of the stellarator type, provided that the helical period of the plasma column is much smaller than the circumference of the torus. In this case, neglecting all toroidal effects, the relevant physical quantities depend only on the coordinates r and $u = \varphi + kz$ (r, φ , and z are the usual cylindrical coordinates, k is the helical wave-number). The equilibrium problem of ideal magnetohydrodynamics can be then reduced to finding the solutions of the equation (1/1, 1/2).

$$\mathcal{L} F = 2 k q f(F) - \frac{df^2/2}{dF} - \frac{\mu_0}{q} \frac{dp(F)}{dF} \quad 1.1$$

$$\mathcal{L} F \equiv \frac{1}{qr} \frac{\partial}{\partial r} q r \frac{\partial}{\partial r} + \frac{1}{qr^2} \frac{\partial^2}{\partial u^2}, \quad q = (1+k^2 r^2)^{-1}. \quad 1.2$$

Here, F is the sum of longitudinal and azimuthal magnetic flux per length, $(F(o) - f(F)) / \mu_0$ the corresponding electric current per length, $p(f)$ the plasma pressure and μ_0 the permeability constant of vacuum.

Given the functional dependence of p and f on F , it is in some cases possible to solve equation 1.1 exactly (1/1, 1/2). In the following,

however, the calculations will for simplicity be limited to the neighbourhood of the magnetic axis.

In /2/, Mercier's necessary criterion /3/ and a sufficient criterion derived in /4/ and /5/ were evaluated on the magnetic axis of helically symmetric equilibrium configurations. The results of those calculations are used here to study, in more detail, the dependence of the plasma β on the geometrical parameters of the equilibrium and on the steepness of the pressure profile. The validity of the results is, of course, limited by the fact that the calculations are restricted to the neighbourhood of the magnetic axis, where the second- and higher-order terms in the expansion of $p(F)$ and $f(F)$ with respect to F are considered to be negligible.

II. Helically Symmetric Equilibria

In a coordinate system (σ, θ, ℓ) associated with the magnetic axis (Fig. II.1) the equilibrium of helically symmetric

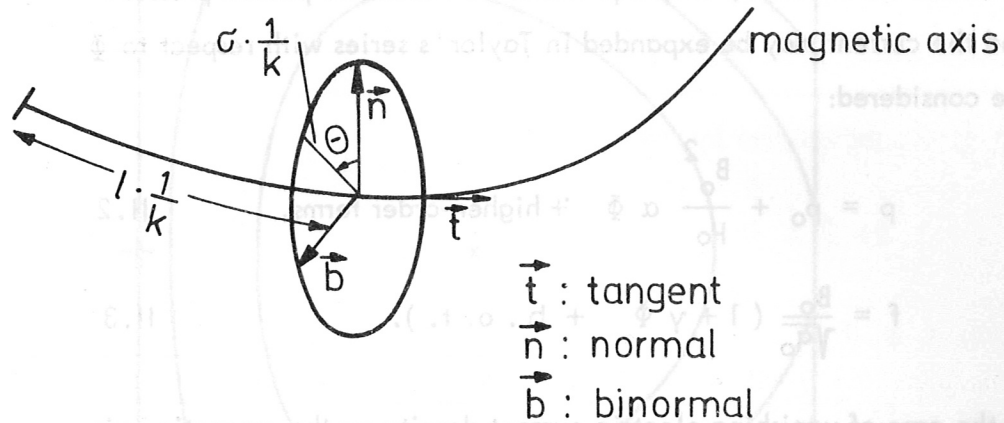


Fig. II.1

configurations is described by the equation

$$\frac{D}{q} \left(\frac{\partial}{\partial \sigma} \frac{q}{D} g^{\sigma\sigma} \frac{\partial F}{\partial \sigma} \right) + \frac{\partial}{\partial \theta} \frac{q}{D} g^{\theta\theta} \frac{\partial F}{\partial \theta} =$$

$$= 2kqf - f \frac{df}{dF} - \frac{\mu_0}{q} \frac{dp}{dF}, \quad \text{II.1}$$

$$D = \frac{k^3}{\sigma(1 - q_0 \rho_0 \sigma \cos \theta)}, \quad q = \frac{q_0 \sigma^2 D^2}{k^6 + \sigma^4 D^2 q_0^2},$$

$$g^{\sigma\sigma} = k^2, \quad g^{\theta\theta} = \frac{q_0 D^2}{k^4 q}.$$

Here, $\rho_0 = r_0 \cdot k$, r_0 being the radius of the cylinder on which the magnetic axis is situated. $q_0 = (1 + \rho_0^2)^{-1}$. The meaning of k , F , $f(F)$, and $p(F)$ is

explained above.

Introducing a dimensionless fluxfunction Φ by means of the relation $\Phi = (k \sqrt{q_0} F) / B_0$, where B_0 is the magnetic field strength on the magnetic axis, the cross-sections of the magnetic surfaces with the planes $\ell = \text{const.}$ (which are perpendicular to the magnetic axis) are described by the equation $\Phi(\sigma, \theta) = \text{const.}$ In the following, $\sigma \ll 1$ is assumed. Furthermore, only equilibria for which the plasma pressure and the current may be expanded in Taylor's series with respect to Φ are considered:

$$p = p_0 + \frac{B_0^2}{\mu_0} \alpha \Phi + \text{higher-order terms}, \quad 11.2$$

$$f = \frac{B_0}{\sqrt{q_0}} (1 + \gamma \Phi + \text{h. o. t.}). \quad 11.3$$

In the case of vanishing electric current density on the magnetic axis and symmetry of the cross-sections with respect to the osculating plane (corresponding to symmetry with respect to the midplane in axial symmetry) the solution of equations 11.1 - 3 (correct to third order in σ) is given by

$$\Phi(\sigma, \theta) = \frac{q_0}{1 + \epsilon^2} (\epsilon^2 \cos^2 \theta + \sin^2 \theta) \sigma^2 + (\lambda_3 \cos^3 \theta + \lambda_4 \cos \theta \sin^2 \theta) \sigma^3, \quad 11.4$$

$$\lambda_4 = -3\lambda_3 + q_0^2 \rho_0 \frac{2 + \epsilon^2}{1 + \epsilon^2} + \alpha q_0 \rho_0, \quad 11.5$$

$$\alpha + \gamma = 0. \quad 11.6$$

The parameter α describes the steepness of the pressure profile. The relation $\alpha + \gamma = 0$ ensures that the current density vanishes on the magnetic axis (this allows for a closer analogy to stellarators).

ϵ is the ratio of the semiaxes of the plasma cross-sections, which are elliptical to second order in σ . Taking into account the influence of the third-order terms in 11.4, the cross-sections are slightly triangular and shifted relative to the magnetic axis, the triangularity being described by λ_3 and λ_4 and the shift depending only on λ_3 (s. Fig. 11.2).

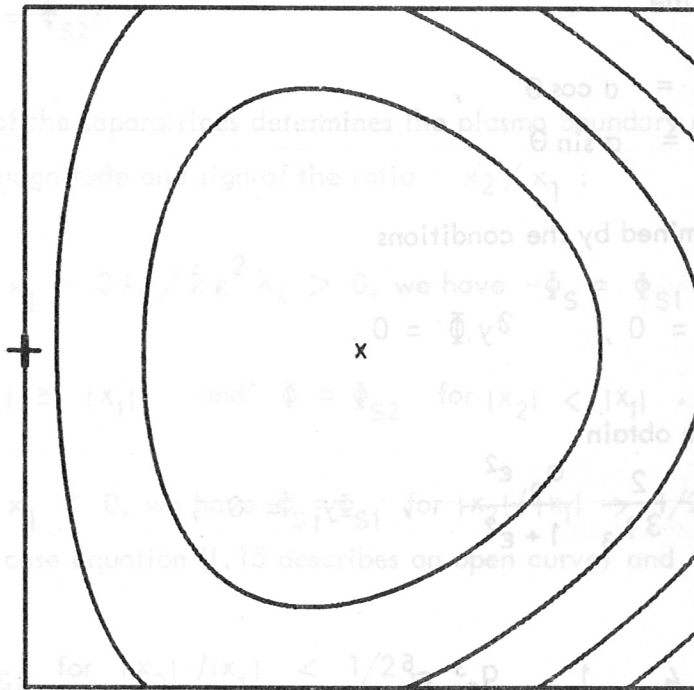


Fig. 11.2

Intersection lines of a plane perpendicular to the magnetic axis with the magnetic surfaces of a helical equilibrium.

x : magnetic axis. $+$: center of the cylinder on which the magnetic axis is situated. \vec{n} , \vec{b} : normal and binormal to the magnetic axis, respectively. Here, $\rho_0 = 0.15$, $\epsilon = 1.1$, $\lambda_3 = 0.4$, $\lambda_4 = -2.0$.

