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TOTALITY OF WAVES IN A HOMOGENEOUS VLASOV PLASMA

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IPP IV/73 August 1974

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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Abstract

Counting through correspondence, it is shown that the total number N, of wave pairs for a given frequency ω in a non-relativistic, magnetized, homogeneous Vlasov plasma is countably infinite and is given by $N = 3n + 2$, where n is the number of cyclotron harmonics. For reason of simplicity, the counting is performed for waves propagating perpendicular to the static magnetic field. The branches of the extraordinary and the ordinary modes lying immediately below the cyclotron harmonic frequencies found by Dnestrovskij and Kostomarov are identified respectively as "magnetostatic" and "zero field" waves in the limit of large propagation constant.

1. Introduction

The hot plasma dispersion relation can be written as

$$as^2 + bs + c = o \qquad (1)$$

where $s = n_x^2 = ck_x^2/\omega^2$ is the square of the refractive index in the direction perpendicular to the magnetic field. The coefficients a, b, and c are functions of plasma parameters, the longitudinal refractive index n_z, and involve six tensor components, each of which can be expressed as a convergent infinite series expansion in ascending powers of s. To each root s of (1) belongs a pair of oppositely directed waves with refractive indices $\pm n_x$. The total number of roots is evidently infinite (see Appendix A).

In a set of two papers, remarkable both in breadth and originality, Dnestrovskij and Kostomarov (1961, 1962) profess to identify all the real roots of the dispersion relation for the waves propagating perpendicular to the static magnetic field in a non-relativistic, homogeneous Vlasov plasma. They conclude that for a given ω, since there are only a finite number of real roots, the number of complex (including pure imaginary) roots is infinitely great. The other significant contribution of these papers involves the discovery of propagating waves just below the cyclotron-harmonic frequencies both for the ordinary and the extraordinary modes. These analytical findings are backed up by accurately computed dispersion curves without invoking the electrostatic approximation.

In this paper, it will be shown that the set of real solutions of (1) found by Dnestrovskij and Kostomarov may constitute the totality of waves in a Vlasov plasma. The counting operation is simpler if performed on waves propagating perpendicular to the static magnetic field, for in this case the dispersion relation can be factored into the ordinary and the extraordinary modes.

In the following analysis, the symmetric formulation of the Maxwell's equations in terms of the dielectric and diamagnetic tensors developed by Derfler and Omura (1967), will be employed. This procedure will lead us to the straightforward recognition of the new modes discovered by Dnestrovskij and Kostomarov as being either of a "magnetostatic" or of a "zero field" character in the limit $n_x \rightarrow \infty$.

2. The dispersion relation

Assuming an exp i $(k_x x + k_z z - \omega t)$ dependence of the field quantities, the Maxwell's equations using the symmetric formulation in-terms-of the dielectric and diamagnetic tensors become

$$\underline{k} \times \underline{E} = \omega \mu_0 \underline{\underline{\mu}} \cdot \underline{H} , \tag{2}$$

$$\underline{k} \times \underline{H} = - \omega \epsilon_0 \underline{\underline{\epsilon}} \cdot \underline{E} , \tag{3}$$

$$\underline{k} \cdot \underline{\underline{\epsilon}} \cdot \underline{E} = o , \tag{4}$$

$$\underline{k} \cdot \underline{\underline{\mu}} \cdot \underline{H} = o , \tag{5}$$

where

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ -\epsilon_{xy} & \epsilon_{xx} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}, \tag{6}$$

and

$$\underline{\underline{\mu}} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & 0 \\ -\mu_{xy} & \mu_{xx} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix}. \tag{7}$$

For an isotropic Maxwellian particle velocity distribution, the dielectric and diamagnetic tensor components are given by (Omura 1967)

$$\epsilon_{xx} = 1 + \sum_j \frac{\Pi^2}{\Omega} \rho_0 e^{-\lambda} \sum_{-\infty}^{\infty} \frac{n}{\lambda} I_n(\lambda) Z_n , \tag{8}$$

$$\epsilon_{xy} = -i \sum_j s_j \frac{\Pi^2}{\Omega} \rho_0 e^{-\lambda} \sum_{-\infty}^{\infty} \left[I_n(\lambda) - I_n'(\lambda) \right] Z_n , \tag{9}$$

$$\epsilon_{zz} = 1 - \sum_j \Pi^2 \rho_0^2 e^{-\lambda} \sum_{-\infty}^{\infty} I_n(\lambda) Z_n' , \tag{1o}$$

$$\underline{\underline{\mu}} = (\underline{\underline{I}} - \underline{\underline{\chi}})^{-1} , \tag{11}$$

where

$$\chi_{xx} = \sum_j \frac{\Pi^2}{2\Omega n_z^2} e^{-\lambda} \sum_{-\infty}^{\infty} \frac{n}{\lambda} I_n(\lambda) Z_n' , \tag{12}$$

$$\chi_{xy} = -i \sum_j s_j \frac{\Pi^2}{2\Omega n_z^2} e^{-\lambda} \sum_{-\infty}^{\infty} \left[I_n(\lambda) - I_n'(\lambda) \right] Z_n' , \tag{13}$$

