Stability of boundaries of tokamaks with
respect to rigid displacements.

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Abstract

The stability of axially symmetric equilibria with respect to rigid
displacements is studied by means of the energy principle. For
horizontal displacements and flipping the vacuum term is neglected,
thus yielding sufficient stability criteria. For vertical displacements
the vacuum term is kept, thus yielding a necessary and sufficient
condition. The resulting stability criteria are applied to simple
algebraic equilibria and numerical results are presented.

General considerations and stability criteria.

Because of the negative effect of impurities in recent tokamak research
there is a trend towards confinement without the use of material
limiters. With regard to MHD stability, in purely magnetically confined
plasmas perturbations become possible which move the whole plasma
column quasi-rigidly towards the wall. In this paper, the energy
principle [1] is used to study the stability of axisymmetric configurations
with respect to rigid perturbations. Consistently with the use of the
energy-principle it is assumed that any conductors in the vacuum region
surrounding the plasma are either superconducting or completely
permeable (i.e. non-existent) for the perturbational fields.
Real physical perturbations must satisfy two boundary conditions at the
plasma-vacuum interface, namely

$$-\gamma p \, div \, \underline{\xi} + \underline{B} \cdot \left[curl(\underline{\xi} \times \underline{B}) + \underline{\xi} \cdot \nabla \underline{B} \right] = \tag{1}$$
$$\underline{B}_v \cdot \left[\delta \underline{B}_v + \underline{\xi} \cdot \nabla \underline{B}_v \right]$$

$$\underline{n} \times \delta \underline{A} = - \underline{n} \cdot \underline{\xi} \, \underline{B}_v \tag{2}$$

For rigid perturbations, $\underline{\xi}$ is a given function and thus the
perturbational vacuum field $\delta \underline{B}_v = curl \, \delta \underline{A}$ is completely
determined by (2). Condition (1) cannot be additionally fulfilled
and thus rigid perturbations are no physically possible motions but may
only be used as test functions. However, this implies using a form of
the energy principle where $\delta^2 W$ splits into plasma and vacuum

contributions [1]. According to [2] the form

$$\delta^2 W = \delta^2 W_{Pl} + \delta^2 W_{vac} \tag{3}$$

$$\delta^2 W_{Pl} = \frac{1}{2} \int_{Pl} \left\{ \left[\underline{B} \cdot \nabla \underline{\xi} - \underline{B} \, div \, \underline{\xi} \right]^2 + \underline{B} \cdot \nabla \underline{B} \cdot \left[\underline{\xi} \, div \, \underline{\xi} - \underline{\xi} \cdot \nabla \underline{\xi} \right] \right.$$
$$\left. + \frac{5}{3} p (div \, \underline{\xi})^2 \right\} d\tau + \frac{1}{2} \int_S \underline{\xi} \cdot \nabla p^* \, \underline{\xi} \cdot d\underline{S}$$

$$\delta^2 W_{vac} = \frac{1}{2} \int_{vac} \delta \underline{B}_v^2 \, d\tau = \frac{1}{2} \int_S \underline{B}_v \cdot \delta \underline{B}_v \, \underline{\xi} \cdot d\underline{S}$$

may be taken. (The surface integral in the expression for $\delta^2 W_{Pl}$ has been omitted by mistake in [2]). In (3) it is assumed that in equilibrium there are no surface currents. As a consequence, we have

$$\underline{B} = \underline{B}_v \tag{4}$$

at the plasmaboundary. Furthermore, the equilibrium condition

$$\nabla p^* = \nabla \left(p + \frac{B^2}{2} \right) = \underline{B} \cdot \nabla \underline{B} \tag{5}$$

has been used. For the most general rigid perturbation

$$\underline{\xi} = \underline{\xi}_0 + \underline{\omega} \times \underline{X} , \quad \underline{\xi}_0 = const , \quad \underline{\omega} = const \tag{6}$$

we have

$$div \, \underline{\xi} = 0 , \quad \underline{B} \cdot \nabla \underline{\xi} = \underline{\omega} \times \underline{B} , \quad \underline{\xi} \cdot \nabla \underline{\xi} = \underline{\omega} \times \underline{\xi}$$

and

$$(\underline{\omega} \times \underline{\xi}) \cdot (\underline{B} \cdot \nabla \underline{B}) = (\underline{\omega} \times \underline{B})^2 + div \, \underline{B} \left[\underline{B} \cdot (\underline{\omega} \times \underline{\xi}) \right]$$

Thus, after partial integration and with use of (4) it is seen that $\delta^2 W$ reduces to

$$\delta^2 W = \frac{1}{2} \int_S \underline{\xi} \cdot \nabla p^* \, \underline{\xi} \cdot d\underline{S} + \frac{1}{2} \int_S \underline{B} \cdot \delta \underline{B}_v \, \underline{\xi} \cdot d\underline{S} \tag{7}$$

Using Cartesian coordinates x, y ,z and cylindrical coordinates R, Θ , z , the most general non-trivial rigid perturbation of axially symmetric configurations may be written

$$\underline{\xi} = \xi \underline{e}_x + \eta \underline{e}_y + \zeta \underline{e}_z + \omega \underline{e}_x \times \underline{x}$$

$$= (\xi \cos\theta + \eta \sin\theta - \omega z \sin\theta) \underline{e}_R + \tag{8}$$

$$(-\xi \sin\theta + \eta \cos\theta - \omega z \cos\theta) \underline{e}_\theta + (\zeta + \omega R \sin\theta) \underline{e}_z$$

where $\underline{e}_x, \underline{e}_y, \cdots$ are unit vectors parallel to the $x, y, \cdots$ -axis and $\underline{x}$ is the position vector.