Considering only configurations for which the higher-order terms in the expansion 11.2 can be neglected and assuming that the plasma pressure vanishes on the separatrix (in this case we have $p_0 = -(B_0^2 \alpha \Phi_{sep}) / \mu_0$, the plasma β (on the magnetic axis) is given by

$$\beta_0 = \frac{P_0}{P_0 + B_0^2 / 2\mu_0} = \frac{2 |\alpha| \Phi_{sep.}}{1 + 2 |\alpha| \Phi_{sep.}} \quad \text{II.7}$$

In order to estimate β_0 , it is necessary to know the position of the separatrix. This can be obtained from the coordinates of the singular points which, setting

$$\begin{aligned} x &= \sigma \cos \theta, \\ y &= \sigma \sin \theta \end{aligned} \quad \text{II.8}$$

in II.4, are determined by the conditions

$$\partial x \Phi = 0, \quad \partial y \Phi = 0. \quad \text{II.9}$$

From II.4, 8, 9 we obtain

$$x_1 = -\frac{2}{3\lambda_3} \frac{q_0 \epsilon^2}{1 + \epsilon^2}, \quad y_1 = 0, \quad \text{II.10}$$

corresponding to

$$\Phi_{S1} = \frac{4}{27} \cdot \frac{1}{\lambda_3^2} \cdot \frac{q_0^3 \epsilon^6}{(1 + \epsilon^2)^3} \quad \text{II.11}$$

and

$$x_2 = \frac{q_0}{(1 + \epsilon^2) \lambda_4(\rho_0, \epsilon, \lambda_3, \alpha)}, \quad y_2^2 = \frac{q_0^2}{(1 + \epsilon^2)^2 \lambda_4^2(\rho_0, \epsilon, \lambda_3, \alpha)} \quad \text{II.12}$$

corresponding to

$$\Phi_{S2} = \frac{q_0^3 \epsilon^2}{(1 + \epsilon^2)^3 \lambda_4^2(\rho_0, \epsilon, \lambda_3, \alpha)} \left(1 - \frac{\lambda_3}{\epsilon^2 \lambda_4(\rho_0, \epsilon, \lambda_3, \alpha)} \right) \quad \text{II.13}$$

the equations of the separatrices are then

$$y^2 = \frac{1}{3} \epsilon^2 x_1^2 \left(1 - \frac{x}{x_1} \right) \frac{\left(1 + \frac{2x}{x_1} \right)}{\left(1 - \frac{x}{x_2} \right)} \quad \text{II.14}$$

(for $\Phi = \Phi_{S1}$)

and

$$y^2 = \varepsilon^2 x_2^2 \left[\left(1 + \frac{x}{x_2}\right) \left(1 - \frac{2}{3} \frac{x_2}{x_1}\right) - \frac{2}{3} \frac{x^2}{x_1 x_2} \right], \quad \text{II.15}$$

$$x = x_2 \quad \text{II.16}$$

(for $\Phi = \Phi_{S2}$).

Which of the separatrices determines the plasma boundary depends only on the magnitude and sign of the ratio x_2/x_1 :

If $x_2/x_1 = 3\lambda_3/2\varepsilon^2\lambda_4 > 0$, we have $\Phi_S = \Phi_{S1}$

for $|x_2| \geq |x_1|$ and $\Phi = \Phi_{S2}$ for $|x_2| < |x_1|$.

If $x_2/x_1 < 0$, we have $\Phi_S = \Phi_{S1}$ for $|x_2|/|x_1| > 1/2$
(in this case equation II.15 describes an open curve) and

$\Phi = \Phi_{S2}$ for $|x_2|/|x_1| < 1/2$.

If $|x_2|/|x_1| = 1/2$, the two separatrices coincide.

The dependence of the plasma β on the equilibrium parameters has thus the form

$$\beta_0 = \frac{2|\alpha| \Phi(\rho_0, \varepsilon, \lambda_3, \alpha)}{1 + 2|\alpha| \Phi_S(\rho_0, \varepsilon, \lambda_3, \alpha)} \quad \text{II.17}$$

For a given set of values ρ_0 , ε , and λ_3 , the range of values which α can assume is limited by the stability criteria (see next paragraph).

The investigations carried out in this report can be described as a numerical study of the influence which the parameters ρ_0 , ε , λ_3 , and the stability criteria have on the plasma β , as given by equation II.17.

III. Stability Criteria

In /4/ and /5/, a sufficient criterion for the stability of periodic equilibria was derived. Applied to helical equilibria and evaluated on the magnetic axis, this criterion reads

$$q_0 \rho_0^2 + q_0 \frac{(1-\epsilon^2)^2}{(1+\epsilon^2)^2} + \frac{3\rho_0}{q_0} \frac{1-\epsilon^2}{\epsilon^2} \lambda_3 -$$

$$- \alpha \rho_0^2 \left[\frac{(1-\epsilon)(2+\epsilon)}{\epsilon(1+\epsilon)} + \frac{(1+\epsilon^2)^3}{2^6 \epsilon^7} (5\epsilon^4 - 4\epsilon^2 + 7) \right] \leq 0 \quad \text{III.1}$$

while Mercier's necessary criterion /3/ reduces to

$$q_0 \rho_0^2 + q_0 \frac{(1-\epsilon^2)^2}{(1+\epsilon^2)^2} + \frac{3\rho_0}{q_0} \frac{1-\epsilon^2}{\epsilon^2} \lambda_3 -$$

$$- \alpha \rho_0^2 \frac{(1-\epsilon)(2+\epsilon)}{\epsilon(1+\epsilon)} \leq 0 \quad \text{III.2}$$

(ref. /2/).

The corresponding critical α -values are thus

$$\alpha_{cr. (1)} = \frac{1}{\rho_0^2} \frac{\left[q_0 \rho_0^2 + q_0 \frac{(1-\epsilon^2)^2}{(1+\epsilon^2)^2} + \frac{3\rho_0}{q_0} \frac{(1-\epsilon^2)}{\epsilon^2} \lambda_3 \right]}{\frac{(1-\epsilon)(2+\epsilon)}{\epsilon(1+\epsilon)} + \frac{(1+\epsilon^2)^3}{2^6 \epsilon^7} (5\epsilon^4 - 4\epsilon^2 + 7)} \quad \text{III.3}$$

and

$$\alpha_{cr. (2)} = \frac{\epsilon(1+\epsilon)}{\rho_0^2(1-\epsilon)(2+\epsilon)} \left[q_0 \rho_0^2 + q_0 \frac{(1-\epsilon^2)^2}{(1+\epsilon^2)^2} + \frac{3\rho_0}{q_0} \frac{(1-\epsilon^2)}{\epsilon^2} \lambda_3 \right], \quad \text{III.4}$$

respectively.

The pressure term in III.1 ($\sim \alpha$) can be shown to be always destabilizing,

the sufficient condition for stability thus being

$$0 \geq \alpha \geq \alpha_{cr.}(1) \quad \text{III.5}$$

In III.2, however, the effect of the pressure depends on the ellipticity of the cross-sections and the necessary criterion can be written as

$$0 \geq \alpha \geq \alpha_{cr.}(2) \quad \text{for } \epsilon < 1 \quad \text{III.6}$$

and

$$\left. \begin{array}{l} \alpha \leq \alpha_{cr.}(2) \quad (\text{if } \alpha_{cr.}(2) \leq 0) \\ \alpha \leq 0 \quad (\text{if } \alpha_{cr.}(2) \geq 0) \end{array} \right\} \text{for } \epsilon > 1 \quad \text{III.7}$$

Thus, if $\epsilon > 1$, III.7 allows for an arbitrarily large $|\alpha|$ and, in our approximation, the corresponding plasma β is only limited by the position of the separatrix.

It is obvious that for $\rho_0 = 0$ and $\epsilon = 1$ (theta-pinch) the criteria III.1,2 are marginally satisfied for any α . Helically symmetric equilibria which differ but little ($\rho_0 \ll 1, |\epsilon - 1| \sim \rho_0, \lambda_3 \sim \rho_0$) from the theta-pinch configuration are, however, unstable. As shown in /8/, these equilibria do satisfy Mercier's criterion in the plasma regions where the pressure gradient is sufficiently large. If ρ_0 and $|\epsilon - 1|$ are not too small, and if λ_3 and λ_4 do not vanish simultaneously, Mercier's criterion can be satisfied on the magnetic axis. Taking into account the results obtained in /8/, one might expect that these configurations satisfy Mercier's criterion also on magnetic surfaces which are not in the proximity of the magnetic axis.

Clearly, in the general case ($\rho_0 \neq 0, \epsilon \neq 1$), the sufficient criterion III.1

can be satisfied only by shifting the elliptical cross-sections along the normal to the magnetic axis, thereby making them slightly triangular (λ_3 and $\lambda_4 \neq 0$). The sign of the shift depends on the ellipticity of the cross-sections: an outward shift ($\lambda_3 > 0$) is favourable for $\epsilon > 1$ (major axis along the binormal) and an inward shift ($\lambda_3 < 0$) for $\epsilon < 1$ (major axis along the normal). However, if a large triangularity is required for stability, the separatrix will approach the magnetic axis, thus reducing the plasma β . If we set $\lambda_3 = \lambda_4 = 0$ (no triangularity), α is determined by equation II.5 as a function of φ_0 and ϵ . Setting the resulting expression for α in III.2 one can easily verify that Mercier's criterion is violated.

If $\lambda_4 = 0$ but $\lambda_3 \neq 0$, the necessary criterion can be satisfied only if $\epsilon < 1$, i.e., if the major axis of the elliptical cross-sections and the normal to the magnetic axis are parallel to each other. If $\lambda_4 \neq 0$ but $\lambda_3 = 0$ (triangularity but not shift), Mercier's criterion can be satisfied only if $\epsilon > 1$.

In order to satisfy the sufficient criterion III.1 both, λ_4 and λ_3 , must be different from 0.

If $\epsilon < 1$, the sufficient and the necessary criterion can be satisfied only by shifting the elliptical cross-sections toward the axis of symmetry ($\lambda_3 < 0$).

If $\epsilon = 1$, the necessary criterion is violated (except when $\varphi_0 = 0$, in which case both criteria are marginally satisfied).

If $\epsilon > 1$, it is possible to satisfy the sufficient criterion III.1 only if $\lambda_3 > 0$. Mercier's criterion III.2, however, can be satisfied for any λ_3 (in this case condition III.2 imposes a restriction only on α). Thus, if the major semiaxis of the elliptical cross-section is parallel to the binormal of the magnetic axis, Mercier's criterion allows for a shift of the cross-sections in

either direction of the normal or opposite to it.

For different values of ρ_0 and λ_3 , fig. 1 - 15 show the dependence of the plasma β on the ratio ε of the semi-axes of the elliptical cross-sections. The curves show the highest β -values which are obtained from II.17, taking into account the restrictions imposed by III.1 (curves labeled with L) and III.2 (curves labeled with M). As observed before, in order to satisfy the sufficient criterion it is necessary to choose $\lambda_3 > 0$ if $\varepsilon > 1$ and $\lambda_3 < 0$ if $\varepsilon < 1$. Mercier's criterion, on the contrary, allows for $\varepsilon > 1$ and $\lambda_3 < 0$.