$$\chi_{zz} = \sum_j \frac{\Pi^2 V^2}{\Omega^2 n_z} e^{-\lambda} \sum_{-\infty}^{\infty} \left[I_n(\lambda) - I_n'(\lambda) \right] Z_n , \tag{14}$$

$$\chi_{xz} = \chi_{zx} = \chi_{yz} = \chi_{zy} \equiv 0 ,$$

$$\xi_n = \frac{1 - n\Omega}{n_z V} , \tag{15}$$

$$\lambda = \frac{s V^2}{2\Omega^2} , \tag{16}$$

$$I_n'(\lambda) = dI_n(\lambda)/d\lambda , \tag{17}$$

$$Z_n' = d Z_n / d\xi_n , \tag{18}$$

$\Pi = \omega_p/\omega$, $\Omega = \omega_c/\omega$, $V = v/c$, v is the particle thermal velicity, $\underline{\underline{I}}$ is the identity tensor, j denotes the particle type (unless an ambignity exists, this subscript will be dropped), $s_j = \pm 1$ is the sign of the charge carried by the particle, $I_n(\lambda)$ is the modified Bessel function in the notation of Watson (1922) and $Z_n = Z(\xi_n)$ is the plasma dispersion function defined as

$$Z(\xi_n) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - \xi_n} \, dx \, , \quad \mathrm{Im}\, \xi_n > 0 . \tag{19}$$

This tensor formulation has been previously used (Puri et al. 1973) for obtaining the generalized cyclotron-harmonic modes and was shown to be essentially identical to the more common Stepanov (1958) tensor.

Substituting (6) and (7) in (2) - (3) and upon subsequent elimination of $\underline{E}$ and $\underline{H}$ yields the dispersion relation (1), where

$$a = \epsilon_{xx} \mu_{xx} , \tag{20}$$

$$b = n_z^2 (\epsilon_{xx} \mu_{zz} + \epsilon_{zz} \mu_{xx}) - \epsilon_{xx} \epsilon_{zz} (\mu_{xx}^2 + \mu_{xy}^2) - \mu_{xx} \mu_{zz} (\epsilon_{xx}^2 + \epsilon_{xy}^2) , \tag{21}$$

$$c = \epsilon_{zz} \mu_{zz} \left[n_z^4 + 2 n_z^2 (\epsilon_{xy} \mu_{xy} - \epsilon_{xx} \mu_{xx}) + (\epsilon_{xx}^2 + \epsilon_{xy}^2)(\mu_{xx}^2 + \mu_{xy}^2) \right] . \tag{22}$$

For propagation perpendicular to the static magnetic field, $n_z = o$, and (1) can be cast into a product of two symmetric factors

$$\left[s\mu_{xx} - \epsilon_{zz}(\mu_{xx}^2 + \mu_{xy}^2)\right]\left[s\epsilon_{xx} - \mu_{zz}(\epsilon_{xx}^2 + \epsilon_{xy}^2)\right] = 0 \;, \qquad (23)$$

giving rise to the two uncoupled branches

$$s\mu_{xx} = \epsilon_{zz}\,(\mu_{xx}^2 + \mu_{xy}^2)\;, \qquad (24)$$

and

$$s\,\epsilon_{xx} = \mu_{zz}\,(\epsilon_{xx}^2 + \epsilon_{xy}^2)\;, \qquad (25)$$

commonly referred to as the ordinary and the extraordinary branches, respectively. Further simplification of (24) - (25) is effected by replacing $\underline{\underline{\mu}}$ by $\underline{\underline{\chi}}$, which then become

$$s\,(1 - \chi_{xx}) = \epsilon_{zz}\;, \qquad (26)$$

and

$$s\,(1 - \chi_{zz})\,\epsilon_{xx} = \epsilon_{xx}^2 + \epsilon_{xy}^2\;. \qquad (27)$$

3. The asymptotic solutions ($s \to \infty$)

In the limit $s \to \infty$, the extraordinary mode (27) possesses the two solutions

$$\epsilon_{xx} = o , \tag{28}$$

and

$$1-\chi_{zz} = o . \tag{29}$$

The first of these is the familiar electrostatic mode studied in detail by Gross (1951). The otherwise excellent treatment of Gross contained two erroreous conclusions, one regarding the damping of the wave when the wavelength approaches the Debye shielding distance and the other with respect to the occurance of forbidden propagation gaps just below the cyclotron-harmonic frequencies. The first of these errors was rectified in the mathematically impeccable analysis of Bernstein (1958). The second error was corrected by Dnestrovskij and Kostomarov (1961, 1962) who found propagating waves immediately below the cyclotron-harmonic frequencies. The remainder of this section is addressed to the discussion of these propagating modes.

One such mode is given by the asymptotic solution (29). In the limit $s \to \infty$, this is a magnetostatic mode. The reason that Dnestrovskij and Kostomarov failed to recognize the magnetostatic nature of this mode is directly attributable to the fact that they worked with the Stepanov (1958) tensor formulation in which the diamagnetic contribution remains concealed. Since this wave exists in the extraordinary branch, we shall henceforth refer to it as the "extraordinary magnetostatic mode".