In axisymmetric equilibrium $\underline{B}$ may be represented by a flux function $\psi(R, z)$

$$\underline{B} = \nabla\theta \times \nabla\psi + \Lambda(\psi) \nabla\theta \tag{9}$$

At the plasma-vacuum interface we have ψ = const and therefore with $dl = \sqrt{dR^2 + dz^2}$,

$$\underline{\xi} \cdot d\underline{S} = \underline{\xi} \cdot \nabla\psi \, \frac{R \, d\theta \, dl}{|\nabla\psi|} \tag{10}$$

Combining equations (8) and (10) we obtain after little calculation

$$\delta^2 W_{PR} = \frac{1}{2} \int_S \underline{\xi} \cdot \nabla p^* \, \underline{\xi} \cdot d\underline{S}$$

$$= \frac{\pi}{2} (\xi^2 + \eta^2) \oint \frac{R \psi_R}{|\nabla \psi|} p_R^* \, d\ell + \pi \zeta^2 \oint \frac{R \psi_z}{|\nabla \psi|} p_z^* \, d\ell +$$

$$\frac{\pi}{2} \omega^2 \oint \frac{R(R\psi_z - z\psi_R)}{|\nabla \psi|} (R p_z^* - z p_R^*) \, d\ell +$$

$$\frac{\pi}{2} \omega \eta \oint [(R\psi_z - 2z\psi_R) p_R^* + R\psi_R p_z^*] \frac{R}{|\nabla \psi|} \, d\ell$$

In the following, we shall assume that the configurations under consideration are symmetric with respect to the $z = 0$ plane, i.e.

$$\psi(R, -z) = \psi(R, z) \tag{12}$$

Then, the integral with $\omega \eta$ in front has an antisymmetric integrand and vanishes.

We shall now turn to the vacuum contribution in equ. (7). $\delta \underline{A}$ must satisfy $\Delta \delta \underline{A} = 0$ in the vacuum region and equation (2) at the boundaries. Since the right-hand side of (2) is a linear function of ξ, η, ζ and ω, for $\delta \underline{B}_v = \text{curl} \, \delta \underline{A}$ the following decomposition is possible:

$$\delta \underline{B}_v = \xi \, \underline{b}_\xi + \eta \, \underline{b}_\eta + \int \underline{b}_\zeta + \omega \, \underline{b}_\omega \tag{13}$$

If $\underline{b}_\xi, \underline{b}_\eta, \ldots$ are now Fourier-analyzed with respect to Θ, it follows from (2) and (8) that $\underline{b}_\xi, \underline{b}_\eta$ and $\underline{b}_\omega$ are linear functions of $\sin\Theta$ and $\cos\Theta$, whereas $\underline{b}_\zeta$ is independent of Θ. Thus using the orthogonality relations of angular functions, we have

$$\delta^2 W_v = \frac{1}{2} \delta^2 W_v^\zeta + \frac{1}{2} \int_{Vac} (\xi \underline{b}_\xi + \eta \underline{b}_\eta + \omega \underline{b}_\omega)^2 d\tau \tag{14}$$

$$\delta^2 W_v^\zeta = \frac{1}{2} \int^2 \int_{Vac} b_\zeta^2 \, d\tau = \frac{1}{2} \int^2 \int_S \underline{B} \cdot \underline{b}_\zeta \, \underline{e}_z \cdot d\underline{S}$$

It is seen from (7), (11) and (14) that the vertical displacement separates completely from the other rigid perturbations. These, in turn, separate from each other, which, however, will not be proven here since we shall not need this fact below.

Collecting the terms with $\int^2$ in (11) and (14) yields a necessary and sufficient stability condition for vertical displacements

$$\oint \frac{R \psi_z}{|\nabla \psi|} p_z^* \, dl + \frac{1}{2\pi} \int_S \underline{B} \cdot \underline{b}_\zeta \, \underline{e}_z \cdot d\underline{S} \geqslant 0 \tag{15}$$

For the remaining perturbations we shall be satisfied with a sufficient stability condition by omitting the positive vacuum contribution. From (11) we obtain the sufficient criteria

$$\oint \frac{R\,\psi_R}{|\nabla \psi|}\, p^*_R \, dl \;\geqslant\; 0 \qquad (16)$$

$$\oint \frac{R(R\psi_z - z\psi_R)}{|\nabla \psi|}\left(Rp^*_z - z\,p^*_R\right) dl \;\geqslant\; 0 \qquad (17)$$

If (16) or (17) is fulfilled, stability with respect to horizontal displacements or flipping is achieved. If (15) - (17) are satisfied, stability will respect to any arbitrary rigid perturbation is achieved. In the following, we shall investigate the criteria (15) - (17) for a class a very simple analytic axisymmetric equilibria.

Equilibrium

The flux function ψ defined in (9) obeys the equation

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} \;=\; -R^2 p'(\psi) - \Lambda \Lambda'(\psi) \qquad (18)$$

For $p' = p_1 = const$ and $\Lambda\Lambda' = g_1 = const$ Solovev' has given the following exact solution [3]:

$$\psi = \left(-g_1 + cR^2\right)\frac{z^2}{2} - \frac{(p_1 + c)}{8}\left(R^2 - R_0^2\right)^2 \qquad (19)$$

Besides p_1 and g_1, ψ contains the integration constants c and R_0 as parameters. Integration of the relation $\Lambda\Lambda' = g_1$ yields an integration constant Λ_0 :

$$\Lambda(\psi) = \sqrt{\Lambda_o^2 + 2 g_1 \psi} \tag{20}$$

A last parameter of equilibrium is given by the value ψ^* of the flux function at the plasma-vacuum boundary. Consistently with our assumption that in equilibrium no surface currents are present, we must have $p(\psi^*) = 0$ and, therefore,

$$p(\psi) = p_1 (\psi - \psi_*) \tag{21}$$

In the following, we shall pass over to other parameters and to dimensionless quantities in a similar manner to Lortz and Nührenberg [4], thus reducing the number of free parameters. If ι is the value of the rotational transform on the magnetic axis at $R = R_o$, $z = 0$, if e is the ratio $\frac{b}{a}$ of the vertical axis b and the horizontal axis a of the elliptical flux surfaces in the neighbourhood of the magnetic axis and if β_p is defined by

$$\beta_p = \lim_{\bar{\psi} \to 0} \frac{\int_o^{V(\bar{\psi})} p_1 (\psi - \bar{\psi}) \, d\tau}{\int_o^{V(\bar{\psi})} (\nabla \theta \times \nabla \psi)^2 \, d\tau}$$