In our approximation, high β -values are obtained with small values of ρ_0 and $|\varepsilon - 1|$. In this case $|\alpha| \sim 1/\rho_0$ follows from the stability criteria. Thus, either λ_3 or λ_4 , or both, must be finite (see eq. II.5). Therefore, these equilibria may not be considered as small helical deviations from the theta-pinch.

Conclusion

Configurations which have a nearly straight magnetic axis and almost circular cross-sections in the neighbourhood of the magnetic axis, are unstable, i.e., small helically symmetric deviations from the theta-pinch do not satisfy Mercier's criterion, evaluated on the magnetic axis.

If the deviations from the theta-pinch are finite, but still quite small, it is possible to satisfy both sufficient and necessary criteria in the neighbourhood of the magnetic axis by introducing a certain triangularity and a shift of the cross-sections along the normal to the magnetic axis, the sign of the shift depending on the ellipticity of the cross-sections.

D-shaped plasma cross-sections appear to be favourable for stability and high β -values.

Acknowledgements

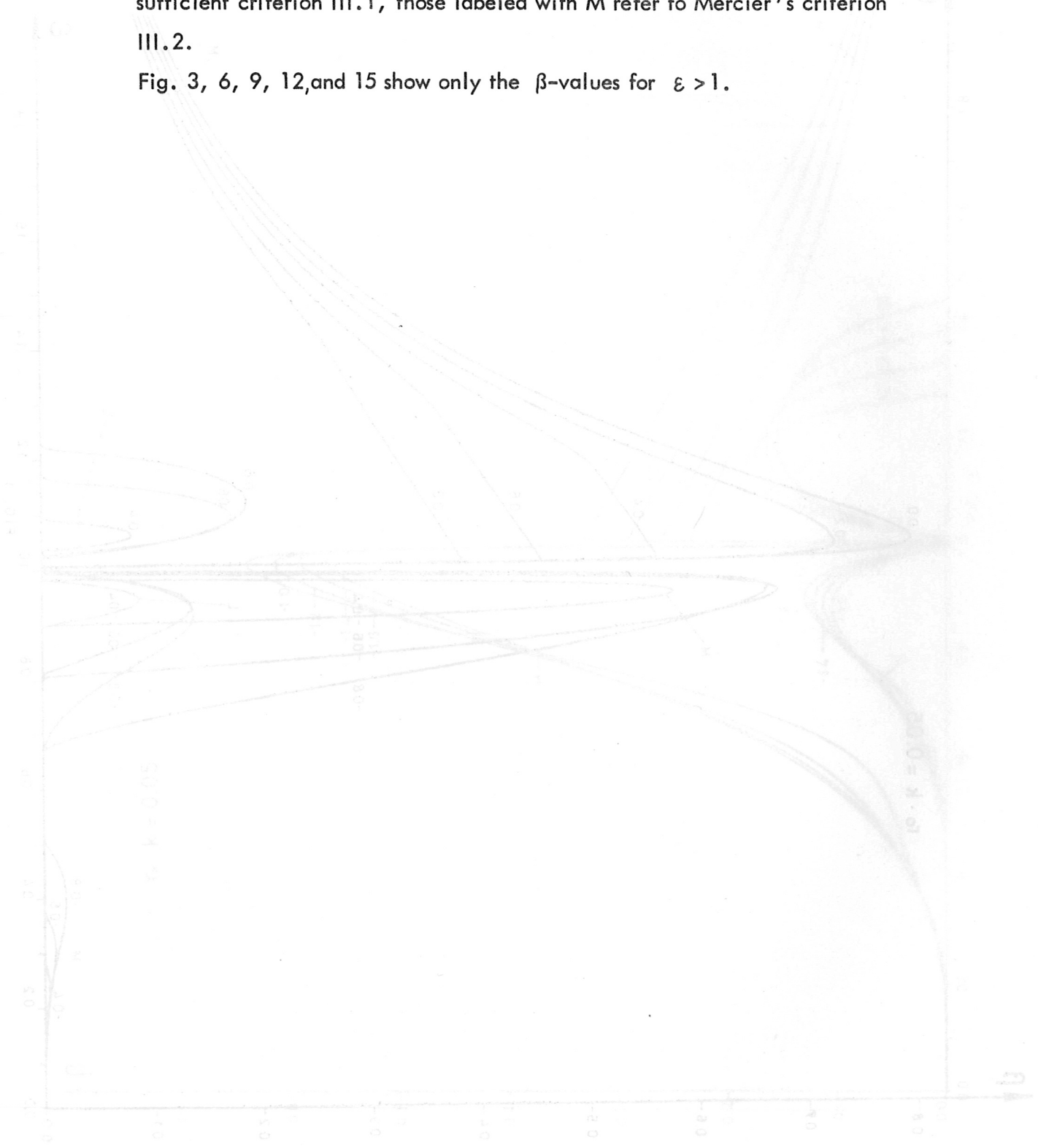
The author would like to thank Dr. J. Nührenberg for many helpful discussions and suggestions.

Frequent dialogues with Dr. M. Kaufmann were also of great assistance.

Caption for Fig. 1 - 15

The figures 1 - 15 show the dependence of the plasma β on the ellipticity ϵ of the plasma cross-sections for different values of $\rho_0 = r_0 \cdot k$ and λ_3 . The values of ρ_0 are written on the upper left part of the figures, those of λ_3 near the corresponding curve. Curves labeled with L refer to the sufficient criterion III.1, those labeled with M refer to Mercier's criterion III.2.

Fig. 3, 6, 9, 12, and 15 show only the β -values for $\epsilon > 1$.



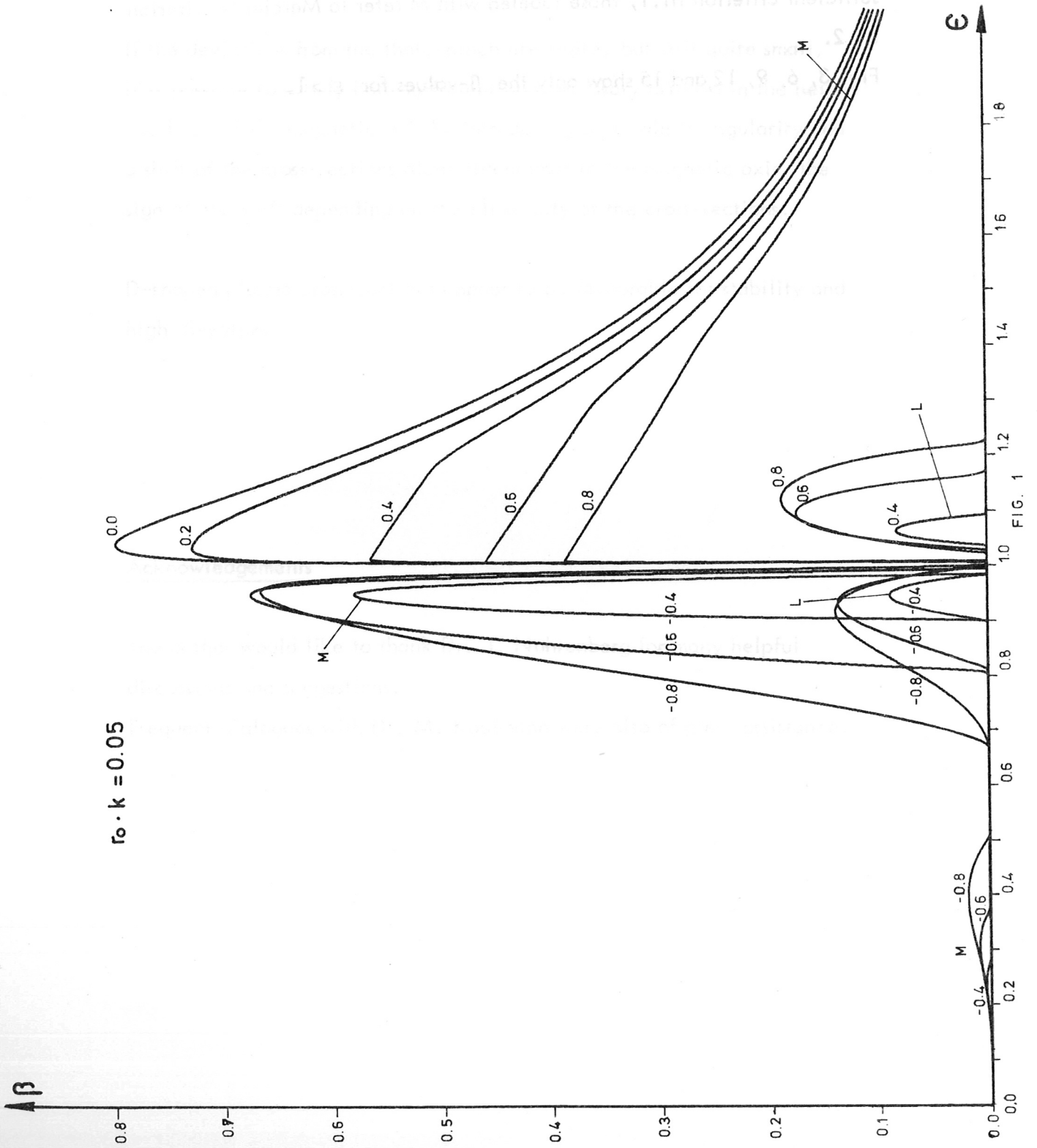


FIG. 1

The figures 1 - 13 show the dependence of the plasma β on the ellipticity ϵ of the plasma cross-section for different values of $\Gamma_0 \cdot k$ and Γ_0 . The values of Γ_0 are written on the upper part of the figures. The curves near the corresponding curves. Curves labeled with 'L' refer to the sufficient criterion II.1, those labeled with 'M' refer to the criterion II.2.

