Plasma Transport Equations for High Flow Velocities

A. Salat

IPP 6/125 June, 74

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Plasma Transport Equations for High Flow Velocities

A. Salat

IPP 6/125 June, 74

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Abstract

Collision terms for plasma transport equations are derived from the Fokker-Planck equation without the usual linearization in U = flow velocity / thermal velocity. A 13 - moment ansatz for the distribution functions includes the linearized dependence of the collision terms on viscous pressure and heat flow in the transport equations for momentum, energy, viscous pressure and heat flow. The masses, densities, flow velocities and temperatures of the species are arbitrary. Simplified results for small mass ratio or equal masses are given. For moderate velocities the collision terms are expanded in powers of U. Some of the resulting lowest order terms are nonlinear. These usually neglected collision terms may be important for moderate U.

1. Introduction

The present investigation was prompted by difficulties arising in hydrodynamic computer calculations of laser pellet heating. An appreciable fraction of the energy deposited may go into the creation of high energy super thermal electrons (FREIDBERG et al., 1972; FORSLUND et al., 1973; MORSE and NIELSON, 1973). Simple two fluid hydrodynamic equations are not general enough to take care of this non-Maxwellian tail of the electron distribution function. Temperature gradients are obtained, for example, which are so large that the heat flow $\underline{q}$ calculated from $\underline{q} = - \kappa \nabla T$ is incompatible with a microscopic plasma description (SALZMANN, 1972; BICKERTON, 1973). In practice, therefore, the heat flux is artificially limited in computer calculations to some physically reasonable upper bound, essentially to the product of the average electron energy and the electron or ion thermal velocity; see, for example, ASHBY and CHRISTIANSEN (1973).
Clearly, it would be good to have a more detailed description of the electron component. One possibility might be to treat the super thermal electrons as a species of its own, with density, temperature, heat flow, etc. independent of the bulk electrons, and to follow its dissipation in space and time after its creation from local source terms.

However, a further complication arises from the fact that the macroscopic flow velocity of the fast electrons propagating from the region where they originated is initially of the same order of magnitude as their thermal velocity. Transport coefficients like friction, etc., however are usually calculated only in the limit $U \equiv$ flow velocity / thermal velocity $\ll 1$, e.g. BRAGINSKII, 1965. Therfore an extension of existing transport theory is required.

Collision terms for large U have been obtained from the Fokker-Planck equation in the theory of electron runaway. DREICER (1959) considered friction due to electron motion. HERTWECK (1965) derived collision terms for momentum, energy, viscous pressure and heat flow based on a 13-moment ansatz of the distribution function of electrons and ions. Arbitrarily large pressure anisotropy was allowed. Owing to the generality of the ansatz, however, his results are expressed in terms of complicated integrals which are not readily amenable to approximations. More explicit results were obtained from the Boltzmann equation with a similar approach by YEN (1968), who assumed both pressure anisotropy and heat flow to be small corrections in the ansatz for the distribution function. He restricted him self, however, to nonlinear collision terms for electrons only. For arbitrary species nonlinear collision terms based on a five-moment ansatz of the distribution function were derived in the book on transport theory by BURGERS (1969).

The present paper is devoted to the derivation of moment equations for density, flow velocity, temperature, viscous pressure and heat flow of a multicomponent plasma with collision terms in which the flow velocities need not be small compared to the thermal velocities. The number and type of species are arbitrary, so that the result may be applied to, for example, the two-electron-species model of laser plasma mentioned above. The method closely follows that of BURGERS, except that a 13-moment ansatz is made throughout in the derivation of the nonlinear collision terms. The analytic evaluation of the integrals in the collision terms was performed with the algebraic computer language REDUCE 2 (HEARN, 1972), which facilitates the book-keeping of the many resulting terms.

Section 2 starts with the Fokker-Planck collision term. A 13-moment ansatz for the distribution functions which closes the set of moment equations is made. For completeness a short review of the collisionless terms of the moment equations is given. In Section 3 the collision terms are linearized with respect to viscous pressure and heat flow, as usual. For each particle species a set of twelve collision terms is obtained each of which has contributions from differences in flow velocity or temperature, from viscous pressure $\underline{\underline{P}}$ or from heat flow $\underline{q}$ and is a nonlinear function of the flow velocity difference $U = |\underline{u}_s - \underline{u}_t|/\alpha$ of species s and t, ($\alpha^2 = v^2_{thermal,s} + v^2_{thermal,t}$). Simplified expressions for small mass ratio or equal masses are derived.

For U not too close to 1 an expansion in powers of U is made in Section 4 for each collision term. Apart from the usual linear collision terms, additional terms are obtained which contain powers of U or products of U with viscous stresses or heat currents.

2. Thirteen-moment approximation

Let F_s ($\underline{r}$, $\underline{v}$, t) be the distribution function of the particles of arbitrary species s colliding with particles of other (or equal) species t. The Fokker-Planck equation for F_s (MONTGOMERY and TIDMAN, 1964) is

$$\frac{\partial}{\partial t} F_s + v_h \frac{\partial}{\partial x_h} F_s + \frac{1}{m_s} f_{sh} \frac{\partial}{\partial v_h} F_s = R_s$$

$$R_s \equiv -\frac{\partial}{\partial v_h}(a_h F_s) + \frac{1}{2}\frac{\partial^2}{\partial v_h \partial v_k}(b_{hk} F_s) \tag{1}$$

where the summation convention is applied with respect to vector and tensor components, $\underline{f}_s$ are the electric and magnetic forces, and the friction and diffusion coefficients a_h , b_{hk} are given by

$$a_h = \sum_t \frac{m_s + m_t}{m_t} \cdot \frac{4\pi e_s^2 e_t^2 \ln \Lambda}{m_s^2} \int d^3v' \, F_t(\underline{r}, \underline{v}', t) \frac{g_h}{g^3} \tag{2}$$

$$b_{hk} = \sum_t \frac{4\pi e_s^2 e_t^2 \ln \Lambda}{m_s^2} \int d^3v' \, F_t(\underline{r}, \underline{v}', t)(\delta_{hk} g^2 - g_h g_k) \frac{1}{g^3} \tag{3}$$

$$g = |\underline{g}|; \qquad \underline{g} = \underline{v}' - \underline{v} . \tag{4}$$

