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Abstract - The MHD equilibria of helically symmetric, current-carrying plasmas are investigated. Configurations such as straight helical pinches or high-β Stellarators (where the plasma column is surrounded by a force-free region) are considered in the sharp-boundary, flat-pressure-profile approximation. The plasma current and pressure can be large. The appropriate boundary conditions are imposed by solving a self-consistent double-free-boundary problem in the approximation of an almost circular, otherwise arbitrary cross-section. The helical field can be an arbitrary superposition of small helical field components with different ℓ-numbers and equal winding period length. The plasma boundary determines the external currents or vice versa.

1. INTRODUCTION

The MHD equilibrium configurations of current-carrying, helically symmetric plasmas, such as those contained in, for example, straight Stellarators or in straight helical pinches, are investigated. If the discharge takes place on a time scale longer than that required for the magnetic field to diffuse through the wall, the latter is not a magnetic surface. Wall stabilization of fast oscillations is possible, however. All external currents flow on the cylinder C (see Fig. 1).

In a previous paper (BARBERIO-CORSETTI, 1973), which will be referred to with the abbreviation FFHE, the force-free helical equilibrium of a current-carrying low-β (zero-pressure) plasma contained in a straight ℓ=2 Stellarator was calculated.

We list here three cases, in order of increasing complexity, which generalize the problem solved in FFHE:

1) low-β equilibrium: the plasma and current region is delimited by the magnetic surface S tangent to the limiter or wall (see Fig. 1), and the magnetic fields inside and outside S have to match on S;

2) high-β (finite-pressure) equilibrium with plasma current, flat pressure profile and sharp boundary: the boundary condition involves the field and pressure discontinuity at the boundary surface (GROSSMANN, 1972), which is again a magnetic surface;

3) high-β plasma column with plasma current, flat pressure profile and sharp boundary, surrounded by a low-β region with plasma current (which may be important for stability): there are two boundaries (see Fig. 1), the magnetic surface S° (delimiting the high-β region) with a jump condition, and the magnetic surface S (tangent to the limiter or wall) with a field-matching condition.

Case 3 is reduced to the two preceding cases by appropriately specializing the parameters. We shall therefore solve case 3, and the solution can be specialized to the two other cases. The results of FFHE can thus be recovered as a particular solution of case 1.

In all three cases the pressure profile is assumed to be flat everywhere except for the jump at the boundary of the high-β column. The whole configuration is force-free, except for the boundary between the high- and low-β regions, where the pressure discontinuity is balanced by surface currents. A finite plasma current can be present in each region. The field is given by $\underline{\nabla} \times \underline{B} = s\underline{B}$, where s can be assumed to be a constant in each region for the reasons stated in Sec. 1 of FFHE. A force-free-field solution of the type found in Sec. 3 of FFHE is thus valid in each region. The solution is Fourier expanded in helical components, each component being multiplied by an eigenfunction of the radius and by a coefficient. The coefficients are to be determined by imposing the appropriate boundary conditions.

The boundary conditions have to be imposed on free plasma (magnetic) surfaces (see also Sec. 1 of FFHE), thus confronting us with a self-consistent double-free-boundary problem. This is solved analytically (see Sec. 4) by applying the same method used in FFHE, viz. by assuming that the z = const. sections of the boundary surfaces S and S^o (see Fig. 1) are approximately circles centered on the z-axis (i.e. that the helical field is smaller than the main straight field), and expanding to first order in the deformation (i.e. in the ratio of helical to main field). This approximation yields a separate system of linear equations (see Sec. 5) for the coefficients of each helical or Fourier component of the solution field and of the external currents (the latter are the known terms in the equations). Our solution is therefore valid for an arbitrary superposition of small helical fields with different field periodicity (ℓ-numbers), but with the same winding period length. This implies that the boundary cross-sections can be arbitrary, as long as the deformation from a circle is small. For a finite deformation, or ratio of helical to main field, the free-boundary problem is non-linear and cannot be solved analytically: the solution is always of the form (5.1), (5.2), (5.3), but the coefficients have to be calculated numerically. Our solution can be used as a starting point for the numerical calculation in the non-linear case.

The external currents (2.4), the vacuum field (2.2), the solution field (Sec. 3 of FFHE), the vacuum surfaces (2.5) and those of the solution field (5.1), (5.2), (5.3) are Fourier expanded and their corresponding Fourier components are proportional. The phase of each helical component is arbitrary and independent of the others. In our calculation the phases are omitted for simplicity, but the phase of each helical component of the external current and of the vacuum field can be shifted together with the phase of the corresponding term in the solution field with no need of any other change.