Caption for Fig. 1 - 13

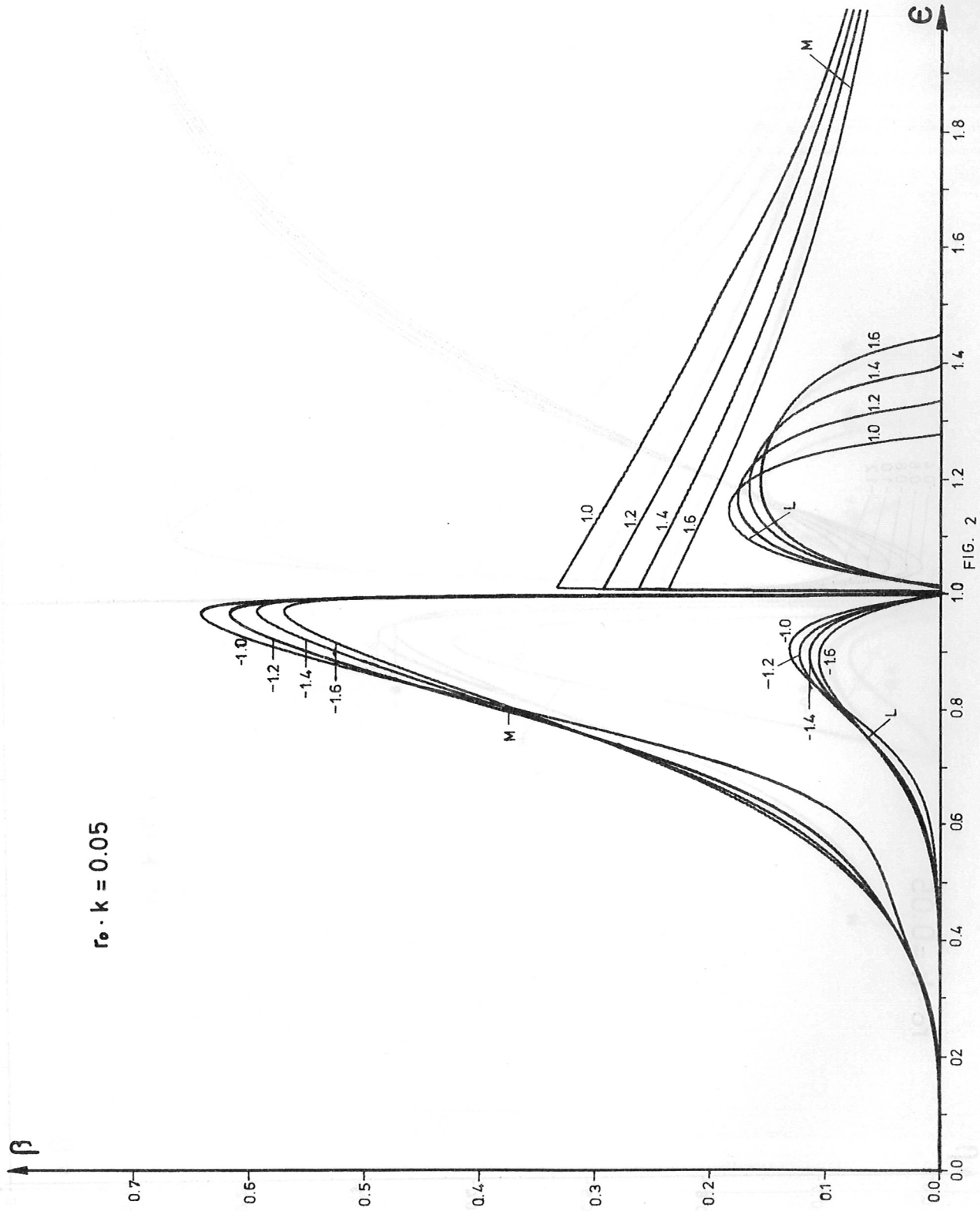


FIG. 2

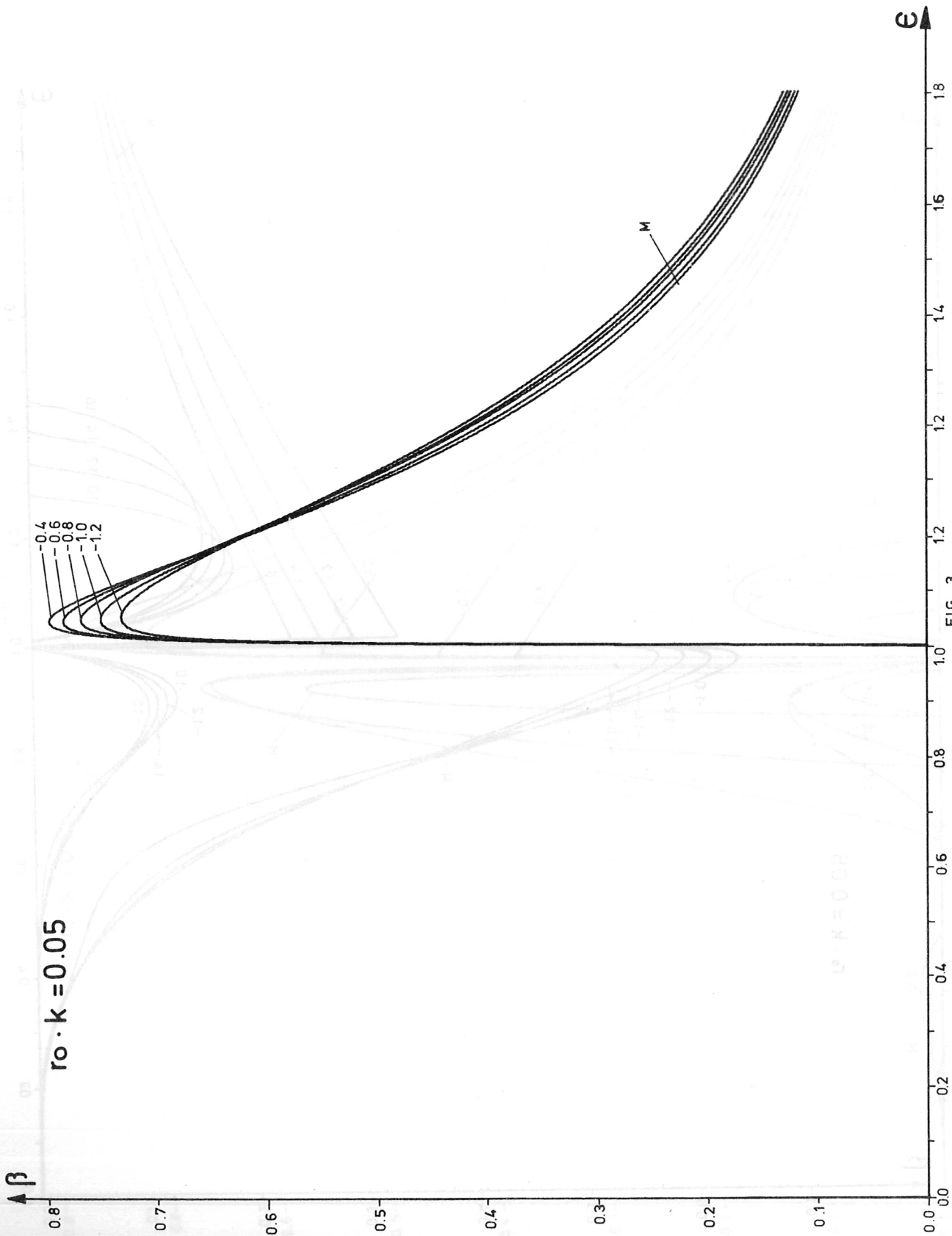


FIG. 3

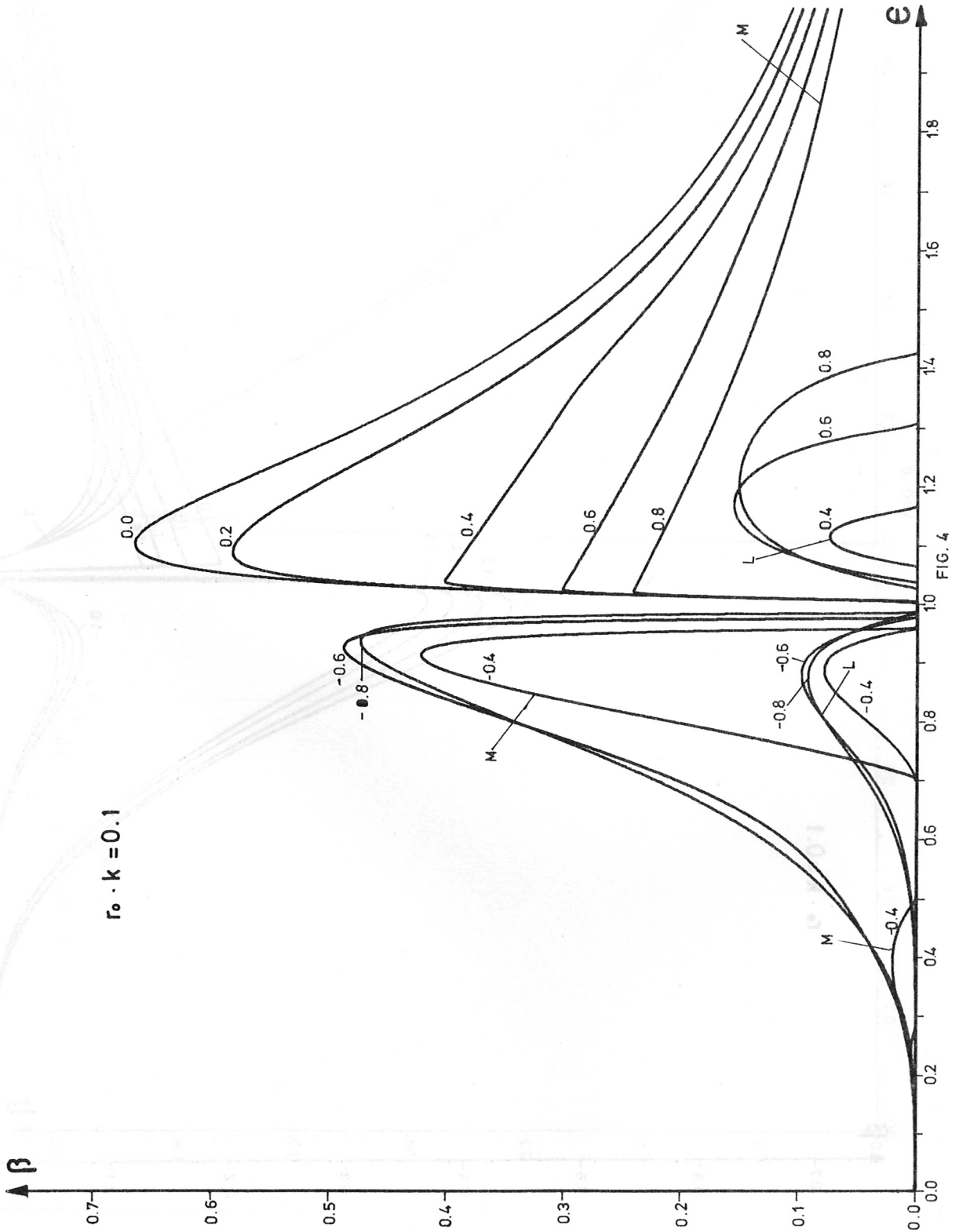
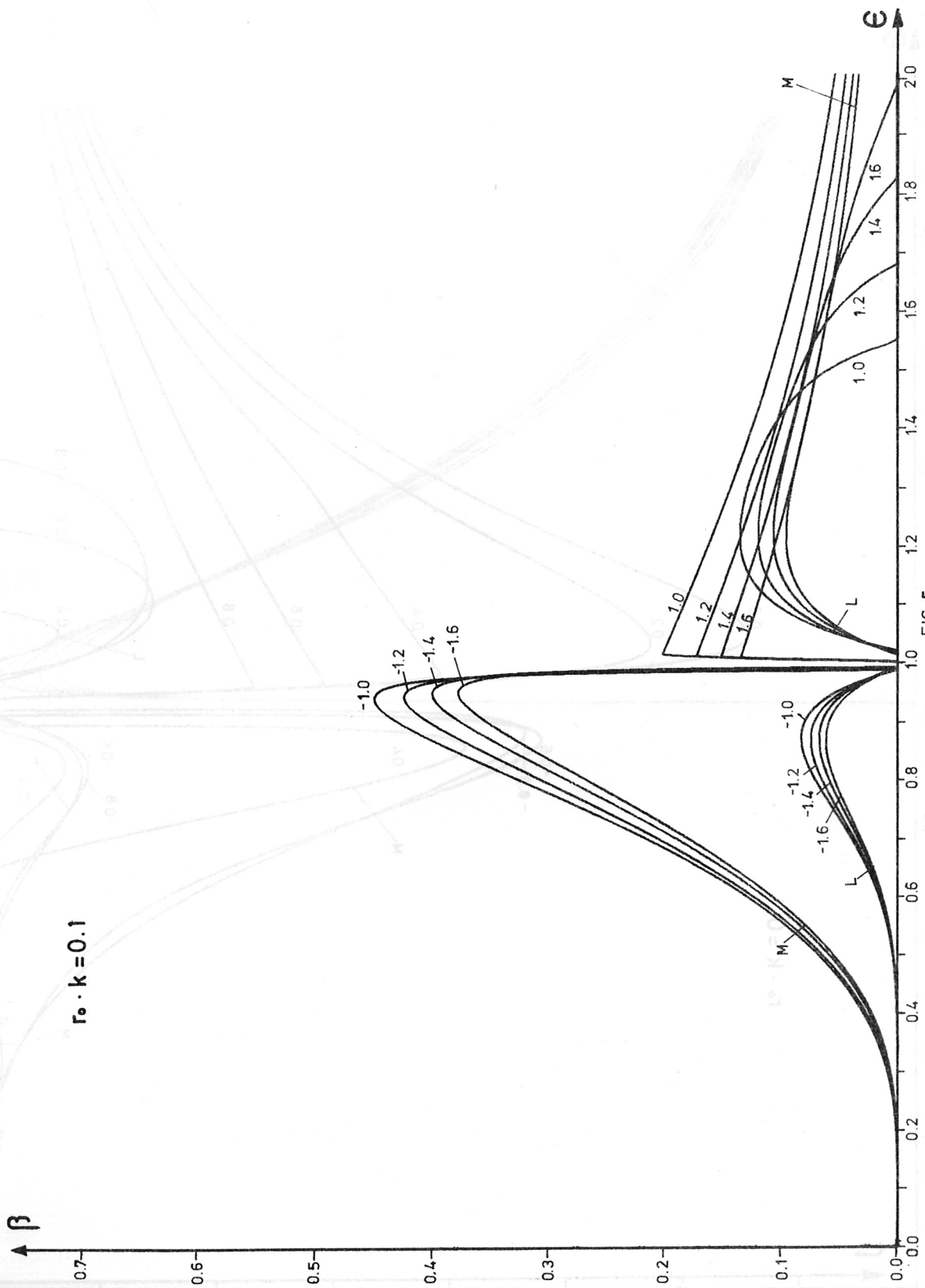