Here m_s , e_s are the mass and the charge of particle species s, and $\ln\Lambda$ is the Coulomb logarithm. The sum over t extends over all species present, including t = s.

Macroscopic moments such as density N_s, flow velocity $\underline{u}_s$, temperature T_s , pressure tensor $\underline{\underline{p}}_s$, traceless viscous pressure $\underline{\underline{P}}_s$ and heat flow $\underline{q}_s$ are obtained in the usual way from integration of F_s with appropriate factors:

$$
\begin{aligned}
N_s &= \int d^3v \; F_s \\
N_s u_{sh} &= \int d^3v \; v_h \; F_s \\
p_s &= N_s k T_s = \frac{1}{3} p_{shh} = \frac{1}{3} m_s \int d^3v \; c_s^2 \; F_s \\
p_{shk} &= p_s \delta_{hk} + P_{shk} = m_s \int d^3v \; c_{sh} c_{sk} \; F_s \\
P_{shk} &= m_s \int d^3v \left(c_{sh} c_{sk} - \frac{1}{3} \delta_{hk} c_s^2 \right) F_s \; ; \qquad P_{shh} = 0 \\
q_{sh} &= \frac{1}{2} m_s \int d^3v \; c_s^2 \, c_{sh} \; F_s
\end{aligned}
\tag{5}
$$

with

$$
c_s^2 = \underline{c}_s^2 \; ; \qquad c_{sh} = v_h - u_{sh} \, . \tag{6}
$$

The evolution of the moments is described by moment equations which follow from integration of the Fokker-Planck equation. The result is:

Equation of continuity:

$$\frac{D_s}{Dt} N_s + N_s \varepsilon_s = 0 \tag{7}$$

Momentum equation:

$$N_s m_s \frac{D_s}{Dt} u_{sh} + \frac{\partial}{\partial x_k} p_{shk} - N_s e_s \hat{E}_{sh} = R_{s1h} \tag{8}$$

Energy equation:

$$N_s \frac{D_s}{Dt}\left(\frac{3}{2} k T_s\right) + p_s \varepsilon_s + \frac{1}{2} P_{shk} \varepsilon_{shk} + \frac{\partial}{\partial x_h} q_{sh} \tag{9}$$

$$= R_{s2}$$

Equation for viscous pressure:

$$\frac{D_s}{Dt} P_{shk} + P_{shk} \varepsilon_s + p_s \varepsilon_{shk} + P_{shi} \frac{\partial}{\partial x_i} u_{sk}$$

$$+ P_{ski} \frac{\partial}{\partial x_i} u_{sh} - \frac{2}{3} \delta_{hk} P_{sij} \frac{\partial}{\partial x_i} u_{sj} + \frac{\partial}{\partial x_i} p_{shki} \tag{10}$$

$$- \frac{2}{3} \delta_{hk} \frac{\partial}{\partial x_i} q_{si} - \frac{e_s}{m_s c} \left(\delta_{hij} P_{ski} + \delta_{kij} P_{shi} \right) B_j$$

$$= R_{s3hk}$$

Heat flow equation:

$$\frac{D_s}{Dt} q_{sh} + q_{sh} \varepsilon_s + q_{si} \frac{\partial}{\partial x_i} u_{sh} + p_{shij} \frac{\partial}{\partial x_i} u_{sj}$$

$$+ \frac{1}{2} \frac{\partial}{\partial x_i} p_{shkki} - \frac{1}{N_s m_s} \left(\frac{5}{2} p_s \frac{\partial}{\partial x_i} p_{shi} + P_{shj} \frac{\partial}{\partial x_i} p_{sij} \right) \tag{11}$$

$$+ \frac{1}{N_s m_s} \left(\frac{5}{2} p_s M_h + P_{shi} M_i \right) - \frac{e_s}{m_s c} \delta_{hij} q_{si} B_j$$

$$= \hat{R}_{s4h} .$$

Here

$$\frac{D_s}{Dt} = \frac{\partial}{\partial t} + u_{si} \frac{\partial}{\partial x_i} \; ; \qquad \hat{E}_{sh} = E_h + \frac{1}{c} \delta_{hkl} u_{sk} B_l . \tag{12}$$

$\underline{E}$ und $\underline{B}$ are the electric and magnetic fields, respectively. The rate of volume expansion, ε_s, and the rate of deformation of the field of flow, ε_{sij}, are defined by

$$\varepsilon_s = \frac{\partial}{\partial x_i} u_{si} \; ; \qquad \varepsilon_{sij} = \frac{\partial}{\partial x_i} u_{sj} + \frac{\partial}{\partial x_j} u_{si} - \frac{2}{3} \delta_{ij} \varepsilon_s . \tag{13}$$

The third and fourth-order pressures p_{shkl}, p_{shklm} are defined as moments of F_s taken with the factors $m_s c_{sh} c_{sk} c_{sl}$ and $m_s c_{sh} c_{sk} c_{sl} c_{sm}$. The quantity M_{sh} is equal to the left-hand side of the momentum equation, equation (8), for $N_s m_s u_{sh}$. δ_{hij} is the antisymmetric Ricci tensor with values -1,0,+1. Source terms may be added in equations (7) - (11) whenever required.