The external currents are the known terms in the equations for the coefficients (one system for each Fourier component). Knowledge of the coefficients in the solution for the field, i.e. of the plasma boundary, yields directly the outside currents (see Sec. 5). This allows us to impose a plasma boundary (which can have an arbitrary almost circular cross-section) and see what currents are needed to obtain it. If, instead, the external currents are known, the linear systems of equations can be solved to derive the solution field and the plasma boundary from the outside currents. It is therefore possible to see how plasma boundaries and external current distributions are related to one another, and how varying one affects the other. The same free-boundary method can be used with non-force-free fields (CORREA and LORTZ, 1973). The plasma current and pressure can be large and our solution is of practical interest.

WEIMER et al. (1970) considered the case of a small current in the low-β region, and no current in the high-β region. WEITZNER (1971) studied the case of a long wave-length, $\ell=1$ equilibrium. NUEHRENBERG (1970), FREIDBERG and MARDER (1971) and FREIDBERG (1971) studied the $\ell=1$ configuration, and FREIDBERG (1973) the $\ell=2$ case.

We recall some definitions and properties of vacuum helical fields in Sec. 2, and of MHD helical equilibria in Sec. 3. In Sec. 4 we analyze the boundary and jump conditions, and in Sec. 5 we apply them to obtain the linear systems of equations which give the coefficients in the solution of case 3.

2. VACUUM HELICAL FIELDS

We use a cylindrical coordinate system (r, ϕ, z), where the z-axis coincides with the magnetic axis (see Fig. 2). All the relevant quantities depend only on the radius r and on

$$\theta = \phi - az, \qquad (2.1)$$

where $a = 2\pi/L$, and L is the period length (wave-length) of the helical windings in the z-direction. For ℓ=n, L is n times the field-period length.

The vacuum helical field (MOROZOV and SOLOV'EV, 1966, p. 42-58) can be written in terms of Bessel functions:

$$\underline{B} = B^o \underline{e}_z + \frac{1}{a} \sum b_n \underline{\nabla} I_n (nar) \sin n\theta, \qquad (2.2)$$

where the sum starts with n=1 and

$$B^o = i^o, \quad b_n = \frac{n}{2} a I^*_n \sin\gamma K'_n(naA). \qquad (2.3)$$

This field is generated by the surface currents

$$i_\phi = i^o + i^* \sin\gamma, \quad i_z = i^* \cos\gamma, \quad i^* = \sum i^*_n \cos n\theta, \qquad (2.4)$$

which flow on the cylinder C (see Fig. 1). Here $tg\gamma = aA$, $i^*_n = nI^*_n/2A$, $i^o = I^o/d = I^o N/L$, I^*_n is the current in each ℓ=n helical winding, I^o is the current in each main field coil, d is the distance between main field coils, N is the number of main field coils per length L, A is the radius of C. Any helical surface-current distribution on C can be Fourier expanded by introducing phases in (2.4).

These currents can be carried by main field coils and helical windings, or by twisted coils (WOBIG and REHKER, 1972), wound on the cylinder C. For any current distribution (2.4) the geometry of the twisted coils is determined by the equation

$$dz/d\theta = A\, i_z/i^o = g'(\theta).$$

Integrating, we easily obtain $z = g(\theta) = g(\phi - az)$, and, if $|ag| \ll 1$ and $|ag'| \ll 1$, we can solve approximately $z(\phi)$ =

$g(\phi)[1-ag'(\phi)]$.

We have modified the formulae of MOROZOV and SOLOV'EV (1966, p. 56-57) by introducing the factor $\cos\gamma$ in (2.4) and by setting $4\pi/c = 1$ in our units (c.g.s. non-rationalized e.m.u.). Our current unit is $10/4\pi$ ampere.

The vacuum-field magnetic surfaces are given by ψ = const., where (see next section)

$$\psi = B^o \frac{ar^2}{2} - r \sum_n b_n I'_n(nar) \cos n\theta. \qquad (2.5)$$

Since $K'_n < 0$, b_n and I^*_n (2.3) have opposite signs. Hence, as can be seen by expanding (2.5) for $ar \ll 1$, the surface "bumps" are found in correspondence to the helical windings with the z-component of the current flowing opposite to the main field. With, for example, a ℓ=2 field the surface (Eq. (2.6) and (5.1) of FFHE) is positioned with respect to the helical windings as shown in Fig. 1.

3. HELICAL EQUILIBRIA

We recall here some properties of helical MHD equilibria (SOLOV'EV, 1967).