A similar asymptotic solution occurs in the ordinary branch (26),

$$1 - \chi_{xx} - \frac{\epsilon_{zz}}{s} = o. \qquad (3o)$$

It will be presently seen that unlike the extraordinary branch, thr RHS of (26) does not tend to a finite limit as $s \rightarrow \infty$ and must be included in the asymptotic solution. Since no component of $\underline{\underline{\epsilon}}$ or $\underline{\underline{\chi}}$ becomes zero in this case, all the field components must vanish. We shall refer to this wave as the "ordinary zero field mode".

In order to obtain the asymptotic soluti ns in an explicit form, we rewirte the tensor components (8) - (14) for the case $n_z = o$ as

$$\epsilon_{xx} = 1 - 2 \sum_j \Pi^2 e^{-\lambda} \sum_1^{\infty} \frac{n^2}{\lambda} \frac{I_n(\lambda)}{\Delta_n} , \qquad (31)$$

$$\epsilon_{xy} = -i \sum_j s_j \frac{\Pi^2}{\Omega} e^{-\lambda} \left\{ I_1(\lambda) - I_o(\lambda) - 2 \sum_1^{\infty} \left[I_n(\lambda) - I_n'(\lambda) \right] \frac{1}{\Delta_n} \right\} , \qquad (32)$$

$$\epsilon_{zz} = 1 - \sum_j \Pi^2 e^{-\lambda} \left\{ I_o(\lambda) + 2 \sum_1^{\infty} I_n(\lambda) \frac{\Gamma_n}{\Delta_n^2} \right\} , \qquad (33)$$

$$\chi_{xx} = 2 \sum_j \Pi^2 V^2 e^{-\lambda} \sum_1^\infty \frac{n^2}{\lambda} \frac{I_n(\lambda)}{\Delta_n^2} , \qquad (34)$$

$$\chi_{xy} = -\frac{i}{2} \sum_j s_j \frac{\Pi^2 V^2}{\Omega} e^{-\lambda} \left\{ I_0(\lambda) - I_1(\lambda) - 2 \sum_1^\infty \left[I_n(\lambda) - I_n'(\lambda) \right] \frac{\Gamma_n}{\Delta_n^2} \right\} , \qquad (35)$$

and

$$\chi_{zz} = \sum_j \frac{\Pi^2 V^2}{\Omega^2} e^{-\lambda} \left\{ I_1(\lambda) - I_0(\lambda) - 2 \sum_1^\infty \left[I_n(\lambda) - I_n'(\lambda) \right] \frac{1}{\Delta_n} \right\} , \qquad (36)$$

where $\Gamma_n = 1 + n^2\Omega^2$ and $\Delta_n = 1 - n^2\Omega^2$. Substituting (31) - (36) in the asymptotic solutions (28) - (3o) gives, respectively

$$\lambda^{3/2} = \frac{1}{\sqrt{2\pi}} \frac{n \Pi^2}{\Omega \, \delta\Omega} , \qquad (37)$$

$$\lambda^{3/2} = -\frac{1}{\sqrt{8\pi}} \frac{n \Pi^2 V^2}{\Omega \, \delta\Omega} , \qquad (38)$$

and

$$\lambda^{3/2} = -\frac{1}{\sqrt{8\pi}} \frac{n \Pi^2 V^2}{\Omega \, \delta\Omega} , \qquad (39)$$

where $\delta\Omega = 1 - n\Omega$. Thus the extraordinary electrostatic (37) and magnetostatic (38) modes have propagating roots just above and below the cyclotron harmonics, respectively. The ordinary zero-field mode (39) too has propagating solutions just below the cyclotron harmonics. By direct substitution of (37) and (38), in turn, in (27) it may be readily confirmed that the right-hand-side of (27) indeed remains finite as mentioned earlier in this section, thereby providing the aposteriori justification for obtaining the asymptotic solutions (28) and (29).

4. Real roots of the dispersion relation

Adopting the technique developed by Dnestrovskij and Kostomarov (1961) we proceed to identify the real roots of (1) in a three component plasma.

a) Ordinary mode

Writing (26) as

$$D(s,\Pi,\Omega,V) = 1 - s - \sum_j \Pi^2 e^{-\lambda}\left[I_0(\lambda) + 2\sum_1^\infty \frac{I_n(\lambda)}{\Delta_n}\right], \tag{40}$$

one may readily verify that

$$D(0,\Pi,\Omega,V) = 1 - \sum_j \Pi^2, \tag{41}$$

$$\lim_{s\to\infty} D(s,\Pi,\Omega,V) < 0, \tag{42}$$

$$\lim_{s\to-\infty} D(s,\Pi,\Omega,V) > 0. \tag{43}$$

From (41) - (43) we find at least one positive real root for $\Pi<1$ and a negative real root for $\Pi > 1$. This, of course, is the ordinary cold-plasma mode. Furthermore (41) - (43) admit of an additional even number of positive (or negative) roots. Existence of the ordinary magnetostatic mode (39) then requires the presence of at least two positive roots lying immediately below the harmonics. The other positive root approaches a cut-off solution for $\delta\Omega \rightarrow o$. The two roots remain propagating in a narrow region near the harmonics, merge together (Fig.1) and thereafter must split into a complex conjugate pair. Further discussion regarding this point will be postponed to § 6 with the observation that the warm-plasma effects contribute at least two real roots at each of the cyclotron-harmonic frequencies.