then from (19) - (20) the following relations are obtained

$$\iota = \frac{R_o^2}{\Lambda_o} \sqrt{(p_1 + c)(g_1 - c R_o^2)} \tag{22}$$

$$e = \sqrt{\frac{(p_1 + c)\, R_0^2}{(g_1 - c\, R_0^2)}} \tag{23}$$

$$\beta p = \frac{R_0^2\, p_1}{g_1 + p_1 R_0^2} \tag{24}$$

Instead of ι we shall also use the safety factor

$$q = 1/\iota \tag{25}$$

Finally, with the definition

$$A = R_0 / a(\psi^*)$$

for the aspect ratio A, where $2a(\psi^*)$ is the diameter of the flux surface $\psi = \psi^*$ at $z = 0$, we obtain

$$\psi^* = -\frac{1}{2}(p_1 + c)\, R_0^4\, \frac{(A^2 - 1)}{A^4} \tag{26}$$

We shall now change our notation for all dimensioned quantities thus, that $R \to \tilde{R}$, $\psi \to \tilde{\psi}$ etc., in order to introduce dimensionless quantities R, ψ etc. by

$$R = \tilde{R}/\tilde{R}_0 \quad , \quad z = \tilde{z}/\tilde{R}_0 \tag{27}$$

$$\psi = -2\tilde{\psi}/[(\tilde{g}_1 - \tilde{c}\,\tilde{R}_0^2)\, e\, \tilde{R}_0^2]$$

$$\Lambda = \tilde{\Lambda}/\tilde{\Lambda}_0$$

With this normalization and using equations (22) - (24), we obtain

from (19) and (20) the relations

$$\psi = \frac{1}{e}\left[Q + (1-Q)R^2\right]z^2 + \frac{e}{4}(R^2-1)^2 \tag{28}$$

$$\Lambda = \sqrt{1 - \frac{Q}{e}\iota^2\psi} \tag{29}$$

where

$$Q = (1+e^2)(1-\beta p) \tag{30}$$

Acoording to (26), the normalized boundary value of ψ is given by

$$\psi^* = e\,\frac{A^2-1}{A^4} \tag{31}$$

Since βp must be non-negative, it follows from (30) that

$$Q \leq 1+e^2 \tag{32}$$

Furthermore, since Λ should be real, we obtain the condition

$$q \geq \frac{1}{A^2}\sqrt{Q(A^2-1)} \tag{33}$$

from (25), (29) and (31).

The flux function given by (28) has stagnation points at $R=0, z=0$
and at $R=\sqrt{Q/(Q-1)}, z = \pm\, e/[\sqrt{2}(Q-1)]$. Since these could

at best lie on the plasma boundary, the following limitations for the

aspect ratio are obtained:

$$A \geqslant \sqrt{2(Q-1)\left[Q-1-\sqrt{(Q^2-2Q)}\right]} \qquad \text{for} \qquad Q < 0 \qquad (34)$$

$$A \geqslant \sqrt{2} \qquad \text{for} \qquad 0 \leq Q \leq 2$$

$$A \geqslant \sqrt{2(Q-1)\left[Q-1+\sqrt{(Q^2-2Q)}\right]} \qquad \text{for} \qquad Q \geqslant 2$$

Roughly speaking, Q is a shape parameter which characterises the triangular deformation of the magnetic flux surfaces. If Q has a value between 0.2 and 0.5, the cross-sections look rather ellipse-like, depending somewhat on A. More precisely, if

$$Q = \frac{2A^2 - 3}{4A^2 - 3} \qquad (35)$$

the top point of the plasma boundary lies vertically above the middle between the outermost and innermost points. If Q is larger or smaller, a triangular deformation appears. Plots of different plasma shapes are shown in Fig. 1 (see also ref. [3]). For later purposes, we finally give an expression for the volume enclosed by the flux surface ψ^*:

If $Q \neq 1$, we have

$$V(\psi^*) = \frac{4\pi}{3} \sqrt{2} \frac{e}{(1-Q)^2} \sqrt{1+\gamma} \left[E(k^*) - \gamma K(k^*)\right] \qquad (36)$$

where

$$\gamma = \sqrt{1 - 4(1-Q)^2(A^2-1)/A^4}$$

$$k^* = \sqrt{(1-\gamma)/(1+\gamma)}$$

and K (k) and E (k) are the complete elliptic integrals of the

first and second kinds.

For $Q = 1$, we have

$$V(\psi^*) = 2\pi^2 e \, (A^2 - 1)/A^4 \tag{37}$$

By analytic continuation it is always possible, in principle, to find a vacuum field which matches the plasma boundary given by (28) and (31). However, we shall not need any detailed information about the equilibrium vacuum field since for flipping and horizontal displacements the vacuum term is neglected and for vertical displacements we shall only consider the case that the vacuum field coils or other external conductors are completely permeable to the perturbational field.

Horizontal displacement and flipping.

We shall first evaluate the sufficient criteria (16) and (17). From (5) and (9) we obtain (omitting the $\sim$ for dimensioned quantities)

$$p^*_R = \frac{1}{R^2}\left(\psi_z \psi_{zR} - \frac{\psi_z^2}{R} - \psi_R \psi_{zz}\right) - \frac{\Lambda^2}{R^3} \tag{38}$$

$$p^*_z = \frac{1}{R^2}\left(-\psi_z \psi_{RR} + \frac{\psi_z \psi_R}{R} + \psi_R \psi_{Rz}\right) \tag{39}$$

Inserting this in (16) and (17) and introducing dimensionless quantities with (22) - (26) and (28) - (30), we obtain for horizontal displacements the criterion

$$A_1 + 4\left(q^2 - \frac{Q}{e}\psi^*\right) A_2 \geqslant 0 \tag{40}$$

where

$$A_1 = \oint \frac{\psi_R}{R|\nabla\psi|}\left(\psi_z\psi_{zR} - \frac{\psi_z^2}{R} - \psi_R\psi_{zz}\right)d\ell \tag{41}$$

$$A_2 = -\oint \frac{\psi_R}{R^2|\nabla\psi|}\,d\ell \tag{42}$$

Analogously, from (17) for flipping the criterion

$$B_1 + 4\left(q^2 - \frac{Q}{e}\psi^*\right)B_2 \geqslant 0 \tag{43}$$

is obtained with

$$B_1 = \oint \frac{(R\psi_z - z\psi_R)}{R|\nabla\psi|}\left[R\left(-\psi_z\psi_{RR} + \frac{\psi_z\psi_R}{R} + \psi_R\psi_{Rz}\right)-\right.$$
$$\left. z\left(\psi_z\psi_{zR} - \frac{\psi_z^2}{R} - \psi_R\psi_{zz}\right)\right]d\ell \tag{44}$$