From $\underline{B} = \nabla \times \underline{A}$, where $\underline{A}$ is the vector potential, we define the magnetic-field flux function, which describes the magnetic surfaces:

$$\psi = A_z + arA_\phi = \text{const.}$$

This is the component of $\underline{A}$ parallel to the helix (2.1) θ = const. (see Fig. 2) and it corresponds to the flux of $\underline{B}$ between the helix and the z-axis.

From the definition of ψ it follows that

$$B_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad B_\phi - arB_z = -\frac{\partial\psi}{\partial r}. \tag{3.1}$$

From $\nabla \times B = J$ we define the plasma-current flux function

$$I(\psi) = B_z + arB_\phi . \tag{3.2}$$

It follows that

$$B_\phi = \frac{arI - \partial\psi/\partial r}{1 + a^2r^2}, \qquad B_z = \frac{I + ar\partial\psi/\partial r}{1 + a^2r^2},$$

$$J_r = \frac{1}{r}\frac{\partial I}{\partial\theta}, \qquad J_\phi - arJ_z = -\frac{\partial I}{\partial r},$$

$$J_z + arJ_\phi = -(1+a^2r^2)\Delta^*\psi + 2aI/(1+a^2r^2),$$

where $\Delta^* = (1/r)\{\partial/\partial r[r/(1+a^2r^2)\ \partial/\partial r]\} + (1/r^2)\partial^2/\partial\theta^2$.

From the equilibrium equation $\nabla P = J \times B$ we have the basic equation

$$\Delta^*\psi + (1/2)\ (dI^2/d\psi)/(1+a^2r^2) - 2aI/(1+a^2r^2)^2 +$$
$$+ dP/d\psi = 0, \tag{3.3}$$

where the functions $P(\psi)$ and $I(\psi)$ can be imposed arbitrarily. They determine ψ, and together with ψ all the other quantities in the problem. For the force-free field we shall use the solution obtained in Sec. 3 of FFHE.

Adding the squares of the components of the magnetic field B, we obtain

$$(1+a^2r^2)B^2 = |\nabla\psi|^2 + I^2. \tag{3.4}$$

If we integrate the equilibrium equation

$$\nabla P = J \times B = (\nabla \times B) \times B = (B \cdot \nabla)B - \nabla B^2/2$$

across a pressure discontinuity surface, we obtain the jump condition

$$[P] + [B^2]/2 = 0, \qquad (3.5)$$

where the square brackets indicate the discontinuity.

Rewriting (3.4) in difference form and eliminating $[B^2]$ with (3.5), we obtain the jump condition for a helically symmetric equilibrium:

$$[|\underline{\nabla}\psi|^2] + [I^2] + 2(1+a^2r^2)[P] = 0. \qquad (3.6)$$

Here again, as in (3.3), P and I are arbitrary. At the discontinuity ψ and $\underline{A}$ are continuous but have a knee, P, I, $\underline{B}$, $\underline{\nabla}\psi$ have a jump, $\underline{\nabla}P$, $\underline{J}$, $\Delta^*\psi$, $\underline{\nabla}I$ have a peak described by a δ-function.

4. THE BOUNDARY CONDITIONS AND THE JUMP CONDITION

At a boundary S (between a region 1 with current and a region 2 without current, see the following section) with no pressure jump the magnetic field components (3.1), (3.2) inside and outside the magnetic surface S have to match. $\underline{A}$ and ψ are continuous and smooth on S, while $\underline{\nabla}\psi$, $\underline{B}$, I are continuous but have a knee, and $\underline{J}$ has a jump. As in Sec. 4 of FFHE, we require I and $\underline{\nabla}\psi$ to be continuous on S:

$$I^{(1)} = I^{(2)} \qquad (4.1)$$

$$\partial\psi^{(1)}/\partial r = \partial\psi^{(2)}/\partial r \qquad (4.2)$$

$$\partial\psi^{(1)}/\partial\theta = \partial\psi^{(2)}/\partial\theta. \qquad (4.3)$$

The surface S is a free boundary, i.e. S itself is determined by the solution to the problem. In order to solve it, we assume that the section of S is almost circular and hence is described by $r(\theta) = R + \epsilon r_1(\theta)$, where $\epsilon \ll 1$. This implies $\psi(r, \theta) = p(r) + \epsilon q(r, \theta)$. In this case the problem is reduced to a system of linear equations for the coefficients in the solution for Ψ (see the following section).

