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ELONGATED FREE-BOUNDARY AXISYMMETRIC EQUILIBRIA

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Abstract - The MHD equilibria of axially symmetric, current-carrying low- β plasmas with non-circular cross-section are investigated. The appropriate boundary conditions are imposed by solving a self-consistent free-boundary problem in the approximation of a thin, vertically elongated, otherwise arbitrary cross-section. The plasma boundary determines the external currents or vice versa.

1. INTRODUCTION

Axisymmetric equilibria with a vertically elongated cross-section have raised considerable interest owing to their property that the maximum current density permitted by the Kruskal-Shafranov limit is higher than in the case of a circular cross-section, thus allowing a stronger ohmic heating.

The study of the MHD equilibria of magnetized plasmas is greatly simplified by symmetry, i.e. by the fact that one coordinate is ignorable. In this case the problem reduces to a well known scalar equation for the magnetic flux function ψ (Ref. 1) in the cylindrical varibles (r, ϕ, z). The function ψ describes the magnetic surfaces and the constant plasma-pressure surfaces through the equation ψ = const. The most general symmetry is helical: this reduces to axial or plane symmetry in the limit of vanishing or infinite helical period length respectively.

The general equation for ψ is non-linear and has to be solved numerically. It is possible, however, to solve that equation analytically if the functions $I(\psi)$ and $P(\psi)$ (respectively current flux and pressure) are so chosen as to make the equation for ψ linear.

See Ref. 2, 3, 4 for axial symmetry, Ref. 5, 9 for the helical case. The linear equation separates: ψ is Fourier expanded in the variable $\Theta = \phi - az$ in the helical case, or in z in the axisymmetric case. Each Fourier component is multiplied by an eigenfunction of the radius r, and by a coefficient. These coefficients are to be determined by the boundary conditions.

If the discharge takes place on a time scale shorter than that required for the magnetic field to diffuse through the wall, the latter is a magnetic surface. This is not true if the discharge lasts longer. The former problem has been investigated analytically in the axisymmetric case in Ref. 2, 3, 4, numerically in Ref. 6, and the latter (free-boundary) problem numerically in Ref. 7, 8.

We study here the free-boundary problem analytically in axial symmetry using the same method as in Ref. 9, 10 (where the helical case was considered). The free boundary is assumed to be approximately a r = const. cylinder, which gives in axial symmetry a thin, vertically elongated cross-section (see Fig. 1). Formally expanding to first order for a small deformation yields a separate system of linear equations for the coefficients of each Fourier component (in z) of ψ (i.e. of the field) and of the external currents. The latter are assumed to flow on other r=const. cylinders ($r=R_0$, $r=R_1$, see Fig. 1). Our solution is valid for a thin, arbitrary cross-section, and for an arbitrary superposition (3.4) of Fourier components of the field, smaller than the main field. This superposition is important for stability (see Ref. 11). The phase of each component is arbitrary. The phases are omitted for simplicity, but they can be independently shifted with no need of any other change.

The external currents are the known terms in the equations for the coefficients (one system for each Fourier component) (see Sec. 4). Knowledge of the coefficients in the solution for ψ (3.4), i.e. of the plasma boundary, yields directly the outside currents. This allows us to impose a plasma boundary and see what currents are needed to obtain it. If, instead, the external currents are known, the linear systems of equations can be solved to derive ψ and the plasma boundary from the outside currents. It is therefore possible to see how plasma boundaries and external current distributions are related to one another, and how varying one affects the other. The currents can be expected to be strongly affected by a small change of the boundary.

Since β is low, we make the zero-pressure approximation, which leads to the force-free equation

$$\nabla \times \underline{B} = s\underline{B} \qquad (1.1)$$

where s can be assumed to be a constant for the reasons stated in Sec. 1 of Ref. 9. The same free-boundary method can be used also for the non-force-free case (see Ref. 2, 3). The current density, i.e. s, is finite. Our units are c.g.s. non-rationalized e.m.u. Our current unit equals $10/4\pi$ amps. The aspect ratio is arbitrary.

In Sec. 2 we write the vacuum field, in Sec. 3 we solve for the force-free field (1.1), in Sec. 4 we impose the boundary conditions and determine the coefficients.

2. AXISYMMETRIC FIELDS

We use a cylindrical coordinate system (r, ϕ, z), where the z-axis is our axis of symmetry (see Fig. 1). All quantities depend on the radius r and on z, and are independent of the azimuth ϕ.