and

$$B_2 = \oint \frac{(R\psi_z - z\psi_R)z}{R^2|\nabla\psi|}\,d\ell \tag{45}$$

From (40) and (43) it is possible to derive critical values of q such that both criteria are fulfilled if $q \geqslant q_{cr}$. (The existence of such q_{cr} is to be expected from the fact that horizontal displacements and flipping are special kink modes). For this purpose we first show that both A_2 and B_2 are positive. With $\underline{n} = \nabla\psi/|\nabla\psi|$ and using Gauss's theorem for two dimensions we have

$$A_2 = -\oint \frac{e_R}{R^2}\cdot\underline{n}\,d\ell = -\int \frac{\partial}{\partial R}\frac{1}{R^2}\,df = 2\int \frac{1}{R^3}\,df > 0$$

and

$$B_2 = \oint \frac{z}{R} \, \underline{e_z} \cdot \underline{n} \, dl - \oint \frac{z^2}{R^2} \, \underline{e_R} \cdot \underline{n} \, dl = \int \frac{\partial}{\partial z} \frac{z}{R} \, df - \int \frac{\partial}{\partial R} \frac{z^2}{R^2} \, df$$

$$= \int \frac{1}{R} \, df + 2 \int \frac{z^2}{R^3} \, df \; > \; 0$$

From this and from the fact that according to (31) and (33) the bracket near A_2 is positive it follows that (37) is satisfied if either $A_1 \gtrless 0$ or if

$$|q| \gg \sqrt{\frac{Q}{e} \psi^* - \frac{A_1}{4 A_2}} \; = \; q_h \tag{46}$$

When A_1 goes from negative to positive values, the bounding condition (46) coming from stability matches continuously the condition (33) coming from equilibrium.

Analogously, (40) is satisfied if either $B_1 \gtrless 0$ or if

$$|q| \gg \sqrt{\frac{Q}{e} \psi^* - \frac{B_1}{4 B_2}} \; = \; q_t \tag{47}$$

with analogous continuation of (47) for B_1 changing sign.

It is rather simple to calculate the limiting values of q_h and q_t for large aspect ratio A by expanding ψ in the neighbourhood of the magnetic axis. To lowest order we obtain from (28) with $X = R - 1$

$$\psi \; = \; \frac{1}{e} z^2 + e x^2$$

At the plasma boundary $\psi = \psi^*$ we have

$$d\psi = \psi_R \, dR + \psi_z \, dz = 0$$

and therefore

$$dl = \left| \frac{\nabla \psi}{\psi_z} \, dx \right|$$

Using the symmetry of ψ with respect to the $z = 0$ plane, we obtain from (41) to lowest order

$$A_1 = -8e^2 \int_{-\sqrt{\psi^*/e}}^{+\sqrt{\psi^*/e}} \frac{x^2}{z} \, dx = -4\pi \psi^* \tag{48}$$

Analogously, from (42), (44) and (45) the following results are obtained:

$$A_2 = 2\pi \psi^* \quad , \quad B_1 = -4\pi \psi^* \quad , \quad B_2 = \pi \psi^* \tag{49}$$

Thus, it follows from (31) and (46) – (49)

$$\lim_{A \to \infty} q_L = 1/\sqrt{2}$$

$$\tag{50}$$

$$\lim_{A \to \infty} q_f = 1$$

the limiting values being independent of Q and e.

For finite values of A the limits q_L and q_f also depend on

e and Q besides on A . However, using (28) and (31) it follows
from (41) and (42) that A_1 and A_2 are linear functions of e so
that according to (46) q_h becomes independent of e . In Fig. 2
and Fig. 3 numerical results are presented for q_h and q_f .
In **Fig. 3** the different abscissa intervals of the curves
are due to the conditions (34) connected with the appearence of
separatrices.

It is seen that both q_h and q_f become smaller with decreasing A,
i.e. toroidal effects act as stabilizing factors. For almost all
parameter values we have $q_h < q_f$. With respect to flipping,
cross-sections with vertically elongated shape ($e > 1$) are
more favorable than those with horizontally elongated shape ($e < 1$);
triangular deformation is more stabilizing for $Q \lesssim 0$ than
for $Q \gtrsim 1$.
Furthermore, in the whole parameter regime q_h and q_f are
below the value $q = 1$ (Kruskal - Shafranow limit), which must at least
be kept in order to obtain stability with respect to other kink modes,
and thus no additional precautions need be undertaken. For the same
reason if would not pay off to take into account the neglected vacuum
term in order to obtain more favorable stability boundaries.

Vertical displacement[+)]

For the moment all quantities used this section are to be under-
stood in dimensioned quantities until defined otherwise.
Since for vertical displacements the plasma contribution to $\delta^2 W$

[+)]Vertical displacements have simultaneously been investigated at
Institut für Plasmaphysik in Garching by Lackner (paper to be
published in Nuclear Fusion) using a somewhat different method and
applying it to numerical equilibria.

given in (15) is generally negative ($\psi_z p^*_z < 0$), the vacuum contribution cannot be neglected as for the other rigid displacements. According to (39) the plasma contribution and, as we shall see below, also the vacuum contribution do not depend on the main field represented by Λ. Thus, instability with respect to vertical displacements may not be stabilized by an increase of the main field and must therefore be considered as particularly dangerous. In order to calculate the vacuum contribution in (15), the tangential components of the perturbational vacuum field are needed at the plasma boundary S. These are to be calculated from the equation

$$\Delta \, \delta_{\underline{A}} \; = \; 0$$

and from the boundary condition (2), which is essentially equivalent to prescribing the normal components of $\delta \underline{B}_v$. The problem thus posed is appreciably facilitated by the fact that $\delta \underline{B}_v$ must be axisymmetric for vertical displacements and is thus purely poloidal because of the boundary conditions at infinity ($\delta \underline{B}_v \vec{\rightarrow} 0$). We may therefore make the ansatz

$$\delta \underline{B}_v \; = \; \int \left[\, y \, (\underline{e}_\theta \times \underline{n}) \, + \, w \, \underline{n} \, \right] \tag{51}$$

where $\underline{n} = \nabla \psi / |\nabla \psi|$ and ψ is the flux function of the equilibrium-vacuum field ($\underline{n}$ will later be needed only at the plasma boundary where it is defined by (28)).