The collision terms on the right-hand side follow from taking the appropriate moments of the Fokker-Planck collision term. By partial integration they reduce to

$$
\begin{aligned}
R_{s1h} &= m_s \int d^3v\ F_s\, a_h \\
R_{s2} &= m_s \int d^3v\ F_s \cdot \left(c_{si}\, a_i + \frac{1}{2} b_{ii} \right) \\
R_{s3hk} &= m_s \int d^3v\ F_s \cdot \Big(c_{sh}\, a_k + c_{sk}\, a_h \\
&\quad - \frac{2}{3} \delta_{hk}\, c_{si}\, a_i + b_{hk} - \frac{1}{3} \delta_{hk}\, b_{ii} \Big) \\
R_{s4h} &= \frac{1}{2} m_s \int d^3v\ F_s \cdot \Big(c_s^2\, a_h + 2 c_{sh}\, c_{si}\, a_i \\
&\quad + c_{sh}\, b_{ii} + 2 c_{si}\, b_{hi} \Big).
\end{aligned}
\tag{14}
$$

A closed set of moment equation is obtained from equations (7) - (14) by the following 13-moment ansatz for the distribution function F_s

(and F_t):

$$F_s(\underline{r},\underline{v},t) = F_{s0}(\underline{r},\underline{v},t)\left[1 + \Phi_s(\underline{r},\underline{v},t)\right] \tag{15}$$

where F_{s0} is the Maxwellian

$$F_{s0} = \frac{N_s}{\pi^{3/2} a_s^3} \exp\left(-\frac{c_s^2}{a_s^2}\right); \quad c_s^2 = (\underline{v}-\underline{u}_s)^2 \; ; \quad a_s^2 = \frac{2kT_s}{m_s} \tag{16}$$

and Φ_s is the simplest polynomial in $\underline{c}_s$ compatible with the viscous pressure $\underline{\underline{P}}_s$ and heat current $\underline{q}_s$:

$$\Phi_s = P_{sij} c_{si} c_{sj} \frac{1}{p_s a_s^2} + q_{si} c_{si} \left(c_s^2 - \frac{5}{2} a_s^2\right) \frac{4}{5 p_s a_s^4} \, . \tag{17}$$

By integration of F_s the moments p_{shki} and p_{shkki} which occur in equation (10) and (11) may be expressed as

$$p_{shki} = \frac{2}{5}\left(\delta_{ki} q_{sh} + \delta_{kh} q_{si} + \delta_{hi} q_{sk}\right) \tag{18}$$

$$p_{shkki} = 5\,\delta_{hi} \frac{p_s^2}{N_s m_s} + 7\, P_{shi} \frac{p_s}{N_s m_s} \, .$$

All third-order moment terms on the left-hand side of equation (10) may then be replaced by

$$-\frac{4}{15}\delta_{hk}\frac{\partial}{\partial x_i}q_{si} + \frac{2}{5}\frac{\partial}{\partial x_h}q_{sk} + \frac{2}{5}\frac{\partial}{\partial x_k}q_{sh}\,.$$

The heat flow equation may be simplified in the case of small viscous pressure, $\|P_{shk}\| \ll p_s$, (see next section) by omission of all terms quadratic in P_{sij} and by omission of the small term $P_{shi}\ M_i$. When M_i and p_{shkki} are inserted from equations (8) and (18), the heat flow equation reads

$$\begin{aligned}
&\frac{D_s}{Dt}q_{sh} + \frac{7}{5}q_{sh}\,\varepsilon_s + \frac{7}{5}q_{si}\frac{\partial}{\partial x_i}u_{sh} + \frac{2}{5}q_{si}\frac{\partial}{\partial x_h}u_{si} \\
&+ p_s\frac{\partial}{\partial x_h}\left(\frac{5}{2}\frac{kT_s}{m_s}\right) + P_{shi}\frac{\partial}{\partial x_i}\left(\frac{5}{2}\frac{kT_s}{m_s}\right) + p_s\frac{\partial}{\partial x_i}\left(\frac{P_{shi}}{N_s m_s}\right) \qquad (19)\\
&- \frac{e_s}{m_s c_s}\delta_{hij}\,q_{si}\,B_j \\
&= R_{s4h}
\end{aligned}$$

$$R_{s4h} \equiv \hat{R}_{s4h} - \frac{5}{2}\frac{kT_s}{m_s}R_{s1h}\,. \qquad (19a)$$

Since F_s has been specified as a function of velocity, the integrations in equs. (2) - (3) and (14) may be performed and yield the collision terms R_n as functions of the moments $N_s, \underline{u}_s, T_s, \underline{\underline{P}}_s, \underline{q}_s$ so that a closed set of 13-moment equation is obtained. The next section deals with the evaluation of the collision terms.