We take a length L in the z direction as the half wave-length of our Fourier expansion in z and denote $\eta = \pi/L$. This corresponds to a periodic solution in the z direction, but the analysis is perfectly analogous if we use Fourier integrals instead of series.

From $\underline{B} = \nabla \times \underline{A}$, where $\underline{A}$ is the vector potential, we define the magnetic-field flux function ψ, which describes the magnetic surfaces (ψ = const.):

$$\psi = rA_{\phi}. \qquad (2.1)$$

From the definition of ψ it follows that

$$\partial\psi/\partial r = rB_z, \quad \partial\psi/\partial z = -rB_r. \tag{2.2}$$

From $\nabla\times B = J$ we define the plasma-current flux function

$$I(\psi) = rB_\phi. \tag{2.3}$$

We distinguish the following regions (see Fig.1): (1) $0<r<R_o$, (2) $R_o<r<R^-$, plasma region $R^-<r<R^+$, (3) $R^+<r<R_1$, (4) $R_1<r$.

The vacuum field in regions 1, 2, 3, 4 is given by $\nabla\times B=0$. Hence $B=\nabla\Phi$ and, since $\nabla\cdot B=0$, $\nabla^2\Phi=0$. The solution is

$$B_1 = \sum_n c_{1n} \nabla I_o(n\eta r) \sin n\eta z$$

$$B_2 = e_z B_v + e_\phi B_o R/r + \sum_n c_{2n} \nabla I_o(n\eta r) \sin n\eta z + \sum_n c'_{2n} \nabla K_o(n\eta r) \sin n\eta z$$

$$B_3 = e_z B'_v + e_\phi B_o R/r + \sum_n c_{3n} \nabla I_o(n\eta r) \sin n\eta z + \sum_n c'_{3n} \nabla K_o(n\eta r) \sin n\eta z$$

$$B_4 = \sum_n c'_{4n} \nabla K_o(n\eta r) \sin n\eta z$$

where the sums start with n=1.

There is no net z-current in the plasma, so B_o is the same in B_2 and B_3. A net ϕ-current in the plasma causes $B_v \neq B'_v$.

The external currents are represented by surface currents on the cylinders $r=R_o$ and $r=R_1$:

$$r = R_o \begin{cases} i_z = i_o \\ i_\phi = i_v + \sum_n i_{on} \cos n\eta z \end{cases}$$

$$r = R_1 \begin{cases} i_z = i'_o \\ i_\phi = i'_v + \sum_n i_{1n} \cos n\eta z \end{cases}$$

Any axisymmetric current distribution can be obtained by introducing phases in these formulae.

Peaked currents at $z = \pm L/2$ can simulate the top and bottom boundaries, which are missing in our model. Without these currents the surfaces (3.4) would not be closed, since the Fourier components of the solution (see Sec. 4) are proportional to those of the external currents.

The boundary conditions on $r=R_o$ are

$$B_{1r} = B_{2r};\ B_{1\phi} = B_{2\phi} - i_z;\ B_{1z} = B_{2z} + i_\phi$$

and analogously on $r=R_1$. The results are:

$$i_v = -B_v;\ i'_v = B'_v;\ i_o = RB_o/R_o;\ i'_o = -RB_o/R_1$$

$$c_{1n} - c_{2n} = i_{on} R_o K_1(n\eta R_o)$$

$$c'_{2n} = -i_{on} R_o I_1(n\eta R_o)$$

$$c_{3n} = i_{1n} R_1 K_1(n\eta R_1)$$

$$c'_{3n} - c'_{4n} = i_{1n} R_1 I_1(n\eta R_1)$$

where we have used $I'_o = I_1$, $K'_o = -K_1$, and their Wronskian.

In region 2 we have

$$\psi_2 = B_v r^2/2 + r \sum_n c_{2n} I_1(n\eta r) \cos n\eta z + - r \sum_n c'_{2n} K_1(n\eta r) \cos n\eta z \qquad (2.4)$$

and ψ_3 is analogous. This does not give closed surfaces, of course, since there is no current, which is necessary for equilibrium in axial symmetry. In this case $I=RB_o$ (2.3) is a constant. We shall need (2.4) in Sec. 4.