We expand (4.1), (4.2), (4.3) formally in ε. Eq. (4.1) gives only one condition, of order 1, since I (3.2) is a constant on S. For a force-free field ($\underline{\nabla}\times\underline{B}^{(1)} = s\underline{B}^{(1)}$) with $s\neq 0$ we have $I^{(1)} = c^{(1)} + s\Psi^{(1)}$ (Eq. (2.7) of FFHE), and to order 1

$$I^{(1)} = c^{(1)} + sp^{(1)}. \qquad (4.4)$$

For a vacuum field (s = 0) see (5.4).

Eq. (4.2) to order 1 is

$$p^{(1)\prime} = p^{(2)\prime} \qquad (4.5)$$

and to order ε it can be written in the convenient form

$$q^{(1)}[\log(q^{(1)}/p^{(1)\prime})]' = q^{(2)}[\log(q^{(2)}/p^{(2)\prime})]', \qquad (4.6)$$

as in Eq. (4.5.4) of FFHE. Here the prime means d/dr, and all quantities are calculated for r = R.

Eq. (4.3) gives only one condition, of order ε:

$$q^{(1)} = q^{(2)}. \qquad (4.7)$$

Since $r_1 = -q/p'$, from (4.5) we see that (4.7) is also the condition that S be a common surface for the two fields.

Our conditions on S are (4.1), (4.5), (4.6), (4.7).

At a boundary S^o (between regions 0 and 1, both with current, see the next section) with a pressure jump, the magnetic field, i.e. both I and $\underline{\nabla}\Psi$, are discontinuous. I approaches two different values (constant on each surface) on either side of S^o, and $\underline{\nabla}\Psi$ remains parallel to itself across S^o ($\underline{\nabla}\Psi^{(0)} = \nu\underline{\nabla}\Psi^{(1)}$):

$$I^{(0)} = \mu I^{(1)} \qquad (4.8)$$

$$\partial\psi^{(0)}/\partial r = V\partial\psi^{(1)}/\partial r \quad (4.9)$$

$$\partial\psi^{(0)}/\partial\theta = V\partial\psi^{(1)}/\partial\theta \quad (4.10)$$

where u and V are constant on S^0, and $V = v + \epsilon w(\theta)$.

The surface S^0 is also a free boundary and we expand in ϵ as before. S^0 is described by $r = R^0 + \epsilon r\xi(\theta)$. We have $I^{(0)} = c^{(0)} + s^0\psi^{(0)}$ and Eq.(4.8) gives only one condition, of order 1:

$$c^{(0)} + s^0 p^{(0)} = u(c^{(1)} + sp^{(1)}). \quad (4.11)$$

Eq. (4.9) to order 1 is

$$p^{(0)\prime} = vp^{(1)\prime}, \quad (4.12)$$

and to order ϵ

$$q^{(0)\prime} + p^{(0)\prime\prime}r\xi = v(q^{(1)\prime} + p^{(1)\prime\prime}r\xi) + wp^{(1)\prime} \quad (4.13)$$

where $r\xi = -q/p'$. The prime indicates d/dr, and all quantities are calculated at $r = R^0$ ($< R$).

The jump condition (3.6) can also be expanded. Writing $[g] = g^{(0)} - g^{(1)}$ for the discontinuity of each quantity g, we have

$$(1-1/V^2)|\underline{\nabla}\psi^{(0)}|^2 + (1-1/u^2)I^{(0)2} + 2(1+a^2r^2)P = 0,$$

where $[P] = P$ since the pressure is P = const. in region 0 and vanishes in region 1 (see the next section). To order 1

$$(1-1/v^2)p^{(0)\prime 2} + (1-1/u^2)I^{(0)2} + 2(1+a^2R^{02})P = 0. \quad (4.14)$$

This relates u, v, P. To order ϵ

$$(w/v^3)p^{(0)\prime 2} + (1-1/v^2)(p^{(0)\prime}q^{(0)\prime} - p^{(0)\prime\prime}q^{(0)}) +$$
$$+ 2a^2R^0 r\xi P = 0. \quad (4.15)$$

Eliminating w between (4.13) and (4.15), we obtain

$$vq^{(0)}\{[\log(q^{(0)}/p^{(0)\prime})]' - 2a^2R^0P/p^{(0)\prime 2}\} =$$
$$= q^{(1)}[\log(q^{(1)}/p^{(1)\prime})]'. \quad (4.16)$$

Eq.(4.10) yields only one condition, of order ϵ, which is also the condition that S^0 be a common surface for the two fields:

$$q^{(0)} = vq^{(1)}. \qquad (4.17)$$

Our conditions on S^0 are (4.11), (4.12), (4.16), (4.17).