3. Collision terms

The collision terms R_{sn}, n = 1,2,3,4, from equations (2), (14) and (19a) have the common structure

$$R_{sn} = \sum_t \int d^3v \int d^3v' \, F_s(\underline{v}) F_t(\underline{v}') P_{st}(\underline{v}, \underline{v}') \frac{1}{|\underline{v} - \underline{v}'|^3} \tag{20}$$

where $P_{st}(\underline{v},\underline{v}')$ are polynomials in $\underline{v}$ and $\underline{v}'$ and

$$F_s F_t = F_{so} F_{to} \cdot (1 + \Phi_s + \Phi_t + \Phi_s \Phi_t) . \tag{21}$$

In the following we shall assume that $\Phi_s , \Phi_t \ll 1$ and shall neglect the product $\Phi_s \Phi_t$. Equation (17) shows that this implies small viscous pressure and small heat flow:

$$\| \underline{\underline{P}}_s \| \ll p_s \; ; \qquad |\underline{q}_s| \ll p_s a_s . \tag{22}$$

No linearization, however, is made with respect to the flow velocity $\underline{u}_s$.

The sixfold integrals in equation (20) may be done analytically after some transformation of variables (see Appendix A). However, the book-keeping of the many terms generated becomes increasingly difficult and, in fact, virtually impossible for the higher-order moments. Therefore, the analytic evaluation of the integrals was done by computer with the algebraic computer language REDUCE 2.

As a result, one obtains a list of twelve collision terms; see Table 1. For each moment equation, $n = 1,\cdot\cdot 4$, the collision term R_n (for brevity, the index s will be omitted henceforth) is a sum of three contributions:

$$R_n = R_{n0} + R_{nP} + R_{nQ} . \tag{23}$$

The index zero stands for the effect of differences in flow velocity and temperature. R_{nP} and R_{nQ} are collision terms due to viscous pressure and heat flow, respectively. The terms without a star are the ones usually retained in plasma transport theories, e.g. BRAGINSKI (1965), where apart from viscous pressure and heat flow the flow velocities are also considered small and only linear terms are retained, except for the Joule heating R_{20} (and R_{2Q}), which is nonlinear in the velocity, and which is kept for energy conservation.

collision terms for	driving effects		
	$\underline{u}$, T	$\underline{\underline{P}}$	$\underline{q}$
$d\underline{u}$ / dt	$\underline{R}_{10}$	* $\underline{R}_{1P}$	$\underline{R}_{1Q}$
dT / dt	R_{20}	* R_{2P}	R_{2Q}
$d\underline{\underline{P}}$ / dt	* $\underline{\underline{R}}_{30}$	$\underline{\underline{R}}_{3P}$	* $\underline{\underline{R}}_{3Q}$
$d\underline{q}$ / dt	$\underline{R}_{40}$	* $\underline{R}_{4P}$	$\underline{R}_{4Q}$

Table 1. Collision terms. Starred quantities disappear when nonlinear terms in U or bilinear terms are neglected.

As a result of the REDUCE calculations the following collision terms in vector notation are obtained:

Momentum equation:

$$\underline{R}_{1O} = \sum_t K_{st}\cdot(\underline{u}_t - \underline{u}_s)\,\frac{3}{2}\,\frac{\mathrm{Erf}\,U - U e^{-U^2}}{U^3} \tag{24}$$

$$\begin{aligned} {}^{*}\,\underline{R}_{1P} = {} & \sum_t K_{st}\,\alpha\cdot\left(\underline{\underline{P}}_s\cdot\underline{U}\,\frac{m_t T_s}{p_s} + \underline{\underline{P}}_t\cdot\underline{U}\,\frac{m_s T_t}{p_t}\right)\frac{3}{4SU^5} \\ & \cdot\left[(2U^3 + 3U)e^{-U^2} - 3\,\mathrm{Erf}\,U\right] \\ & + \sum_t K_{st}\,\alpha\cdot\left(\underline{U}\cdot\underline{\underline{P}}_s\cdot\underline{U}\,\frac{m_t T_s}{p_s} + \underline{U}\cdot\underline{\underline{P}}_t\cdot\underline{U}\,\frac{m_s T_t}{p_t}\right)\underline{U}\,\frac{3}{8SU^7} \\ & \cdot\left[-(4U^5 + 10U^3 + 15U)e^{-U^2} + 15\,\mathrm{Erf}\,U\right] \end{aligned} \tag{25}$$

$$\begin{aligned} \underline{R}_{1Q} = {} & \sum_t K_{st}\left\{\left[\underline{q}_s - 2\underline{U}\,(\underline{q}_s\cdot\underline{U})\right]\frac{m_t T_s}{p_s}\right. \\ & \left. - \left[\underline{q}_t - 2\underline{U}\,(\underline{q}_t\cdot\underline{U})\right]\frac{m_s T_t}{p_t}\right\}\frac{3}{5S}\,e^{-U^2} \end{aligned} \tag{26}$$

where $\underline{U}$ is the ratio of flow velocity difference $\underline{u}_t - \underline{u}_s$ between the species t and s to the effective thermal velocity α ,

$$\underline{U} = (\underline{u}_t - \underline{u}_s)\frac{1}{\alpha} ; \qquad U = |\underline{U}| ; \qquad \alpha^2 = \frac{2kT_s}{m_s} + \frac{2kT_t}{m_t} , \tag{27}$$

Erf x $= \int_0^x dt \, \exp(-t^2)$ is the error function, and

$$S = m_s T_t + m_t T_s ; \qquad m_0 = m_s + m_t . \tag{28}$$

The interaction coefficient K_{st},

$$K_{st} = \frac{16\sqrt{\pi}}{3} \frac{N_s N_t e_s^2 e_t^2 m_0 \ln \Lambda}{m_s m_t \alpha^3} = K_{ts} \tag{29}$$

corresponds to the momentum collision frequency for s - t collisions, multiplied by $m_s N_s$.