Only in total vacuum (no plasma current) $\underline{B}_2 = \underline{B}_3$, and $B_v = B'_v$, $c_{2n} = c_{3n}$, $c'_{2n} = c'_{3n}$. In this case the coefficients are fully determined. Otherwise, c_{2n} and c'_{3n} are determined in Sec. 4.

3. FORCE-FREE AXISYMMETRIC FIELDS

We solve now for the field in the plasma and current region ($R^- < r < R^+$, see Fig. 1). Using the same technique as in Sec. 3 of Ref. 9, we solve Eq. (1.1) through

$$\nabla^2 f + s^2 f = 0 \tag{3.1}$$

where $\underline{B} = \underline{\nabla}x\underline{\nabla}x(\underline{e}_z f) + s\underline{\nabla}x(\underline{e}_z f)$. We set $f = \sum_n f_n$, where $f_n(r, z) = g_n(r) \cos n\eta z$. The equation for the g's is

$$g_n'' + g_n'/r + (s^2 - n^2\eta^2)g_n = 0. \tag{3.2}$$

For n=0 the solution is $g_o = J_o(sr)$ or $g_o = Y_o(sr)$. For $n > 0$, since we can expect $|n\eta| > |s|$, we have $g_n = I_o(kr)$, or $g_n = K_o(kr)$, where $k^2 = n^2\eta^2 - s^2$.

For a force-free field (1.1), $\underline{\nabla}x\underline{B} = s\underline{B} = s\underline{\nabla}x\underline{A}$ and, for constant s, $\underline{B} = s\underline{A} + \underline{\nabla}h$. Hence, for $s \neq 0$, from (2.1) and (2.3) we have

$$I = rB_\phi = c_o + srA_\phi = c_o + s\psi. \tag{3.3}$$

We can therefore write the solution for the plasma region in the form

$$I = c_o + s\psi = crJ_1(sr) - c'rY_1(sr) + \\ + sr \sum_n \left[c_n I_1(kr) - c_n' K_1(kr) \right] \cos n\eta z. \tag{3.4}$$

This gives ψ and the surfaces in the plasma region, and the field components through (2.2), (2.3). As $s \to 0$, (3.4) reduces to the vacuum expression (2.4).

The coefficients in the RHS of (3.4) are to be determined by the boundary conditions.

The rotational transform $2\pi\iota$ is defined as the angle by which a magnetic line rotates around the magnetic axis (r=R, z=0, see Fig. 1) as it runs once around the long way ($\Delta\phi = 2\pi$). Since, however, all our quantities are expressed in terms of r and z, and r can in principle be known as a function of z on the magnetic surfaces, it is convenient to have an expression for ι in terms of an integral in z. Along a field line the following equalities hold:

$$dl/B = dr/B_r = r d\phi/B_\phi = dz/B_z .$$

Let us define $\Delta\phi = \oint B_\phi dz/rB_z$, where the integral is calculated once around the short way along a field line. If $\Delta\phi = 2\pi$, then $\iota = 1$. Otherwise we add M such magnetic line segments until we approximate $M\Delta\phi = 2\pi N$, $M = N\iota$ (with M, N large integers). This yields $M/N = 2\pi/\Delta\phi = \iota$.

4. THE BOUNDARY CONDITIONS

There are two fixed boundaries ($r=R_o$, $r=R_1$) and two free boundaries ($r=R^-$, $r=R^+$) where field-matching conditions are to be imposed (see Fig. 1). The conditions on the fixed boundaries have been worked out in Sec. 2. On the free boundaries we require that the force-free field (3.4) in the (low-β) plasma region ($R^- < r < R^+$) match with the vacuum field (2.4) outside the plasma (in regions 2 and 3). This is equivalent to requiring that I and $\nabla\psi$ be continuous (2.2), (2.3), (3.3). Following the same method as in Sec. 4 of Ref. 9, we describe the boundary by $r(z) = R^- + \epsilon r_1(z)$, where $\epsilon \ll 1$, and analogously for R^+. This implies $\psi(r,z) = p(r) + \epsilon q(r, z)$, and $r_1 = -q/p'$. By formally expanding in ϵ, the boundary conditions are translated into the following conditions, for both R^- and R^+: to order 1, I and p' have to be continuous, and, to order ϵ, also q and $q[\log(q/p')]'$, where the prime means d/dr. To each order there are four equations, two for each of the two boundaries ($r=R^-$, $r=R^+$). The equations are linear in the coefficients of (2.4), (3.4), and the known terms are the vacuum field components, i.e. the external currents.