5. SHARP-BOUNDARY HELICAL PINCH

Let us consider the configuration described as case 3 in Sec. 1. It consists of the following regions and boundaries (see Fig. 1):

0) a high-β, current-carrying, flat-pressure-profile plasma column of constant pressure P. The boundary is the magnetic surface S^0, where surface currents balance the pressure discontinuity. The field is force-free ($\underline{\nabla}\times\underline{B}^{(0)} = s^0\underline{B}^{(0)}$) and is given in Sec. 3 of FFHE in terms of Bessel functions. The magnetic surfaces are given by

$$I^{(0)} = c^{(0)} + s^0\psi^{(0)} = c^0J^0 + s^0 \sum_n c^0_n [0]_{1n} \cos n\theta, \qquad (5.1)$$

where $J^0 = J_0(s^0r) + arJ_1(s^0r)$,

$$[0]_n = [s^0 I_n(k^0r) - ak^0r\, I'_n(k^0r)]/na^2,$$

$$k^{02} = n^2a^2 - s^{02}.$$

The sums in (5.1), (5.2), (5.3) start with n=1.

1) a low-β (zero-pressure) plasma region, whose boundaries are S^0 and the magnetic surface S tangent to the limiter or wall (a cylinder of radius R). Also in this region there is plasma current. The field here is also force-free ($\underline{\nabla}\times\underline{B}^{(1)} = s\underline{B}^{(1)}$), is discontinuous at S^0, and is also given in Sec. 3 of FFHE. The Y and K terms have to be

added here owing to the currents flowing inside and on S^o. The magnetic surfaces are given by

$$I^{(1)} = c^{(1)} + s\psi^{(1)} = cJ + c'Y + s\sum_n c_{1n} [1]_n \cos n\theta + $$

$$+ s\sum_n c'_{1n} [1']_n \cos n\theta, \quad (5.2)$$

where $J = J_0(sr) + arJ_1(sr)$,

$Y = Y_0(sr) + arY_1(sr)$,

$[1]_n = [sI_n(kr) - akr\, I'_n(kr)]/na^2$,

$[1']_n = [sK_n(kr) - akr\, K'_n(kr)]/na^2$,

$k^2 = n^2a^2 - s^2$.

2) a vacuum region ($\underline{\nabla}\times\underline{B}^{(2)} = 0$) between the surface S and the cylinder C where the windings lie. The field is described by Eq. (4.1) of FFHE, and the magnetic surfaces by Eq. (4.3) of FFHE:

$$\psi^{(2)} = B^o \frac{ar^2}{2} - c_2 \log(r/R) - r\sum_n b_n I'_n(nar) \cos n\theta +$$

$$- ar\sum_n c_{4n} K'_n(nar) \cos n\theta, \quad (5.3)$$

where B^o, b_n are the vacuum-field coefficients (2.2), (2.3), (2.5). The log and K' terms are due to the currents flowing inside S. For a vacuum field (s=0), I is a constant: from (3.2) we have

$$I^{(2)} = B^o + ac_2. \quad (5.4)$$

3) a vacuum region outside C ($\underline{\nabla}\times\underline{B}^{(3)} = 0$). The field is described by Eq. (4.2) of FFHE:

$$\underline{B}^{(3)} = c_2/r\, \underline{e}_\phi + \sum_n c_{6n} \underline{\nabla}K_n(nar) \sin n\theta,$$

where $c_{6n} = c_{4n} + b_n I'_n(naA)/aK'_n(naA)$, and A is the radius of C.

Eq. (5.1), (5.2), (5.3) are solutions of (3.3) with $I = c + s\psi$ (where $s = s^o$, s, 0 respectively) and P = const. The coefficients in the RHS of (5.1), (5.2), (5.3) are determined by the boundary conditions. The Fourier components of the currents (2.4) in the external windings, or, equivalently, of the vacuum field (2.3), are the known terms in the equations and determine the coefficients. Conversely, knowledge of the coefficients (i.e of the field, surfaces, boundary) yields the currents immediately. In this case the equations are viewed as consistency relations which limit the arbitrariness of the coefficients.

To order 1, from (4.1), (4.4), (5.4), and (4.5) on S ($r = R$), and from (4.11), (4.12) on S^o ($r = R^o$), using (5.1), (5.2), (5.3), remembering that $\psi(r, \theta) = p(r) + \epsilon q(r, \theta)$, we have the following linear system:

$$
\begin{aligned}
cJ &+ c'Y - c_2 a &&= B^o \\
cJ'/s &+ c'Y'/s + c_2/R &&= aRB^o \\
cuJ^+ &+ c'uY^+ - c^oJ^o &&= 0 \\
cvJ'^+/s &+ c'vY'^+/s - c^oJ^{o\prime}/s^o &&= 0
\end{aligned}
$$

where
$J = J\,(r = R)$, $J' = dJ/dr\ (r = R)$,
$J^+ = J\,(r = R^o)$, $J'^+ = dJ/dr\ (r = R^o)$,
$Y = Y\,(r = R)$, $Y' = dY/dr\ (r = R)$,
$Y^+ = Y\,(r = R^o)$, $Y'^+ = dY/dr\ (r = R^o)$,
$J^o = J^o(r = R^o)$, $J^{o\prime} = dJ^o/dr\ (r = R^o)$.