If $\int \underline{A}$ is a solution of the boundary value problem posed above, then according to Kress $\begin{bmatrix} 5 \end{bmatrix} \delta \underline{B}_v$ at the plasma boundary obeys the integral equation

$$\underline{\omega}^* = -\frac{1}{2\pi} \, \underline{n} \times \left[\underline{n} \times \left\{ \int \nabla \frac{1}{r} \times [\underline{n}' \times \underline{\omega}^{*'}] \, dS' + \right.\right.$$

$$\left.\left. \int \nabla \frac{1}{r} \, Div' \, \underline{\gamma}^{*'} \, dS' \right\} \right]$$

(52)

where

$$\underline{\omega}^* = (\underline{n} \times \delta \underline{B}_v) \times \underline{n}$$

$$r = \sqrt{(\underline{x} - \underline{x}')^2}$$

$$\underline{\gamma}^* = -\underline{n} \cdot \oint \underline{B}_v$$

$$Div \, \underline{\gamma}^* = \frac{1}{|\nabla \psi|} \, div(\underline{\gamma}^* |\nabla \psi|)$$

Additional terms due to induced currents would have to be added in (52) if the vacuum field coils or other external conductors were not completely permeable to $\delta \underline{B}_v$.

It is shown in $\begin{bmatrix} 5 \end{bmatrix}$ that the integral equation (52) for $\underline{\omega}^*$ has a unique solution provided a side condition is imposed on $\underline{\omega}^*$. For this we may take the condition that the flux of $\delta \underline{B}_v$ penetrating the circular area S_o around R = 0, which is cut out of the $\underline{z}$ = 0 plane by the plasma boundary should be zero:

$$\int_{S_o} \delta \underline{B}_v \cdot d\underline{S}_o = 0$$

(53)

Since for vertical displacements $\underline{\xi} = \int \varrho \underline{\hat{z}}$ we have

$$\operatorname{Div} \underline{y}^* = -\frac{\xi}{|\nabla \psi|} \underline{B} \cdot \nabla \psi_z = -\frac{\xi}{|\nabla \psi|} (\nabla \Theta \times \nabla \psi) \cdot \nabla \psi_z ,$$

it follows from (52) that also $\delta \underline{B}_v$ does not depend on Λ.
Using (51), $\nabla \frac{1}{r} = -(\underline{x} - \underline{x}')/r^3$, $dS = R\, d\Theta\, d\ell$ and multiplying
(52) by $\underline{e}_\Theta \times \underline{n}$, we obtain after some rearrangement an integral
equation for y

$$y = \frac{1}{2\pi} \int K(\ell, \ell')\, y'\, d\ell' + \frac{1}{2\pi} \int L(\ell, \ell')\, \sigma'\, d\ell' \tag{54}$$

$$K(\ell, \ell') = \int_0^{2\pi} \frac{R'}{r^3 |\nabla \psi|} \Big[-R'\psi_R + \big[R\psi_R + (z - z')\psi_z \big] \cos(\Theta - \Theta') \Big]\, d\Theta' \tag{55}$$

$$L(\ell, \ell') = \int_0^{2\pi} \frac{1}{r^3 |\nabla \psi|} \Big[R\psi_z - (z - z')\psi_R - R'\psi_z \cos(\Theta - \Theta') \Big]\, d\Theta' \tag{56}$$

$$\sigma = \frac{1}{|\nabla \psi|} \Big[\psi_z \psi_{zR} - \psi_R \psi_{zz} \Big] \tag{57}$$

Since the integrals in (54) are extended over the plasma boundary
$\psi(R, z) = \psi^*$, any point R, z on the boundary corresponds to a
certain integration interval ℓ. The functions $K(\ell, \ell')$ and
$L(\ell, \ell')$ are to be understood in this sense. Both may be ex-

pressed in terms of the complete elliptic integrals of the first and second kinds

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\varphi} \; d\varphi, \quad K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\varphi}} \; d\varphi$$

With

$$k^2 = \frac{4RR'}{(R+R')^2 + (z-z')^2} \tag{58}$$

and

$$r_0 = \frac{2}{k} \sqrt{RR'} \tag{59}$$

we have

$$r = r_0 \sqrt{1 - k^2 \cos^2 \varphi/2} \quad , \quad \varphi = \theta - \theta' \tag{60}$$

With the relations

$$\int_0^{2\pi} \frac{1}{r^3} \; d\varphi = \frac{4 E(k)}{r_0^3 (1 - k^2)}$$

and

$$\int_0^{2\pi} \frac{\cos\varphi}{r^3} \; d\varphi = \frac{4}{r_0^3 k^2 (1 - k^2)} \left[(2 - k^2) E(k) - 2(1 - k^2) K(k) \right]$$

we obtain from (55) and (56)

$$K(\ell, \ell') = \frac{4R'}{|\nabla\psi| \cdot r_0^3 (1-k^2)} \left\{ -R'\psi_R E(k) + \frac{1}{k^2} \left[R\psi_R + (z-z')\psi_z \right] \cdot \left[(2-k^2) E(k) - 2(1-k^2) K(k) \right] \right\}$$

(61)

$$L(\ell, \ell') = \frac{4}{|\nabla\psi| \, r_0^3 (1-k^2)} \left\{ \left[R\psi_z - (z-z')\psi_R \right] E(k) - \frac{R'\psi_z}{k^2} \left[(2-k^2) E(k) - 2(1-k^2) K(k) \right] \right\}$$

(62)

Our criterion (15) may now also be expressed in terms of ψ.