Since only the difference $B'_v - B_v$ is determined by the plasma current, to order 1 the equations can be reduced to three:

$$cJ^- - c'Y^- = RB_o/R^-$$

$$cJ^+ - c'Y^+ = RB_o/R^+$$

$$cJ - c'Y - (B'_v - B_v) = 0$$

where $J^\pm = J_1(sR^\pm)$, $Y^\pm = Y_1(sR^\pm)$, $J = J_o(sR^+) - J_o(sR^-)$, $Y = Y_o(sR^+) - Y_o(sR^-)$.

The solution is

$$c = RB_0\,(Y^+/R^- - Y^-/R^+)/\Delta$$

$$c' = RB_0\,(J^+/R^- - J^-/R^+)/\Delta$$

$$B'_v - B_v = -RB_0\,\left[(J^+Y - JY^+)/R^- - (J^-Y - JY^-)/R^+\right]/\Delta$$

where $\Delta = J^-Y^+ - J^+Y^-$.

To order ϵ, equating the Fourier components, we have for each n

$$c_n I_1^- - c'_n K_1^- - c_{2n} I^- = -c'_{2n} K^-$$

$$c_n I_1^+ - c'_n K_1^+ + c'_{3n} K^+ = c_{3n} I^+$$

$$c_n (I^-) - c'_n (K^-) - c_{2n}\, n\eta\, I'^- = -c'_{2n}\, n\eta\, K'^-$$

$$c_n (I^+) - c'_n (K^+) + c'_{3n}\, n\eta K'^+ = c_{3n}\, n\eta\, I'^+$$

where $I_1^{\pm} = I_1(kR^{\pm})$, $K_1^{\pm} = K_1(kR^{\pm})$, $I^{\pm} = I_1(n\eta R^{\pm})$, $K^{\pm} = K_1(n\eta R^{\pm})$,

$I'^{\pm} = I'_1(n\eta R^{\pm})$, $K'^{\pm} = K'_1(n\eta R^{\pm})$,

$$(I^{\pm}) = I_1(kr)\; d/dr \log\left\{I_1(kr)/\left[cJ_0(sr) - c'Y_0(sr)\right]\right\} \quad (r = R^{\pm}),$$

$$(K^{\pm}) = K_1(kr)\; d/dr \log\left\{K_1(kr)/\left[cJ_0(sr) - c'Y_0(sr)\right]\right\} \quad (r = R^{\pm}).$$

The coefficients c'_{2n}, c_{3n} have already been calculated in Sec. 2, whereas c_{2n}, c'_{3n} are determined here. The solution is

$$c_n = \left\{c_{3n}\left[n\eta I'^- K_1^- - (K^-)I^-\right]/R^+ + c'_{2n}\left[(K^+)K^+ - n\eta K'^+ K_1^+\right]/R^-\right\}/\Delta'$$

$$c'_n = \left\{c'_{2n}\left[(I^+)K^+ - n\eta K'^+ I_1^+\right]/R^- + c_{3n}\left[n\eta I'^- I_1^- - (I^-)I^-\right]/R^+\right\}/\Delta'$$

$$c_{2n} = \Big(c'_{2n}\Big\{\left[(I^+)K^+ - n\eta K'^+ I_1^+\right]\left[n\eta K'^- K_1^- - (K^-)K^-\right] + \left[(K^+)K^+ - n\eta K'^+ K_1^+\right]\left[(I^-)K^- - n\eta K'^- I_1^-\right]\Big\} + c_{3n}\left[(I^-)K_1^- - (K^-)I_1^-\right]/R^+\Big)/\Delta'$$

$$c'_{3n} = \Big(c_{3n}\Big\{\left[n\eta I'^- I_1^- - (I^-)I^-\right]\left[n\eta I'^+ K_1^+ - (K^+)I^+\right] + \left[(K^-)I^- - n\eta I'^- K_1^-\right]\left[n\eta I_1^+ I'^+ - (I^+)I^+\right]\Big\} + c'_{2n}\left[(I^+)K_1^+ - (K^+)I_1^+\right]/R^-\Big)/\Delta'$$

where $\Delta' = \left[(I^+)K^+ - n\eta K'^+ I_1^+\right]\left[n\eta I'^- K_1^- - (K^-)I^-\right] +$
$+ \left[n\eta K'^+ K_1^+ - (K^+)K^+\right]\left[n\eta I'^- I_1^- - (I^-)I^-\right].$