b) <u>Extraordinary mode</u>

Writing (27) as

$$D(s,\Pi,\Omega,V) = \left[s(1-X_{zz})\epsilon_{xx} - (\epsilon_{xx}^2 + \epsilon_{xy}^2)\right] \Delta_{1i}\, \Delta_{1e}\,, \qquad (44)$$

one obtains for the cold-plasma limit $V \rightarrow o$,

$$D(s,\Pi,\Omega,o) \approx s(1-\Omega_{lh}^2)(1-\Omega_{uh}^2) - (1-\Omega_{c1}^2)(1-\Omega_{c2}^2)\,, \qquad (45)$$

where Ω_{lh}, Ω_{uh}, Ω_{c1}, and Ω_{c2} are the two hybrid and cut-off frequencies (see Puri et al., 1973), respectively. From (45)

$$D(o, \Pi, \Omega, o) \quad < \quad o, \text{ if } \omega < \omega_{c1}, \tag{46a}$$

$$> \quad o, \text{ if } \omega_{c1} < \omega < \omega_{c2}, \tag{46b}$$

$$< \quad o, \text{ if } \omega > \omega_{c2}, \tag{46c}$$

$$\lim_{s \to \infty} D(s, \Pi, \Omega, o) \quad > \quad o, \text{ if } \omega < \omega_{1h}, \tag{47a}$$

$$< \quad o, \text{ if } \omega_{1h} < \omega < \omega_{uh}, \tag{47b}$$

$$> \quad o, \text{ if } \omega > \omega_{uh}, \tag{47c}$$

and

$$\lim_{s \to -\infty} D(s, \Pi, \Omega, o) \quad < \quad o, \text{ if } \omega < \omega_{1h}, \tag{48a}$$

$$> \quad o, \text{ if } \omega_{1h} < \omega < \omega_{uh}, \tag{48b}$$

$$< \quad o, \text{ if } \omega > \omega_{uh}. \tag{48c}$$

Thus, there is always one real root of the cold-plasma dispersion relation with three well known propagating branches (Fig.2). Inclusion of warm-plasma effects does not materially alter the limiting values of D for $s \to o$ or $s \to \infty$. Hence an additional even set of positive or negative roots are permissible as in the case of the ordinary mode. The presence of the electrostatic mode (37) and the extraordinary magnetostatic mode (38) would demand the existence of at least two positive roots near the cyclotron-harmonic frequencies (Fig.3). It must be borne in mind that the roots lying above and below the harmonics posses different limits for $|\mathcal{E}\,\Omega| \to 0$, would not continue analytically across the harmonics and, therefore, must be regarded as independent modes.

By all appearances,the warm-plasma effects contribute four real roots near each of the harmonics for the extraordinary mode and two for the ordinary mode. There is, in addition, one cold-plasma mode each for the ordinary and the extraordinary branch, respectively. Whether, there exist more such real roots and the precise relationship between the real roots identified here with the infinitely many complex roots will be discussed in the following sections.

5. Counting by correspondence

Expanding (31) - (36) in powers of λ, one obtains

$$\epsilon_{xx} = 1 - \sum_j \Pi^2 \mathcal{S}_1 , \tag{49}$$

$$\epsilon_{xy} = -i \sum_j s_j \frac{\Pi^2}{\Omega} \mathcal{S}_2 , \tag{50}$$

$$\epsilon_{zz} = 1 - \sum_j \Pi^2 \mathcal{S}_3 , \tag{51}$$

$$\chi_{xy} = \sum_j \Pi^2 V^2 \mathcal{S}_4 , \tag{52}$$

$$\chi_{xy} = -\frac{1}{2} i \sum_j s_j \frac{\Pi^2}{\Omega} \mathcal{S}_5 , \tag{53}$$

and

$$\chi_{zz} = \sum_j \frac{\Pi^2}{\Omega^2} V^2 \mathcal{S}_2 , \tag{54}$$

where

$$\mathcal{S}_1 = \frac{1}{\Delta_1} - \lambda\left(\frac{1}{\Delta_1} - \frac{1}{\Delta_2}\right) + \lambda^2\left(\frac{5}{8\Delta_1} - \frac{1}{\Delta_2} + \frac{3}{8\Delta_3}\right) - \lambda^3\left(\frac{7}{24\Delta_1} - \frac{7}{12\Delta_2} + \frac{3}{8\Delta_3} - \frac{1}{12\Delta_4}\right) + \cdots \tag{55}$$

$$\mathcal{S}_2 = \frac{\Omega^2}{\Delta_1} - \lambda\left(\frac{2\Omega^2 + \Gamma_1}{2\Delta_1} + \frac{1}{2\Delta_2}\right) + \lambda^2\left(\frac{6\Omega^2 + 4\Gamma_1 + 1}{8\Delta_1} + \frac{1}{4\Delta_2} + \frac{1}{8\Delta_3}\right) - \lambda^3\left(\frac{20\Omega^2 + 15\Gamma_1 + 6}{48\Delta_1} + \frac{1}{24\Delta_2} + \frac{1}{6\Delta_3} - \frac{1}{48\Delta_4}\right) + \cdots \tag{56}$$