FIG. 4



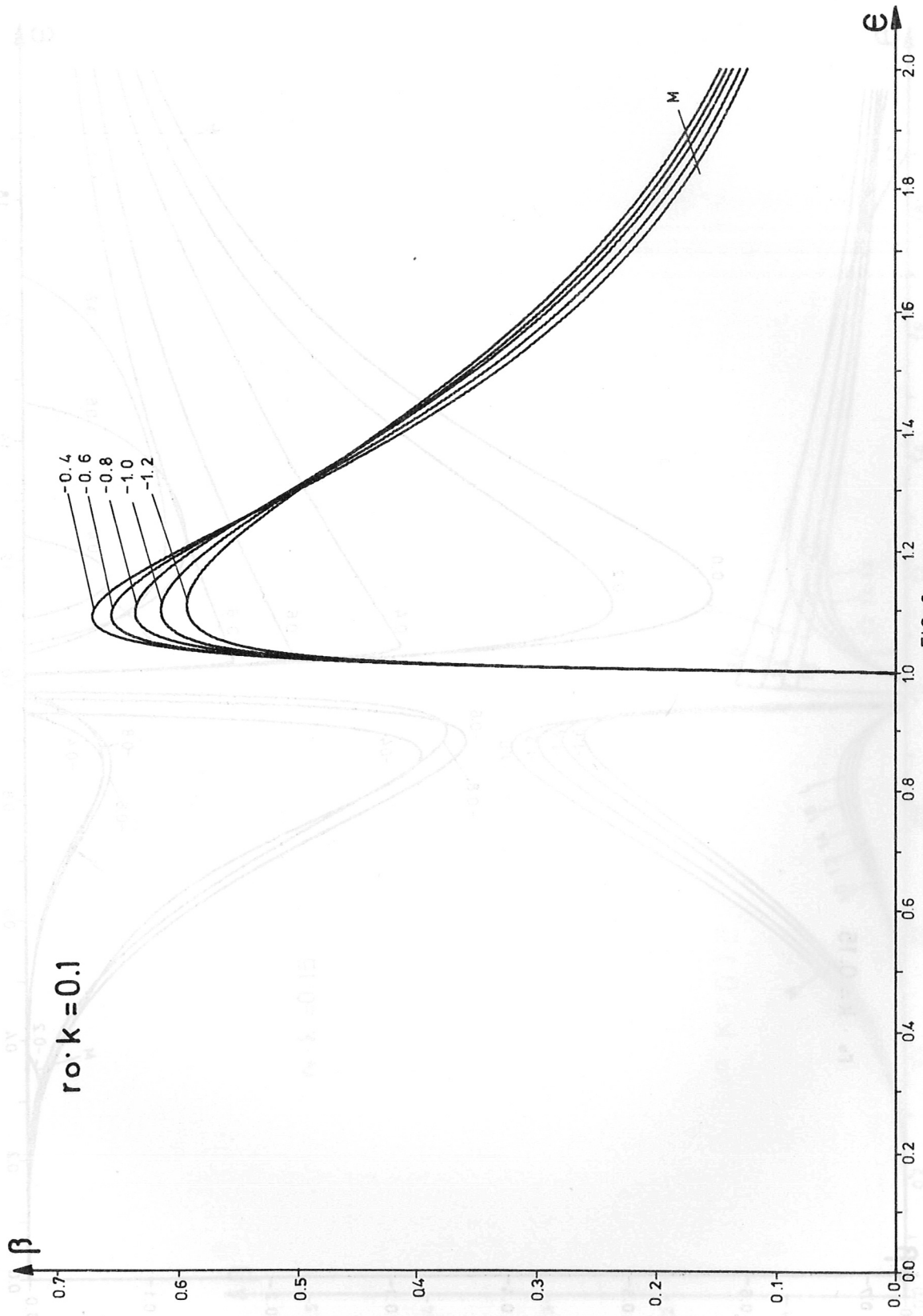


FIG. 6

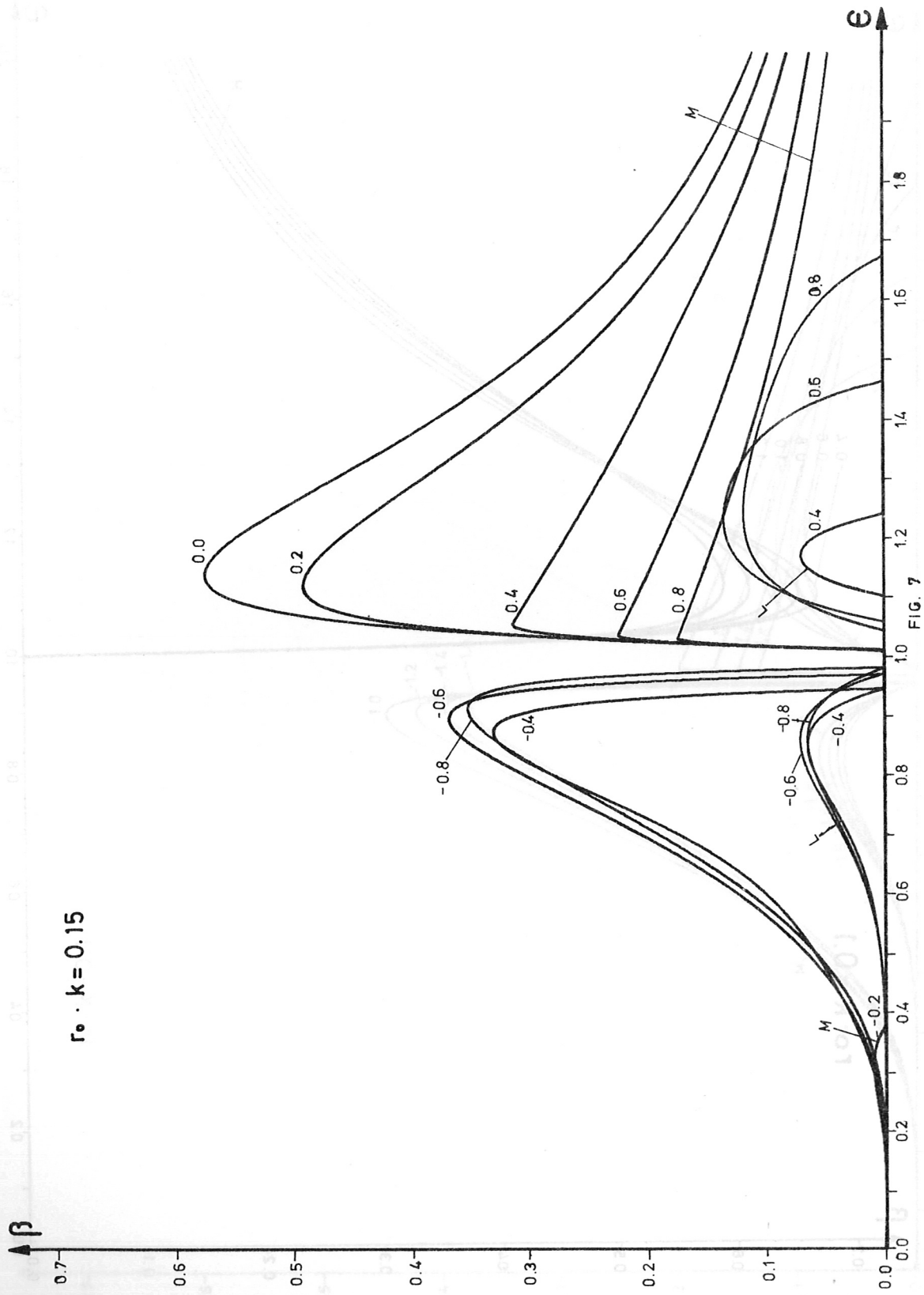
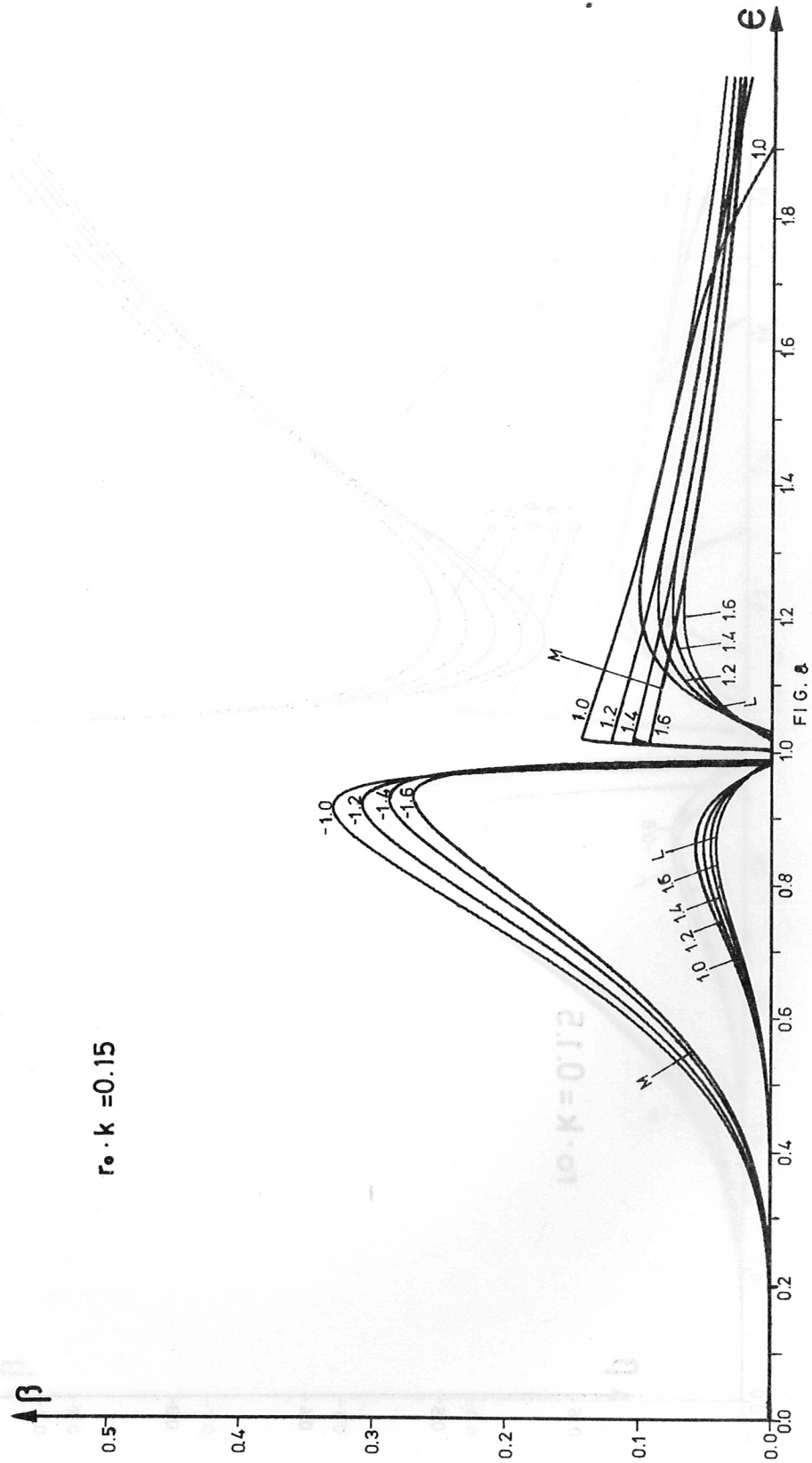