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List of symbols used:

β	beta
∂	partial derivative
Δ	capital delta
$\underline{\nabla}$	del ($\underline{\nabla}$ grad; $\underline{\nabla}\cdot$ div; $\underline{\nabla}\times$ curl)
ϵ	epsilon
η	eta
θ	theta
$\bar{\iota}$	iota bar
π	pi
Σ	sum (capital sigma)
ϕ	phi
Φ	capital phi
ψ	psi

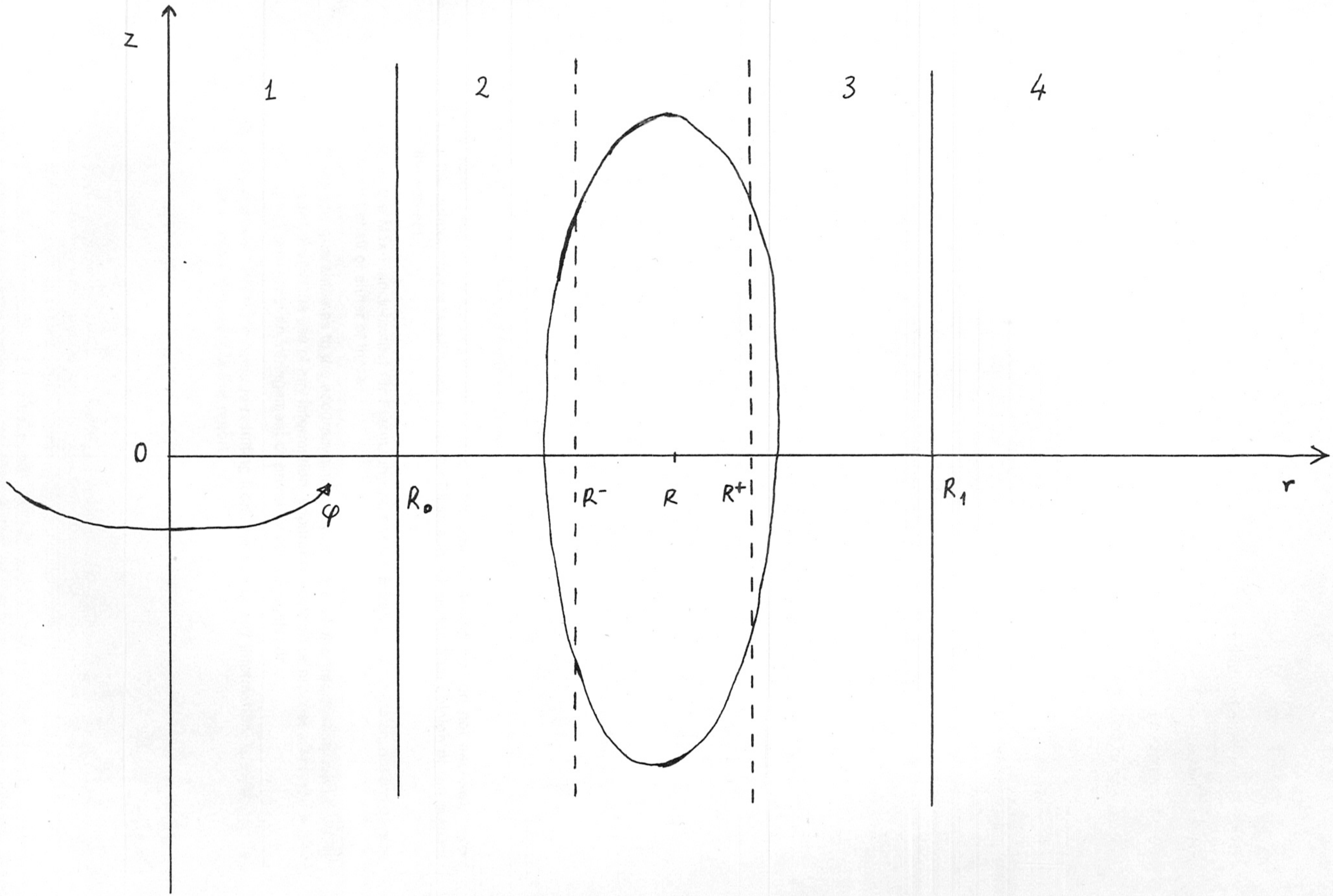

Fig. 1