For small flow velocities, $U \ll 1$, the function $3\,(\mathrm{Erf}\,U - Ue^{-U^2})/2U^3$ goes to 1 so that the classical linear friction term is obtained from equation (24). For $U \gg 1$ the friction goes down $\sim U^{-3}$. Likewise for $U \gg 1$ the contribution of heat flow to the friction, equation (26), decreases exponentially. Moreover, there are nonlinear terms proportional to $\underline{U}$ $(\underline{q}.\underline{U})$ which may reverse the sign of the heat flow contribution $\underline{R}_{1Q}$ if $U > 0.5$.

Energy equation:

$$R_{20} = \sum_t K_{st} \frac{3k}{m_s m_o U} \left(-m_o T_s U e^{-U^2} + S \operatorname{Erf} U\right) \tag{30}$$

$$\begin{aligned} {}^{*}R_{2P} = \sum_t K_{st} \frac{3kT_s}{4m_o m_s p_s S U^5} \, \underline{U}\cdot\underline{\underline{P}}_s\cdot\underline{U} \, \Big\{ & -\left[4m_o m_t T_s U^5 + 2S(m_t+2m_o)U^3 + 3S(m_t+2m_o)U\right] e^{-U^2} \\ & + 3S(m_t+2m_o) \operatorname{Erf} U \Big\} \\ + \sum_t K_{st} \frac{3kT_t}{4m_o p_t S U^5} \, \underline{U}\cdot\underline{\underline{P}}_t\cdot\underline{U} \, \Big\{ & -\left[4m_o T_s U^5 + 2SU^3 + 3SU\right] e^{-U^2} + 3S \operatorname{Erf} U \Big\} \end{aligned} \tag{31}$$

$$\begin{aligned} R_{2Q} = \sum_t K_{st} \frac{3\alpha m_t}{5m_o S^2} e^{-U^2} \Big[& \frac{T_s}{p_s} \underline{q}_s\cdot\underline{U} \left(-m_t S + 2m_o m_t T_s - 3m_o m_s T_t - 2m_o m_t T_s U^2\right) \\ & + \frac{T_t m_s}{p_t} \underline{q}_t\cdot\underline{U} \left(S - 5m_o T_s + 2m_o T_s U^2\right) \Big]. \end{aligned} \tag{32}$$

When R_{20} is expanded in powers of U (see below) the first resulting term is proportional to the temperature difference $T_t - T_s$, while the next term is proportional to U^2 and corresponds to the dissipation of kinetic energy, i.e. corresponds to Joule heating ηj^2 .

Energy is conserved in the sense that the sum of the energy gained by all species, $\sum_s R_{20}$, is equal to the work done by the friction forces, $-\sum_s \underline{R}_{10} \cdot \underline{u}_s$; see Appendix B. Similarly, the work done by $\underline{R}_{1P}$, $\underline{R}_{1Q}$ is equal to the energy gain $\sum_s R_{2P}$, $\sum_s R_{2Q}$.

Equation for viscous pressure:

$$* \underline{\underline{R}}_{30} = \sum_t K_{st} \left(\underline{\underline{1}} - 3 \underline{U}\,\underline{U} \frac{1}{U^2} \right) \frac{k}{2 m_0 m_s U^3} \Big[(3S + 6 m_0 T_s + 4 m_0 T_s U^2) U e^{-U^2} + (-3S - 6 m_0 T_s + 2 S U^2) \operatorname{Erf} U \Big]. \tag{33}$$

Heat flow equation:

$$\underline{R}_{40} = \sum_t K_{st} \underline{U} \alpha \frac{3 k T_s}{4 m_0 m_s S U^3} \Big[-(6 m_t S + 5 m_0 S + 6 m_0 m_t T_s U^2) U e^{-U^2} + (6 m_t S + 5 m_0 S) \operatorname{Erf} U \Big]. \tag{34}$$

The exact expressions for $\underline{\underline{R}}_{3P}$, $\underline{\underline{R}}_{3Q}$, $\underline{R}_{4P}$, $\underline{R}_{4Q}$ are not shown here because they consist of a very large number of terms - more than eighty for $\underline{R}_{4P}$, for example - and it is felt that as such they are not very useful. The four collision terms, however, are included in the subsequent discussion of simplified versions of the collision terms.

In many cases of pratical interest considerable simplification of the collision terms is possible. For electron-ion and ion-electron collisions one may expand in powers of the mass ratio and keep only the lowest order term. For the collision of two groupsof electrons of different velocity, temperature etc. the masses m_s and m_t are equal, which leads to some simplification. Also for the collision of different species of ions $m_s \cong m_t$ may be a good approximation. Finally, in the self-collision terms, $t = s$, there is considerable simplification anyway. In order to present these approximations or special cases in concise form, it is useful to introduce scalar coefficients A (I,J) by the following definitions.

Momentum equation:

$$\underline{R}_{10} = \sum_t K_{st}\, \underline{U}\; A(1,1)$$

$$\begin{aligned} {}^{*}\underline{R}_{1P} = \sum_t \Big[& \frac{1}{U}\, \underline{\underline{B}}_s \cdot \underline{U}\; A(2,1) + \frac{1}{U}\, \underline{\underline{B}}_t \cdot \underline{U}\; A(2,2) \\ & + \frac{1}{U^2}\, \underline{U}\; \underline{U} \cdot \underline{\underline{B}}_s \cdot \underline{U}\; A(2,3) + \frac{1}{U^2}\, \underline{U}\; \underline{U} \cdot \underline{\underline{B}}_t \cdot \underline{U}\; A(2,4) \Big] \end{aligned} \tag{35}$$

$$\begin{aligned} \underline{R}_{1Q} = \sum_t K_{st} \Big[& \underline{\tau}_s\; A(3,1) + \underline{\tau}_t\; A(3,2) \\ & + \frac{1}{U}\, \underline{U}\; \underline{\tau}_s \cdot \underline{U}\; A(3,3) + \frac{1}{U}\, \underline{U}\; \underline{\tau}_t \cdot \underline{U}\; A(3,4) \Big] \end{aligned}$$