$$\mathcal{S}_3 = 1 - \lambda\left(1 - \frac{\Gamma_1}{\Delta_1^2}\right) + \lambda^2\left(\frac{3}{4} - \frac{\Gamma_1}{\Delta_1^2} + \frac{\Gamma_2}{4\Delta_2^2}\right) - \lambda^3\left(\frac{5}{12} - \frac{5\Gamma_1}{8\Delta_1^2} + \frac{\Gamma_2}{4\Delta_2^2} - \frac{\Gamma_3}{24\Delta_3^2}\right) + \cdots \tag{57}$$

$$S_4 = \frac{1}{\Delta_1^2} - \lambda\left(\frac{1}{\Delta_1^2} - \frac{1}{\Delta_2^2}\right) + \lambda^2\left(\frac{5}{8\Delta_1^2} - \frac{1}{\Delta_2^2} + \frac{3}{8\Delta_3^2}\right) - \lambda^3\left(\frac{7}{24\Delta_1^2} - \frac{7}{12\Delta_2^2} + \frac{3}{8\Delta_3^2} - \frac{1}{12\Delta_4^2}\right) + \cdots \tag{58}$$

and

$$S_5 = \left(1 - \frac{\Gamma_1}{\Delta_1^2}\right) - \lambda\left(\frac{3}{2} - \frac{2\Gamma_1}{\Delta_1^2} + \frac{\Gamma_2}{2\Delta_2^2}\right) + \lambda^2\left(\frac{5}{4} - \frac{15\Gamma_1}{8\Delta_1^2} + \frac{3\Gamma_2}{4\Delta_2^2} - \frac{\Gamma_3}{8\Delta_3^2}\right) - \lambda^3\left(\frac{35}{48} - \frac{7\Gamma_1}{6\Delta_1^2} + \frac{7\Gamma_2}{12\Delta_2^2} - \frac{\Gamma_3}{6\Delta_3^2} + \frac{\Gamma_4}{48\Delta_4^2}\right) + \cdots \tag{59}$$

These series expansions are exact in the context of a Vlasov plasma and converge for all λ. Observe that each suceeding power of λ in the expansion involves precisely one more harmonic of the cyclotron frequency in the summations (55) - (59).

In the cold-plasma limit $\lambda \rightarrow 0$, the two pairs of waves, one ordinary and the other extraordinary are recovered on substituting (49) - (59) in (26) and (27), respectively.

Inclusion of each additional term in the series expansions of $\underline{\underline{\epsilon}}$ and $\underline{\underline{\chi}}$ increases the polynomial order of the ordinary branch (26) by one and of the extraordinary branch (27) by two while involving one more harmonic of the cyclotron frequencies of each of the particle species. Thus the totality of wave pairs in a Vlasov plasma is given by

$$N = 3n + 2, \tag{6o}$$

where n is the number of cyclotron harmonics. In deriving (6o) it has been tacitly assumed that the number of roots of the polynomial expansions is given by the degree of the polynomial.

The same result (6o) is obtained if (26) and (27) are expressed in the form

$$s(e^{\lambda} - e^{\lambda} \chi_{xx}) = e^{\lambda} \epsilon_{zz}, \tag{61}$$

and

$$s(e^{\lambda} - e^{\lambda} \chi_{zz}) \, e^{\lambda} \epsilon_{xx} = (e^{\lambda} \epsilon_{xx} + ie^{\lambda} \epsilon_{xy}) \, (e^{\lambda} \epsilon_{xx} - ie^{\lambda} \epsilon_{xy}), \tag{62}$$

Note that, although, the reduction to the form (61) and (62) is possible only for a plasma with a single hot specie, the digression serves to show that the errant factor e^{λ}, capable of distorting the order of the polynomial without any net addition to its roots, plays no significant role in the mechanics of "correspondence counting".

6. Discussion

In section 4 it was shown that the hot-plasma effects give rise to six real roots near each of the harmonic frequencies. Of these six roots, the three cut-off solutions are joined together from one harmonic to the other (see Figs. 1 and 2, Dnestrovskij and Kostomarov, 1961, 1962) while the three asymptotic solutions apparently originate at the harmonics themselves. Such an assertion, though difficult to prove rigorously, acquires a degree of plausibility from the analysis of section 5 where it was indeed shown that three new roots of the dispersion relation correspond to each of the cyclotron harmonic frequencies. Since (6o) is independent of the particle species, it would be further necessary to assume that the asymptotic solutions of all the species are as well joined at the corresponding harmonics! We remind that the result (6o) is valid independently of the accuracy of the conjectures of this paragraph.

The question immediately arises regarding the nature of the infinity of complex solutions. The paradox may be averted by asserting that the complex solutions, when followed on the frequency axis ω, will ultimately merge with one of the real solutions. Such behaviour is not altogether strange for it is well known that the electrostatic cyclotron-harmonic modes split into complex conjugate pairs above the propagation band. Singularities encountered while following the roots across the harmonics can be avoided through analytic continuation e.g. by the addition of an imaginary part to ω during the crossing.