FIG. 7



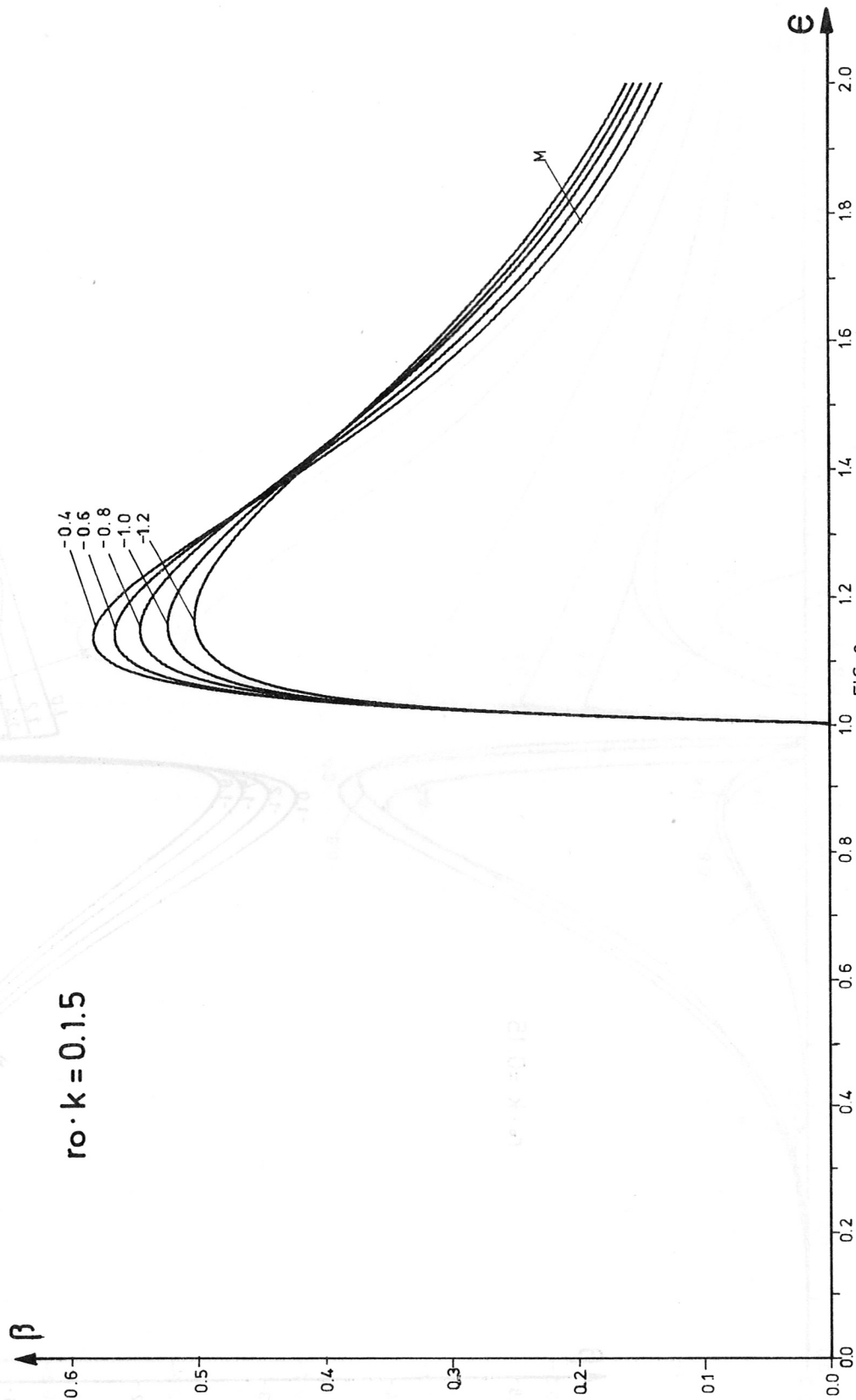
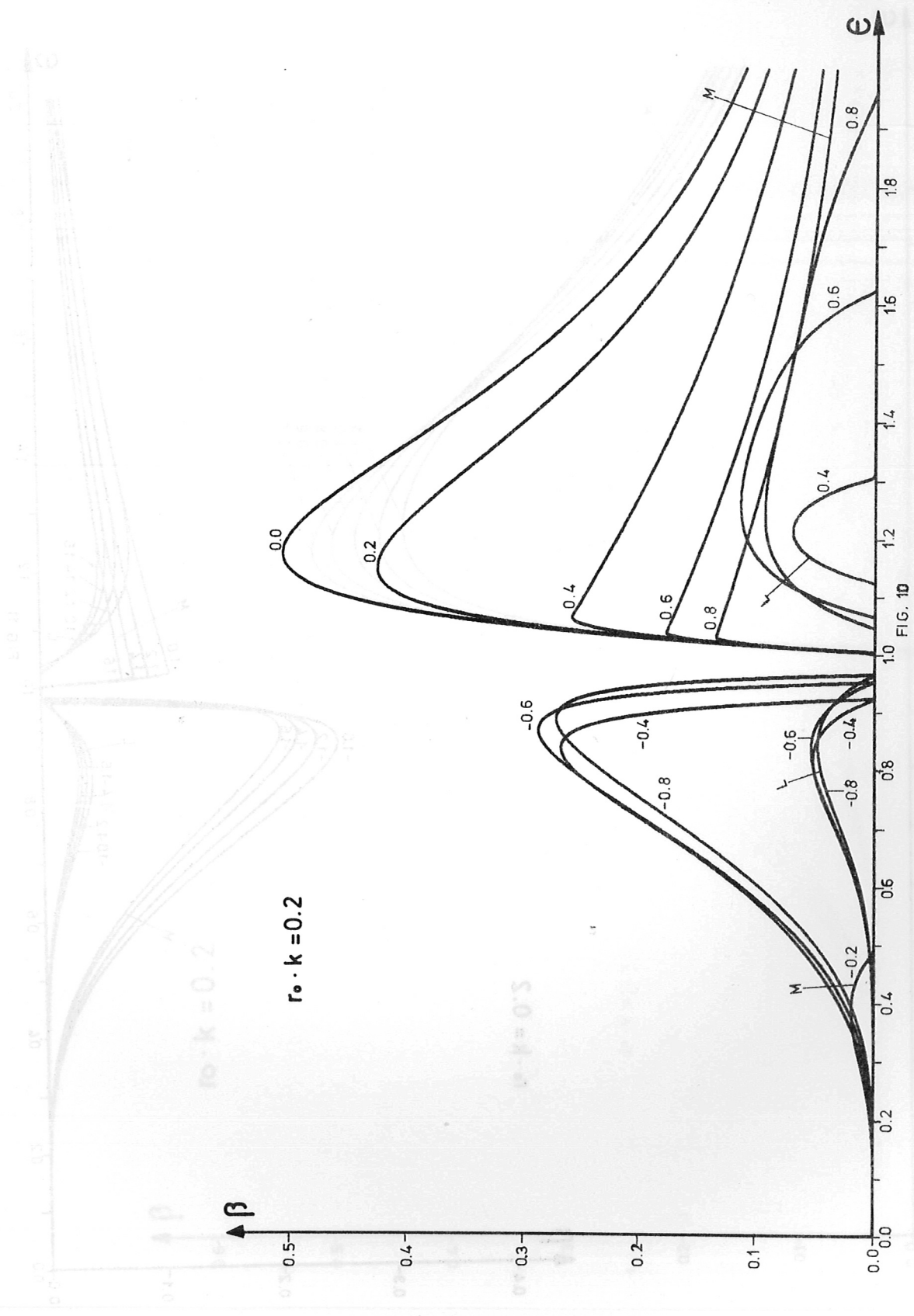