Energy equation :

$$R_{2O} = \sum_t K_{st} \; A(4,1)$$

$$^{*}R_{2P} = \sum_t K_{st} \frac{1}{U^2} \left[\underline{U}\cdot\underline{\underline{B}}_s\cdot\underline{U} \; A(5,1) + \underline{U}\cdot\underline{\underline{B}}_t\cdot\underline{U} \; A(5,2) \right] \tag{36}$$

$$R_{2Q} = \sum_t K_{st} \frac{1}{U} \left[\underline{\tau}_s\cdot\underline{U} \; A(6,1) + \underline{\tau}_t\cdot\underline{U} \; A(6,2) \right]$$

Viscous pressure equation:

$$^{*}\underline{\underline{R}}_{3O} = \sum_t K_{st} \frac{1}{U^2} \left(\underline{\underline{1}} \, U^2 - 3\,\underline{U}\,\underline{U} \right) A(7,1)$$

$$\begin{aligned} \underline{\underline{R}}_{3P} = \sum_t K_{st} \Big\{ & \underline{\underline{B}}_s \; A(8,1) + \underline{\underline{B}}_t \; A(8,2) \\ & + \frac{1}{U^2} \left(\underline{\underline{1}} \, U^2 - 3\,\underline{U}\,\underline{U} \right) \frac{1}{U^2} \left[\underline{U}\cdot\underline{\underline{B}}_s\cdot\underline{U} \; A(8,3) + \underline{U}\cdot\underline{\underline{B}}_t\cdot\underline{U} \; A(8,4) \right] \\ & + \hat{\underline{\underline{B}}}_s \; A(8,5) + \hat{\underline{\underline{B}}}_t \; A(8,6) \Big\} \end{aligned} \tag{37}$$

$$\begin{aligned} ^{*}\underline{\underline{R}}_{3Q} = \sum_t K_{st} \Big\{ & \hat{\underline{\underline{\tau}}}_s \; A(9,1) + \hat{\underline{\underline{\tau}}}_t \; A(9,2) \\ & + \frac{1}{U^2} \left(\underline{\underline{1}} \, U^2 - 3\,\underline{U}\,\underline{U} \right) \frac{1}{U} \left[\underline{\tau}_s\cdot\underline{U} \; A(9,3) + \underline{\tau}_t\cdot\underline{U} \; A(9,4) \right] \Big\} \end{aligned}$$

Heat flow equation:

$$\underline{R}_{40} = \sum_t K_{st}\, \underline{U}\; A(10,1)$$

$$^{*}\underline{R}_{4P} = \sum_t K_{st} \Big[\frac{1}{U} \underline{\underline{B}}_s \cdot \underline{U}\; A(11,1) + \frac{1}{U} \underline{\underline{B}}_t \cdot \underline{U}\; A(11,2)$$
$$+ \frac{1}{U^2} \underline{U}\,\underline{U} \cdot \underline{\underline{B}}_s \cdot \underline{U}\; A(11,3) + \frac{1}{U^2} \underline{U}\,\underline{U} \cdot \underline{\underline{B}}_t \cdot \underline{U}\; A(11,4) \Big] \qquad (38)$$

$$\underline{R}_{4Q} = \sum_t K_{st} \Big[\underline{\tau}_s\; A(12,1) + \underline{\tau}_t\; A(12,2)$$
$$+ \frac{1}{U} \underline{U}\, \underline{\tau}_s \cdot \underline{U}\; A(12,3) + \frac{1}{U} \underline{U}\, \underline{\tau}_t \cdot \underline{U}\; A(12,4) \Big]$$

where

$$\underline{\underline{B}}_s = \frac{1}{p_s} \underline{\underline{P}}_s \; ; \quad \hat{\underline{\underline{B}}}_s = \frac{1}{2U^2} \left(\underline{\underline{B}}_s \cdot \underline{U}\,\underline{U} + \underline{U}\, \underline{\underline{B}}_s \cdot \underline{U} \right) - \frac{1}{U^4} \underline{U}\,\underline{U}\; \underline{U} \cdot \underline{\underline{B}}_s \cdot \underline{U} \qquad (39)$$

$$\underline{\tau}_s = \frac{1}{p_s} \underline{q}_s \; ; \quad \hat{\underline{\underline{\tau}}}_s = \frac{1}{2U} \left(\underline{\tau}_s \underline{U} + \underline{U}\, \underline{\tau}_s \right) - \frac{1}{3U} \underline{\underline{1}}\; \underline{\tau}_s \cdot \underline{U} \; .$$