With some additional effort, it is possible to show that the result (6o) is valid even for the case $n_z \neq o$. This can be readily appreciated by noting in (8) - (14) that the series expansions of $\underline{\underline{\epsilon}}$ and $\underline{\underline{\chi}}$ can be carried out in a manner identical to (49) - (59), with the observation that Z_n which is a function of $(1-n\Omega)$ requires no further unfolding for the arguments required here.

The ability to identify the roots may find applications beyond the pleasures of academic curiousity. In certain limiting conditions when the characteristic gradient length in an inhomogeneous plasma is well in excess of the cyclotron radii, it is possible to use the local dielectric tensors for solving boundary value problems. It then becomes imperative to decide which, if any, of the complex roots carry a significance ?

Acknowledgment

We are grateful to Prof. D.Pfirsch for pointing out possible flaws with regard to identifying the roots of a polynomial with its degree, and of the peculiar difficulty experienced with associating the result (6o) with the asymptotic resonances in a multi-specie plasma, to Drs. R. Croci and R. Saison for drawing our attention to the fact that the asymptotic solution of (26) should also include the contribution due to ϵ_{zz}.

This work has been performed as part of the joint research between Euratom and the Max-Planck-Institut für Plasmaphysik.

Appendix A

The dispersion relation (4o) for the ordinary branch near the mth cyclotron harmonic of the kth specie becomes

$$\Pi_k^2 e^{-\lambda_k} \frac{I_m(\lambda_k)}{1-m\Omega_k} = 1 - \quad s \quad - \sum_j{}' \Pi^2 e^{-\lambda} \left[I_o(\lambda) + 2 \sum_1^{\infty}{}' \frac{I_n(\lambda)}{\Delta_n} \right], \tag{A1}$$

where the prime over the summation denotes the exclusion of the term on the left.

The modified Bessel function $I_m(\lambda_k)$ has an infinite number of complex zeros whose location is unaffected by the remaining expressions multiplying it on the LHS of (A1). By making $(1-m\Omega_k)$ small enough it is possible to make the LHS dominate the RHS in an arbitrarily close vicinity of these zeros. Hence (A1) must have an infinity of roots in the complex plane.

In a similar manner it can be shown that the extraordinary mode, too, possesses an infinite number of solutions.

References

BERNSTEIN, I.B. 1958 Phys. Rev. 1o9, 1o.

DNESTROVSKIJ, YU. N. and KOSTOMAROV, D. P. 1961 Sov. Phys. JETP 13, 968.

DNESTROVSKIJ, YU. N. and KOSTOMAROV, D. P. 1962 Sov. Phys. JETP 14, 1o89.

GROSS, E. P. 1951 Phys. Rev. 82, 232.

OMURA, M. 1967 Electrostatic Waves in Bounded Hot Plasmas, Stanford University IPR Rep. 156.

PURI, S., LEUTERER, F., and TUTTER, M. 1973 J. Plasma Phys. 9, 89.

STEPANOV, K. N. 1958 Sov. Phys. JETP 7, 892.

WATSON, G. N. 1922 Theory of Bessel Functions, Cambridge University Press.

Figure Captions

Fig. 1 Qualitative dispersion curves for the ordinary ion (a) and electron (b) modes. The solutions lying immediately above harmonics have been found as a result of the present analysis which takes into account the full dielectric tensor.

Fig. 2 Cold-plasma dispersion for the extraordinary mode.

Fig. 3 Qualitative dispersion curves for the extraordinary ion (a) and electron (b) modes.