FIG. 9



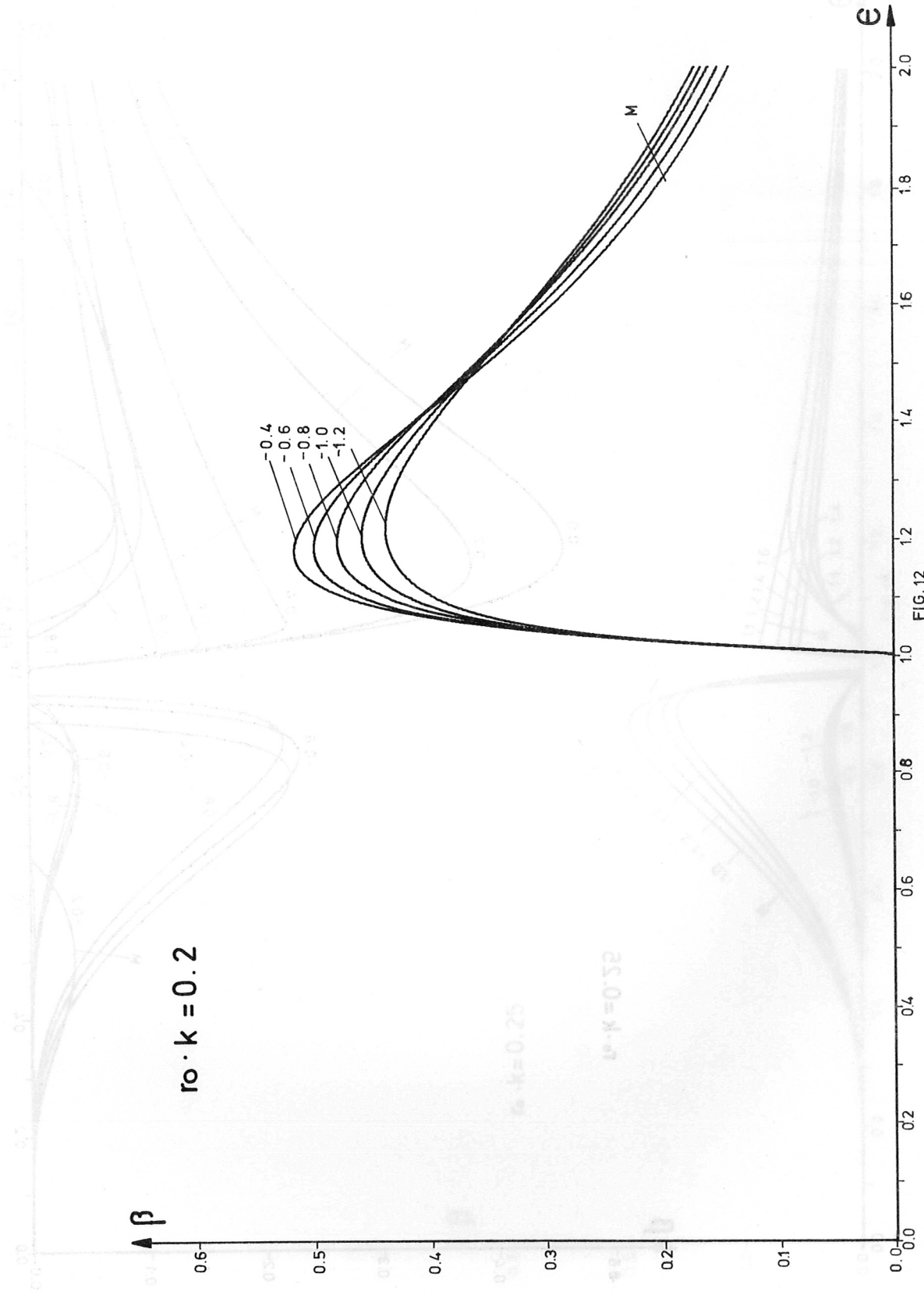
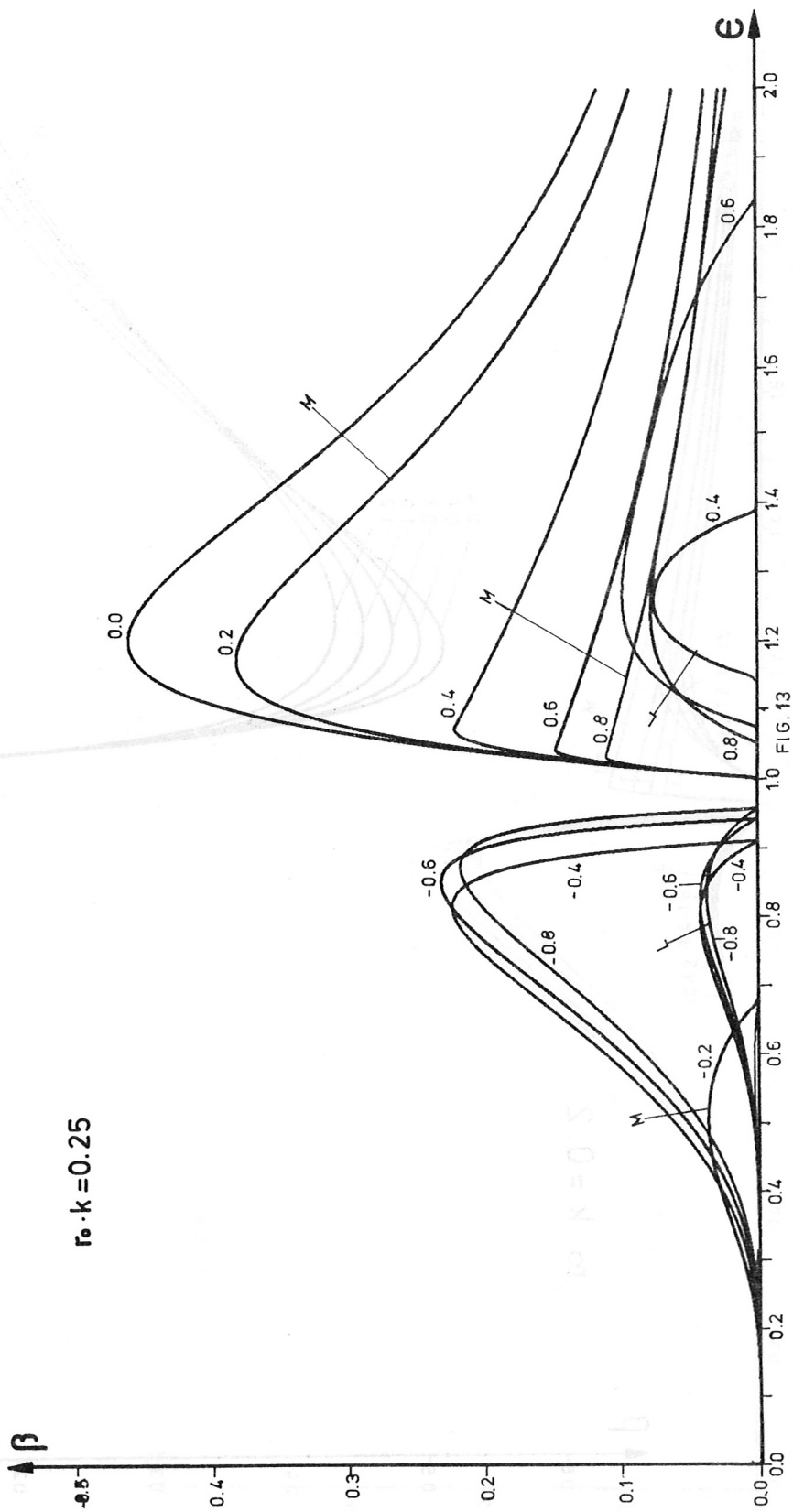