The computer generated list of the A (I,J) is shown in Tables 2 and 3 for small mass ratio, $m_s << m_t$ or $m_t << m_s$, and in Table 4 for equal masses, $m_s = m_t$. The notation ALFA, K, TS, MS, TE, ME, etc. corresponds to α, k = Boltzmann constant, T_s, m_s, T_e, m_e, etc., respectively, and ST = $T_s + T_t$, EXPU = exp $(-U^2)$, ERFU = Erf U. The indices e (electron) and i (ion) have been used for the light and the heavy components. For the expansion in the mass ratio it was assumed that T_i is not much larger than T_e. The reverse however, is not excluded and is compatible with the expansion in m_e/m_i. The coefficients K_{st}, equation (29), in the corresponding cases are

$$K_{ei} = K_{ie} = \frac{8}{3}\sqrt{\frac{\pi}{2}}\,\frac{N_e N_i e_e^2 e_i^2 \sqrt{m_e}\,\ln\Lambda}{(kT_e)^{3/2}}$$

$$K_{st}(m_s = m_t) = \frac{16}{3}\sqrt{\frac{\pi}{2}}\,\frac{N_s N_t e_s^2 e_t^2 \sqrt{m_s}\,\ln\Lambda}{(kT_s + kT_t)^{3/2}} \tag{40}$$

$$K_{ss} = \frac{4}{3}\sqrt{\pi}\,\frac{N_s^2 e_s^4 \sqrt{m_s}\,\ln\Lambda}{(kT_s)^{3/2}}$$

with

$$\Lambda = \Lambda_{st} = \frac{3kSR_D}{|e_s e_t| m_o} \;; \qquad R_D = \left(4\pi \sum_s \frac{N_s^2 e_s^2}{kT_s}\right)^{-1/2}. \tag{41}$$

In the case $m_s \ll m_t$, Table 2, the coefficient A (4,1) has been given both in lowest order with respect to m_s/m_t and in the next higher order for the reasons explained below.

The summation over the species t in equations (35) - (38) includes the self-collisions t = s. In this case, however, almost all right-hand sides disappear expect for the well-known terms

$$\underline{\underline{R}}_{3P}(t = s) = -\frac{6}{5} K_{ss} \frac{kT_s}{m_s p_s} \underline{\underline{P}}_s \tag{42}$$

$$\underline{R}_{4Q}(t = s) = -\frac{4}{5} K_{ss} \frac{kT_s}{m_s p_s} \underline{q}_s .$$

When this is taken into account explicitly, the summation extends only over t with $t \neq s$.

4. Intermediate flow velocities

When the dimensionless flow velocity $U = |\underline{u}_s - \underline{u}_t| / \alpha$, $\alpha^2 = v^2_{thermal,s} + v^2_{thermal,t}$, is not too close to 1, an expansion of the collision terms in powers of U should be a good approximation. This linearization was done by computer and the result is shown in Table 5 for the coefficients A (I,J), equations (35) - (39). For small mass ratios, $m_s \ll m_t$ and $m_t \ll m_s$, the corresponding approximations are given in Tables 6 and 7, respectively, while for $m_s = m_t$ the result is shown in Table 8.

Tables 5-8 show that among the resulting collision terms even in lowest order some (those marked by a star) are nonlinear in U or contain products of the viscous stresses $\underline{\underline{P}}$ or heat currents $\underline{q}$ with powers of U. Hence, although the resulting non-starred terms to lowest order agree with conventional transport theories (e.g. BRAGINSKII, 1965 or, more generally, BURGERS, 1969) the new set of collision terms goes beyond previous theories in that it includes basically nonlinear effects.

In A (4,1), Table 5, apart from the lowest-order term in U, $\sim U^0$, the next higher term $\sim U^2$ has also been retained because it yields the dominant contribution with respect to the expansion in the mass ratio m_s/m_t, Table 6. It has been checked that the term A (4,1) in Table 6 is the only place where the expansions with respect to U and with respect to the mass ratio do not commute.

A further simplification of the collision terms may be obtained if the expansion with respect to U is not made for each A (I,J) separately, but for each R_{nO}, R_{nP}, R_{nQ} as a whole. Tables 5-8 show that in this case all A (I,J $\geqslant$ 3) may be completely neglected because they are of higher order in U than the other coefficients. Note, however, that, in general, the direction of the vector or tensor quantities is different for the neglected and the non-neglected terms, which may complicate the situation.

5. Conclusions

Starting from the Fokker-Planck equation and a 13-moment ansatz for the distribution functions a set of collision terms was derived for the equations of (I) momentum, (II) energy, (III) viscous stress and (IV) heat flow without the usual assumption that the flow velocities of the plasma constituents be small compared to the thermal velocities. Each collision term has contributions from (A) differences in flow velocity or temperature only, (B) viscous stresses and (C) heat currents, so that all together twelve collision terms were obtained. The number, mass, density and temperature of the plasma components was arbitrary. Special results for small mass ratio or equal masses were given.

A simplified set of terms was obtained by expanding the collision terms in powers of U = relative flow velocity / effective thermal velocity and by keeping the lowest non-vanishing terms only in each collision term. Apart from the usual linear terms, some of the twelve collision terms were found to be essentially nonlinear. The relative size of the linear and nonlinear terms, which are usually neglected a priori, may be inferred from Tables 5 - 8 and equations (35) - (38). This comparison was made in detail elsewhere (SALAT, to be published)where also a discussion of the physical content of the nonlinear terms was given.

Appendix A. Evaluation of collision terms.