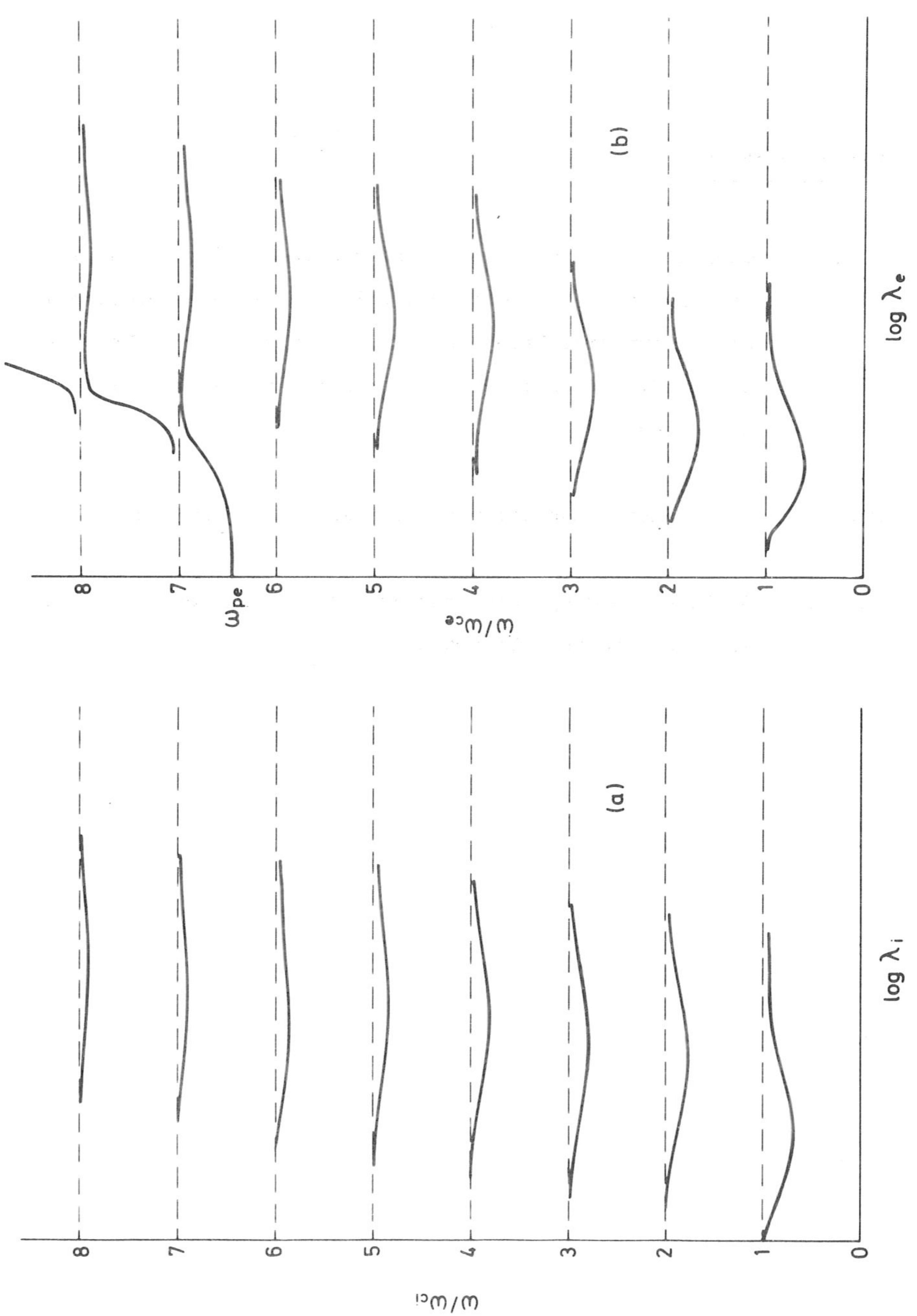

Fig. 1

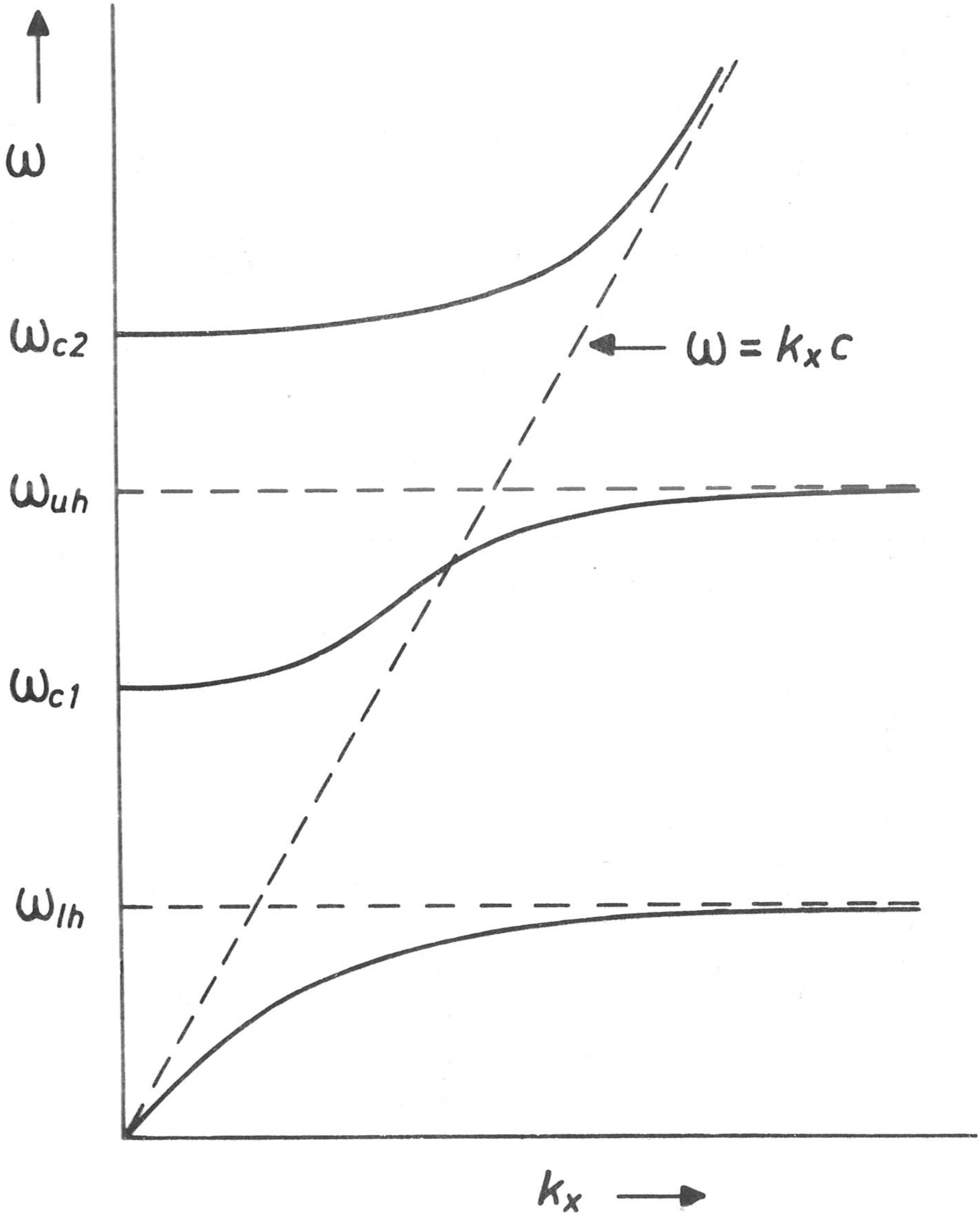