Notice that $dJ/dr = s[arJ_0(sr) - J_1(sr)]$. Analogous equalities hold for all the other J and Y terms. The solution is

$$c = B^o(1+a^2R^2)(uY^+J^{o\prime}/s^o - vY'^+J^o/s)/R\Delta$$

$$c' = -B^o(1+a^2R^2)(uJ^+J^{o\prime}/s^o - vJ'^+J^o/s)/R\Delta$$

$$c_2 = B^o[(J'/s - aRJ)(vY'^+J^o/s - uY^+J^{o\prime}/s^o) + (Y'/s - aRY)(uJ^+J^{o\prime}/s^o - vJ'^+J^o/s)]/\Delta$$

$$c^o = -(B^o/R\Delta)(1+a^2R^2)(uv/s)(2/\pi)(1+a^2R^{o2})/R^o$$

where $\Delta = (aJ'/s + J/R)(uY^+J^{o\prime}/s^o - vY'^+J^o/s) + - (aY'/s + Y/R)(uJ^+J^{o\prime}/s^o - vJ'^+J^o/s)$.

In c^o appears the Wronskian of (3.3) for $I^{(1)} = c^{(1)} + s\psi^{(1)}$ (5.2), $P =$ const., and $\partial/\partial\theta = 0$, in $r = R^o$:

$$J^+Y'^+ - J'^+Y^+ = (2/\pi)(1+a^2R^{o2})/R^o.$$

The constants $c^{(0)}$ (5.1) and $c^{(1)}$ (5.2) are determined by the continuity of ψ on S^o and S.

To order ϵ we equate the Fourier components and for each n, using (5.1), (5.2), (5.3), we have (4.17), (4.16) on S^o ($r = R^o$), and (4.7), (4.6) on S ($r = R$):

$$\begin{aligned}
&c^o_{1n}[0] - c_{1n}v[1]^o - c'_{1n}v[1']^o &= 0\\
&c^o_{1n}\{0\} - c_{1n}\{1\}^o/v - c'_{1n}\{1'\}^o/v &= 0\\
&c_{1n}[1] + c'_{1n}[1'] + c_{4n}aRK' &= -b_n RI'\\
&c_{1n}\{1\} + c'_{1n}\{1'\} + c_{4n}aR(K)K' &= -b_n R(I)I'
\end{aligned}$$

where $[0] = [0]_n \quad (r = R^o)$,

$$\{0\} = [0]_n[(d/dr)\log\{[0]_n/(dJ^o/dr)\} + - 2a^2R^oP/(c^oJ^{o\prime}/s^o)^2] \quad (r = R^o),$$

$$[1] = [1]_n \quad (r = R), \qquad [1'] = [1']_n \quad (r = R).$$

$$[1]^{o} = [1]_{n} \quad (r = R^{o}), \qquad [1']^{o} = [1']_{n} \quad (r = R^{o}),$$

$$\{1\} \;= [1]_{n} \,(d/dr)\log\{[1]_{n} /[(d/dr)(cJ+c'Y)]\} \quad (r = R),$$

$$\{1'\} \;= [1']_{n} \,(d/dr)\log\{[1']_{n} /[(d/dr)(cJ+c'Y)]\} \quad (r = R),$$

$$\{1\}^{o} \;= [1]_{n} \,(d/dr)\log\{[1]_{n} /[(d/dr)(cJ+c'Y)]\} \quad (r = R^{o}),$$

$$\{1'\}^{o} = [1']_{n} \,(d/dr)\log\{[1']_{n} /[(d/dr)(cJ+c'Y)]\} \quad (r = R^{o}),$$

$$I' = I'_{n}(naR), \quad K' = K'_{n}(naR),$$

$$(I) = (d/dr)\log[rI'_{n}(nar)/(B^{o}ar-c_{2}/r)] \quad (r = R),$$

$$(K) = (d/dr)\log[rK'_{n}(nar)/(B^{o}ar-c_{2}/r)] \quad (r = R).$$

Substituting (5.2) and (5.3) in (4.6) and (4.16) we have used the fact that the latter are additive in q. The solution is