With $\underline{e}_z \cdot d\underline{S} = \psi_z R \, d\theta \, d\ell / |\nabla\psi|$ and using the θ - independence of $\underline{b}_f = \psi (\underline{e}_\theta \times \underline{n}) + w\,\underline{n}$ ((13) and (51)), we obtain by means of (9) and (39)

$$\ell_{vert} = \oint \left\{ \frac{1}{R|\nabla\psi|} \left[-\psi_z \psi_{RR} + \frac{\psi_z \psi_R}{R} + \psi_R \psi_{Rz} \right] + \psi \right\} \psi_z \, d\ell \gneqq 0 \qquad (63)$$

In this section so far dimensioned quantities have been used. If with (27) and

$$\psi = \tilde{\psi} \, \tilde{R}_0^3 \, \psi / \tilde{\psi}$$

dimensionless quantities are again introduced, the relations (54) - (63) hold without changes also in dimensionless quantities.

It is now possible to replace the side condition (53) which must be imposed on (54) by a much simpler condition. For this purpose we split ψ into a contribution which is symmetric and into one which antisymmetric in its dependence of z :

$$\psi = \psi_s + \psi_a$$

(64)

It follows from (57) – (59) and (61) – (62) that $\oint K(\ell,\ell')\, y_s'\, d\ell'$ is symmetric and that $\oint K(\ell,\ell')\, y_a'\, d\ell'$ and $\oint L(\ell,\ell')\, \delta'\, d\ell'$ are antisymmetric. Therefore, (54) splits into the two equations

$$y_s = \frac{1}{2\pi} \oint K(\ell,\ell')\, y_s'\, d\ell' \tag{65}$$

$$y_a = \frac{1}{2\pi} \oint K(\ell,\ell')\, y_a'\, d\ell' + \frac{1}{2\pi} \oint L(\ell,\ell')\, \delta'\, d\ell' \tag{66}$$

Since $y_a(R, z=0) = 0$, only the symmetric part of y enters the side condition (53). On the other hand, since ψ_z is antisymmetric according to (28), it is only the antisymmetric part of y which enters our criterion (63). The ambiguity of the solutions of the inhomogeneous integral equation (66) for y_a is removed if the antisymmetry of the solution is required as a side condition. For numerical calculations this has been done in the form

$$\oint y_a\, d\ell = 0 \tag{67}$$

which replaces the former side condition (53).

According to (58) we have $k = 1$ for $R = R'$ and $z = z'$ and hence $K(\ell,\ell')$ as well as $L(\ell,\ell')$ become singular for $\ell = \ell'$. For numerical calculations it is of advantage to remove this singularity following Martensen [6] and [7]. It is shown there that

$$\frac{1}{2\pi} \oint K(\ell,\ell')\, d\ell = 1 \quad , \quad \oint L(\ell,\ell')\, d\ell = 0$$

With this (66) may be transformed into

$$\frac{1}{2\pi} \oint \left[K(\ell',\ell)\, y_a - K(\ell,\ell')\, \dot{y}_a \right] d\ell' =$$

$$\frac{1}{2\pi} \oint \left[L(\ell,\ell')\, \sigma' - L(\ell',\ell)\, \sigma \right] d\ell' \tag{68}$$

Obviously at $\ell = \ell'$ the singular contributions cancel.

The integral equation (68) has been solved numerically together with the side condition (67) by discretisation and simply taking equidistant points on the plasma boundary. The resulting y_a was inserted in (63). Numerical results obtained for the equilibria considered in Section 3 are presented in Figs. 4-6.

In Fig. 4 ζ_{vert}/V is shown as a function of e for several A and for configurations with almost elliptical cross-section ($Q = 0.25$, see Fig. 1). The volume V^* enclosed by the plasma boundary has been calculated from (36) or (37). If e becomes larger than a certain e_{cr} (e.g. $e_{cr} \approx 1.2$ for $A = 3$. This value is in quite good agreement with values obtained by Okabayashi et al [8] by a rather different approach), the plasma becomes unstable ($\zeta_{vert}/V < 0$). The smaller A, the larger is the stable e interval; toroidal effects again acting as stabilizing factors. For $A \to \infty$ a comparison with straight cylindrical plasmas is possible: circular cross-sections ($e = 1$) are marginal, as is expected for reasons of symmetry, while standing ellipses ($e > 1$) are unstable in agreement with the results of Rutherford [8].

In Fig. 5 the critical values of e are shown as functions of the shape parameter Q for different A. (Again the different Q intervals of the curves are due to the conditions (34)). Roughly speaking, triangular

deformation is always stabilizing, and, if A is small enough, by it a considerable vertical elongation of the equilibrium-plasma shape becomes possible. This may be seen in Fig. 6 which shows three different possible plasma shapes corresponding to the points 1, 2 and 3 in Fig. 5.

In case the stability condition $e \leqslant e_{cr}$ is violated, the growth-rate of the expected instability becomes of interest. It may be estimated in the following way. (In the following $\sim$ will be used to indicate dimensioned quantities).

As was mentioned in section 1, rigid displacements are not physical but may only be used as test functions. However, according to [1] there always exist physical perturbations differing only slightly from these test functions and yielding the same value of $\delta^2 W$. Suppose it is possible to decompose into eigenmodes one of the physical perturbations corresponding to vertical displacements:

$$\tilde{\underline{\xi}} \underline{e}_z \approx \tilde{\underline{\xi}} = \sum_k \tilde{\underline{\xi}}_k$$

where

$$- \tilde{\rho}_0 \tilde{\omega}_k^2 \tilde{\underline{\xi}}_k = \tilde{F}(\tilde{\underline{\xi}}_k)$$

and where $\tilde{F}$ is the MHD-operator. Then, the quantity

$$\tilde{\omega}^2 = \delta^2 \tilde{W} / \left[\tilde{\rho}_0 \int \tilde{\underline{\xi}}^2 \int_{Pe} d\tilde{\tau} \right]$$

is an average of the $\tilde{\omega}_\kappa^2$. Since even in unstable

situations some of the $\tilde{\omega}_\kappa^2$ may be positive, $\tilde{\omega}$ may only be considered as

an optimistic estimate for the growth rates observed.

From (11), (14), (21), (23), (24), (26), (27), (51) and (63) we obtain

after eliminating $\tilde{p}_1$, $\tilde{g}_1$ and $\tilde{c}$

$$\tilde{g}_0 \, \tilde{\omega}^2 = \frac{\pi \, A^4 \, \tilde{p}_0}{2 \, \beta_p \, (1+e^2)(A^2-1) \, \tilde{R}_0^2} \cdot \frac{\mathcal{C}_{vert}}{V}$$

Here, $\tilde{p}_0$ is the pressure on the magnetic axis.