FIG. 12



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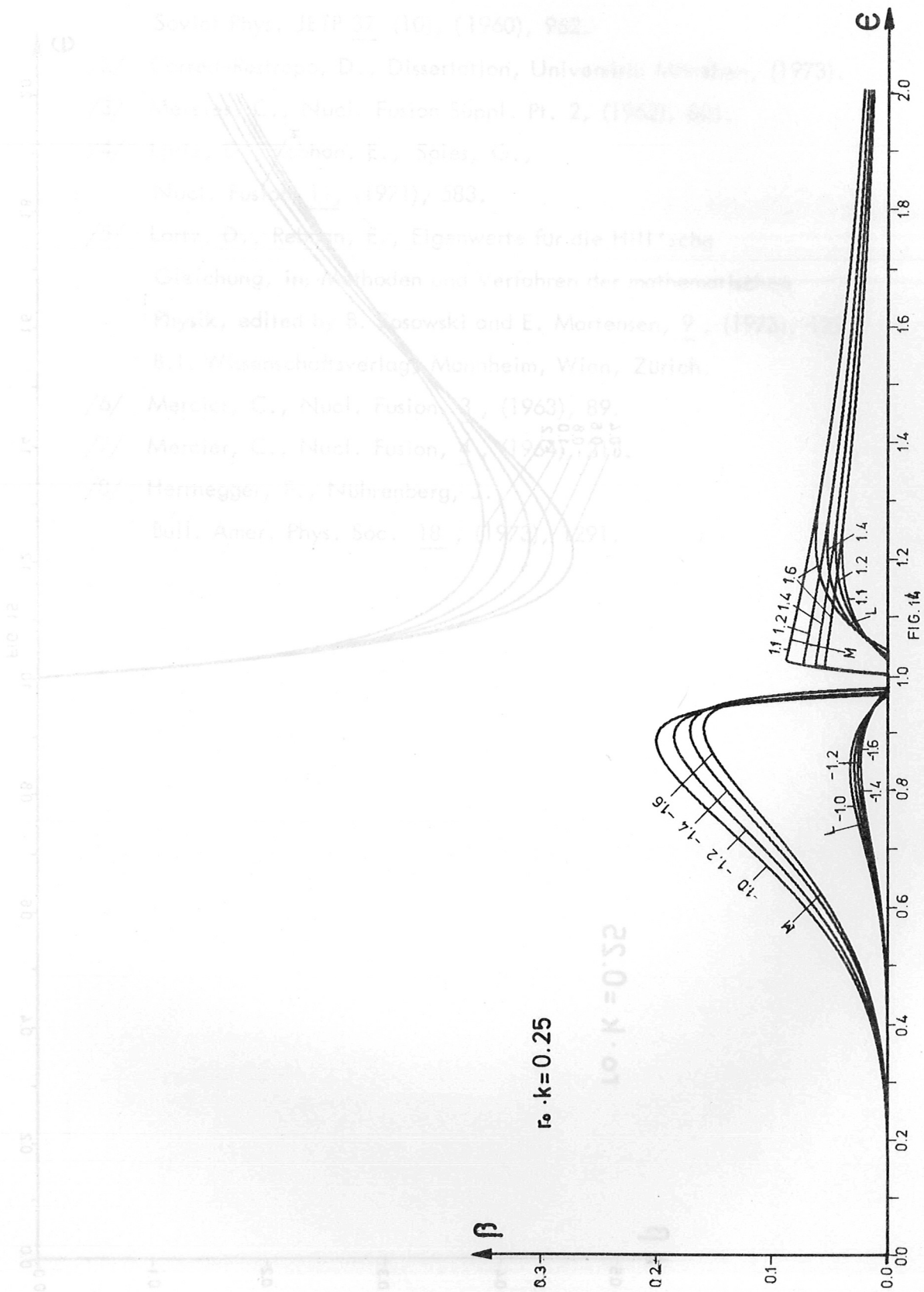


FIG. 14

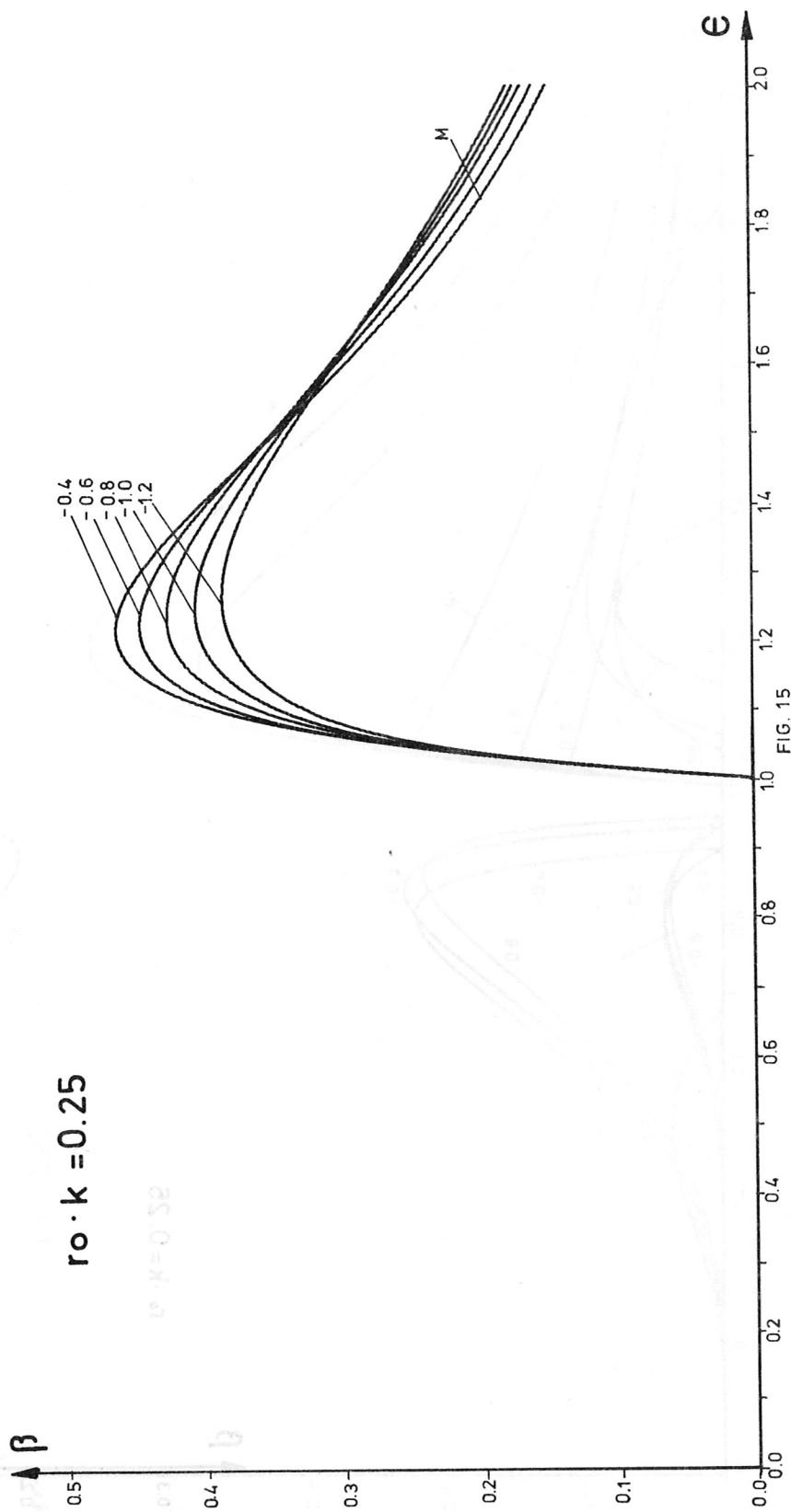


FIG. 15

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