$$c^{o}_{1n} = b_{n} RI'[(I)-(K)]([1]^{o}\{1'\}^{o} - \{1\}^{o}[1']^{o})/\Delta'$$

$$c_{1n} = b_{n} RI'[(I)-(K)]([0]\{1'\}^{o}/v - v\{0\}[1']^{o})/\Delta'$$

$$c'_{1n} = -b_{n} RI'[(I)-(K)]([0]\{1\}^{o}/v - v\{0\}[1]^{o})/\Delta'$$

$$c_{4n} = b_{n} I'\{([1'](I)-\{1'\})([0]\{1\}^{o}/v - v\{0\}[1]^{o}) - ([1](I)-\{1\})([0]\{1'\}^{o}/v - v\{0\}[1']^{o})\}/aK'\Delta'$$

where $\Delta' = ([1](K)-\{1\})([0]\{1'\}^{o}/v - v\{0\}[1']^{o}) +$

$$- ([1'](K)-\{1'\})([0]\{1\}^{o}/v - v\{0\}[1]^{o}).$$

The solution thus found for the coefficients is valid for $b/B^{o} = \epsilon << 1$, and for any value of ar (2.1). The results obtained can be plotted numerically, or expanded for $ar = \delta << 1$ in order to continue the analysis. R, R^{o}, r are comparable, and a, s, s^{o} are of the same order. Three dimensionless parameters enter the problem: we expand to first order in $b/B^{o} = \epsilon$ to solve the free-boundary problem, the rather complicated result can then be expanded in $ar = \delta$ to obtain simpler expressions, while s/a remains finite throughout.

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List of symbols used:

β	beta
γ	gamma
∂	partial derivative
δ	delta
Δ	capital delta
$\underline{\nabla}$	del (grad; $\underline{\nabla}\cdot$ div; $\underline{\nabla}\times$ curl)
ϵ	epsilon
θ	theta
π	pi
Σ	sum (capital sigma)
ϕ	phi
ψ	psi
ℓ	el (handwritten)

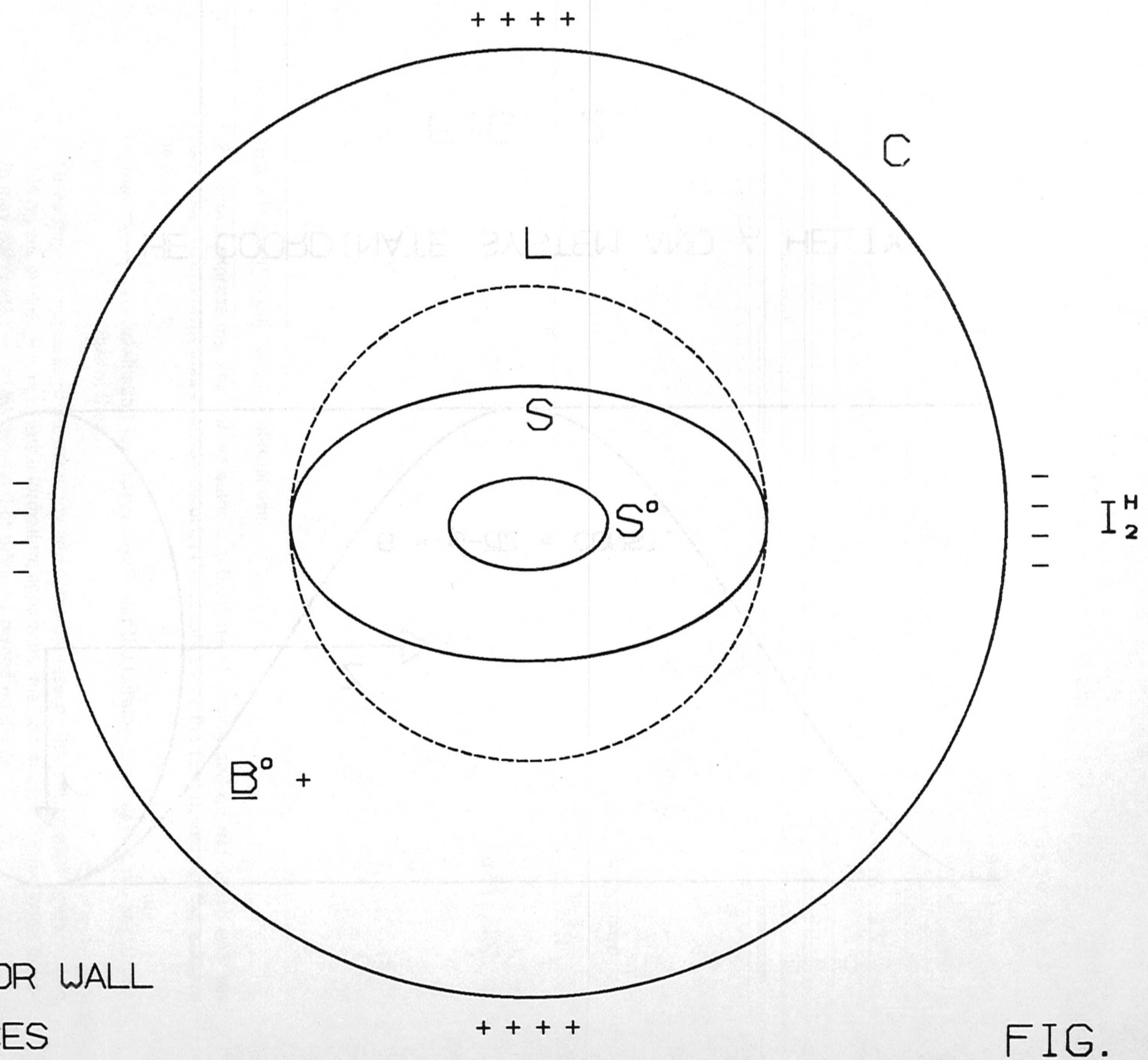
Superscript:

^{H} H

Subscript:

$_1$ 1

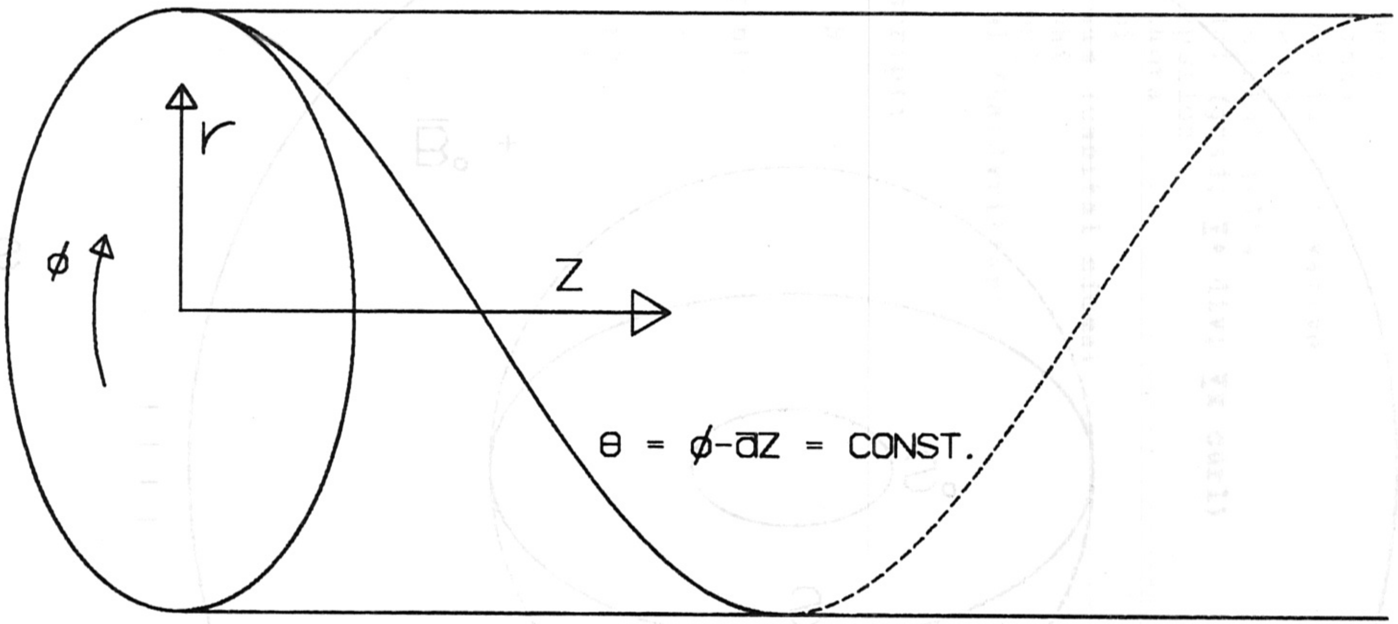
r_1^o r^o_1



SECTIONS
AT Z=0

C: CYLINDER
L: LIMITER OR WALL
S, S°: SURFACES

FIG. 1



THE COORDINATE SYSTEM AND A HELIX

FIG. 2