Assuming quasineutrality and $T_e \approx T_i$, we finally obtain with $\tilde{g}_0 \approx \tilde{m}_i \tilde{n}_i$

and $\tilde{p}_0 \approx 2 \tilde{n}_i \, \tilde{k} \, \tilde{T}_0$.

$$\tilde{\omega}^2 \approx \frac{\pi \, A^4 \, \tilde{k} \, \tilde{T}_0}{\beta_p \, (1+e^2)(A^2-1) \, \tilde{m}_i \, \tilde{R}_0^2} \cdot \frac{\mathcal{C}_{vert}}{V} \tag{69}$$

For a typical situation the growth rate has been calculated (see. Fig.4,

point marked by a small circle): for A = 3, e = 2, Q = 0.25 (β_p = 0.95),

$\tilde{kT}_0$ = 1keV and $\tilde{R}_0$ = 1,5 · 10^2 cm we obtain the growth rate $|\tilde{\omega}| \approx 7.6 \cdot 10^6 /_{sec}$.

Conclusions

Provided the condition $q \gtrsim 1$ is observed (which is necessary

in any case because of other instabilities), then with regard to

rigid displacements precautions must be taken only against the

vertical displacement instability. This is done by keeping the
vertical elongation of the equilibrium plasma shape small enough.
The optimum values for the elongation e lie between 1 and 2
for realistic configurations. Small aspect ratio and strong
triangular deformation are both favourable. However, the results
obtained in this paper may possibly be deteriorated by other
instabilities. For example,for sheared vertical displacements

$$\underline{\xi} = \int(\mathcal{R}) \, \underline{e}_z \qquad \text{as for rigid vertical displacements} \quad \delta^2 W$$

is independent of the main field.

Since the latter are special cases of the first, more stringent
conditions for the stabilization of these are to be expected.
Calculations for this are in progress.

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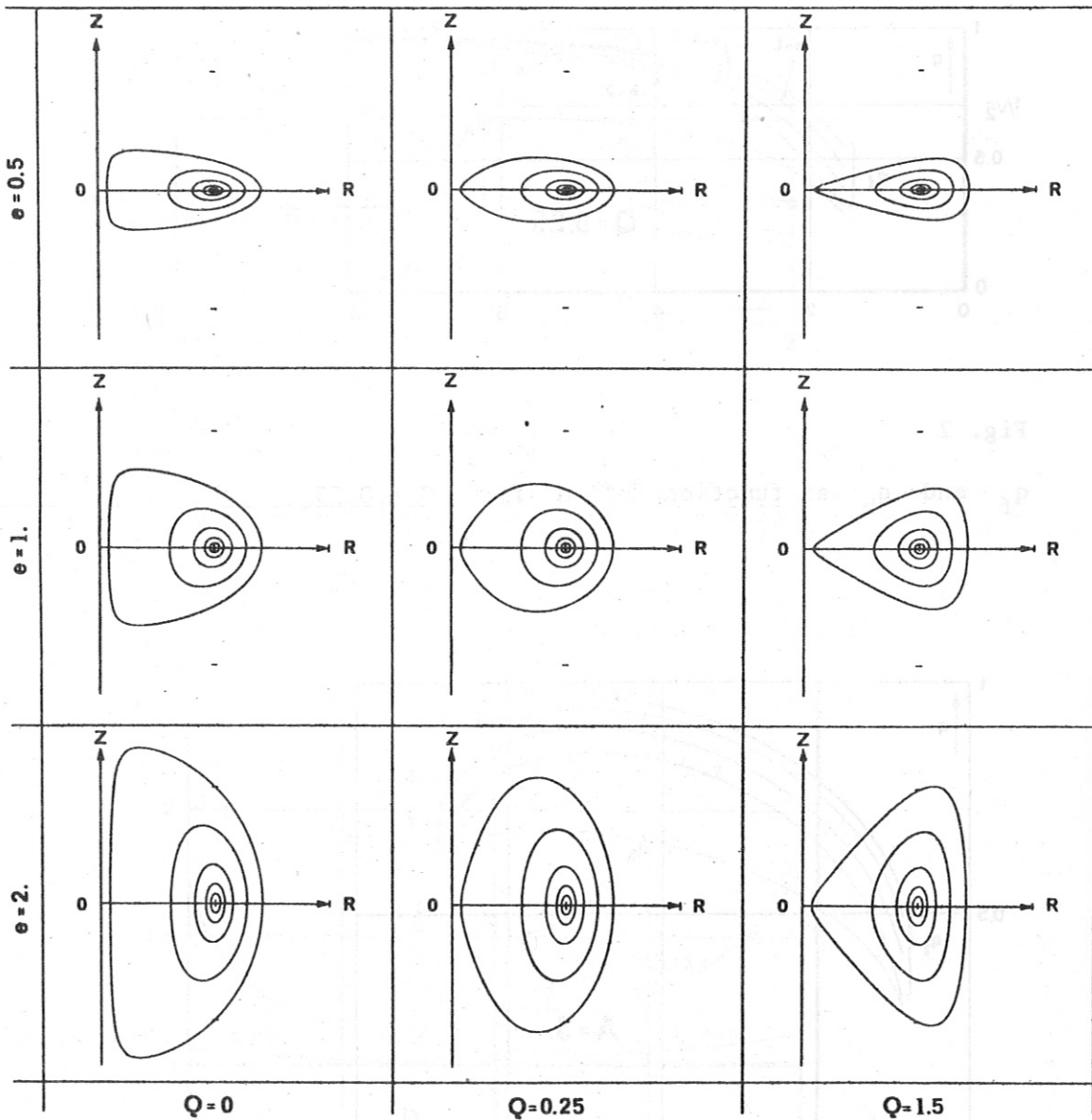
Fig. 1