Fig. 2

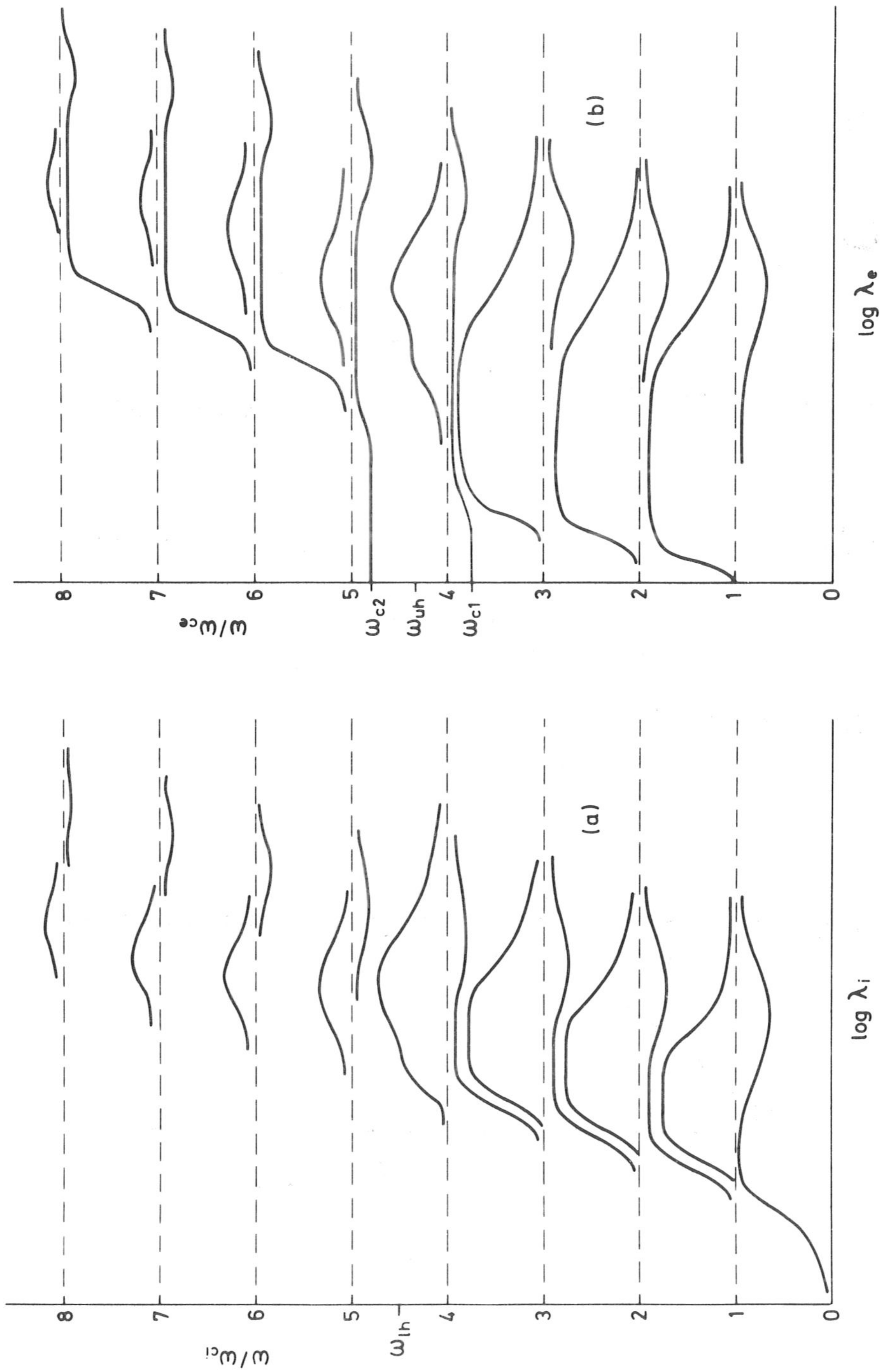

Fig. 3