Cross sections of the magnetic surfaces for several   Q   and   e

and for A = 1.5, 3, 6, 12, 24
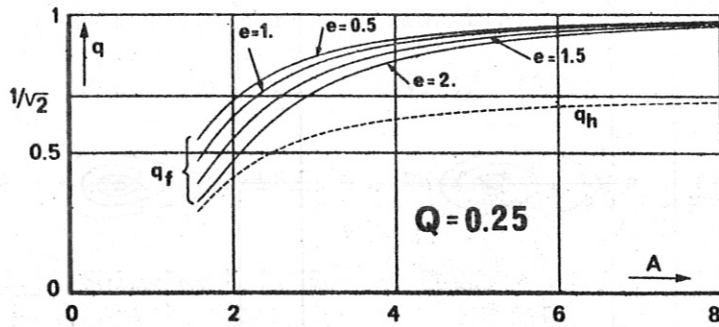
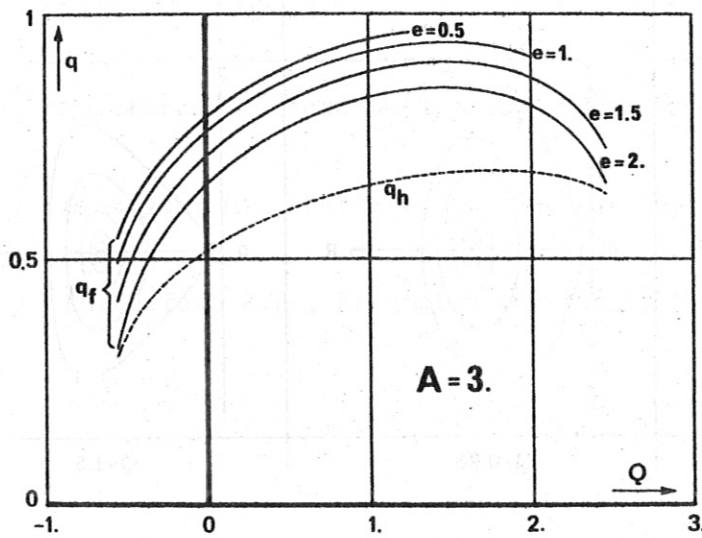Fig. 2

q_f and q_h as functions of A for Q = 0.25

Fig. 3

q_f and q_h as functions of Q for A = 3.
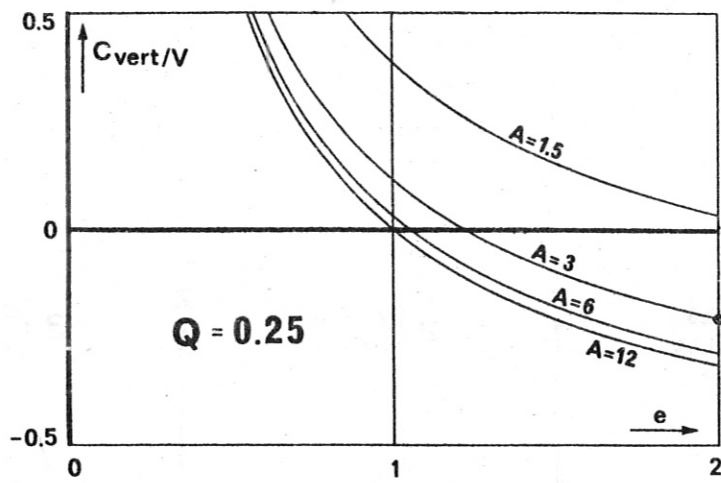
Fig. 4

$C_{ert/V}$ as a function of   e   for   Q = 0.25

Fig. 5
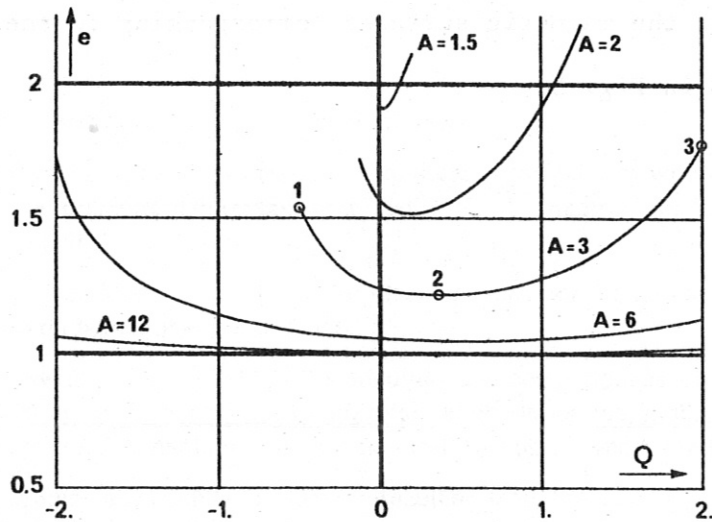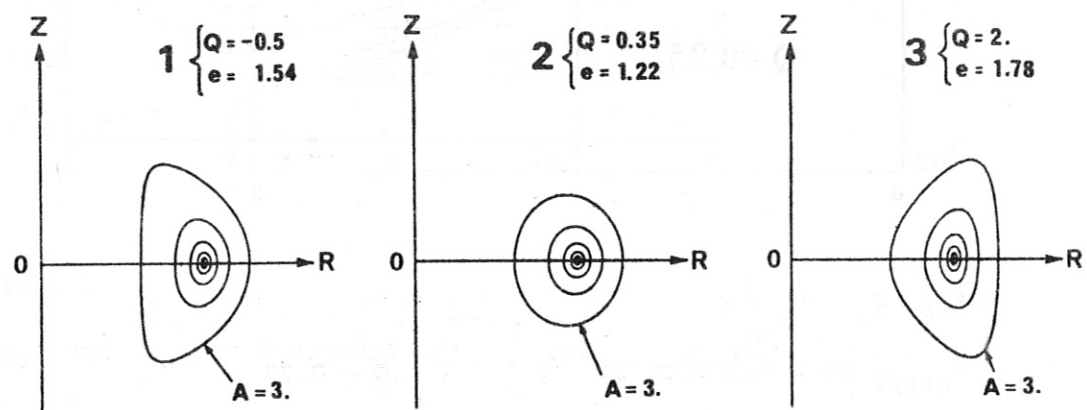
Critical   e   as a function of Q

Fig. 6

Cross-sections of the magnetic surfaces corresponding to the

points 1,2 and 3 in Fig. 5