The collision terms are of the form

$$R \sim \frac{1}{a_s^3 a_t^3} \iint_{-\infty}^{+\infty} d^3v\, d^3v' \exp\left[-\frac{(\underline{v}-\underline{u}_s)^2}{a_s^2} - \frac{(\underline{v}'-\underline{u}_t)^2}{a_t^2}\right] \tag{A1}$$

$$\cdot \frac{1}{|\underline{v}-\underline{v}'|^3} \quad P(\underline{v}, \underline{v}')$$

where P ($\underline{v}$,$\underline{v}'$) are polynomials in $\underline{v}$,$\underline{v}'$. With the transformations

$$\begin{aligned} \underline{v} - \underline{u}_s &= \underline{c} - \psi \underline{g}_* \\ \underline{v}' - \underline{u}_t &= \underline{c} + (1-\psi)\, \underline{g}_* \end{aligned} \tag{A2}$$

$$\underline{g}_* = \underline{g} - \underline{u}\,; \qquad \underline{u} = \underline{u}_t - \underline{u}_s\,; \qquad \underline{g} = \underline{v}' - \underline{v}$$

$$\psi = \frac{m_t T_s}{S}\,; \qquad S = m_s T_t + m_t T_s$$

(A1) goes over into

$$R \sim \frac{1}{a^3 \alpha^3} \iint_{-\infty}^{+\infty} d^3C \, d^3g \; \exp\left[-\frac{C^2}{a^2} - \frac{g_*^2}{\alpha^2} \right] \tag{A3}$$

$$\cdot \; \frac{1}{g^3} \; \tilde{P}(\underline{C}, \underline{g})$$

with

$$a^2 = \frac{2kT_sT_t}{S} \; ; \qquad \alpha^2 = \frac{2kS}{m_s m_t} \tag{A4}$$

The $\underline{C}$-integration involves integrals of the form, in component notation,

$(\sqrt{\pi}\, a)^{-3} \int d^3C \exp(-C^2/a^2)\, C_{i_1} \cdots C_{i_l}$, with

$$\frac{1}{(\sqrt{\pi}\, a)^3} \int d^3C \exp\left(-\frac{C^2}{a^2}\right) C_i = 0$$

$$\frac{1}{(\sqrt{\pi}\, a)^3} \int d^3C \exp\left(-\frac{C^2}{a^2}\right) C_i C_k = \frac{1}{2}\, a^2 \delta_{ik} \tag{A5}$$

etc. For the $\underline{g}$-integration polar coordinates are introduced. The φ - integration gives integrals of the type $\int d\varphi\, g_{i_1} \cdots g_{i_m}$, with

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, g_i = \frac{g z u_i}{u} \quad ; \qquad z = \cos\vartheta \tag{A6}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, g_i g_k = \frac{1}{2} g^2 (1-z^2)\,\delta_{ik} + \frac{1}{2} g^2 (3z^2-1)\frac{u_i u_k}{u^2} \ ,$$

etc. (For more details, see Burgers). The g-integration involves

$$\int_0^\infty dg\, g^n \exp\left[-\frac{g^2-2guz}{\alpha^2}\right]$$

$$= \alpha\left(\frac{\alpha^2}{2z}\right)^n \frac{\partial^n}{\partial u^n} \exp\left(\frac{u^2z^2}{\alpha^2}\right) \mathrm{Erf\,c}\left(\frac{-uz}{\alpha}\right) \tag{A7}$$

where $\mathrm{Erf\ c}\ (x) = \int_x^\infty dt\, e^{-t^2}$. The z-integration, finally, involves integrals of the type $\int_{-a}^{+a} dz\, z^{2t+1} \exp(z^2)\, \mathrm{Erf\,c}(-z)$

By substituting $\exp(z^2) = \exp(s z^2)|_{s=1}$ and taking the derivative with respect to s they may be constructed from

$$\int_{-a}^{+a} dz\, z \exp(z^2)\, \mathrm{Erf\,c}\,(-z) = \exp(a^2)\,\mathrm{Erf}\,(a) - a \tag{A8}$$

where Erf $(x) = \int_0^x dt\, e^{-t^2}$.

A list of integrals (A5) - (A8) l,m,n,r ranging up to 5,4,5,3 respectively, together with some other trivial integrals, was given to the computer. Also the operator definition of the δ-function,

$$\sum_i \delta_{ik}\, a_i = a_k \tag{A9}$$

for arbitrary a_i, and some basic vector rules, like

$$\sum_i g_i^2 = g^2 \tag{A10}$$

were implemented. As a result of the algebraic computer manipulations the equations (24) et seq. were obtained.

Appendix B. Energy conservation

From equations (27) and (30) one obtains

$$\sum_s R_{20} = \sum_{s,t} K_{st} \frac{3k}{m_s m_0 U} \left(-m_0 T_s U e^{-U^2} + S \operatorname{erf} U\right)$$

$$= \sum_{s,t} K_{st} \frac{3k}{m_0 U} \left[-\frac{m_0}{2}\left(\frac{T_s}{m_s} + \frac{T_t}{m_t}\right) U e^{-U^2} + \frac{1}{2}\left(\frac{1}{m_s} + \frac{1}{m_t}\right) S \operatorname{erf} U\right]$$

$$= \sum_{s,t} K_{st} \frac{3kS}{2 m_s m_t U} \left(-U e^{-U^2} + \operatorname{erf} U\right)$$

while from equations (24) and (27) it follows that

$$\sum_s \underline{R}_{10} \cdot \underline{u}_s = \sum_{s,t} K_{st} (\underline{u}_t - \underline{u}_s) \cdot \underline{u}_s \frac{3}{2U^3} (\mathrm{erf}\, U - U e^{-U^2})$$

$$= \sum_{s,t} K_{st} \cdot \frac{-1}{2} (\underline{u}_t - \underline{u}_s)^2 \frac{3}{2U^3} (\mathrm{erf}\, U - U e^{-U^2})$$

$$= - \sum_{s,t} K_{st} \frac{3kS}{2 m_s m_t U} (\mathrm{erf}\, U - U e^{-U^2})$$

$$= - \sum_s R_{20}$$

Acknowledgements: The author is grateful to Dr. H.K. Wimmel for helpful discussions.

TABLE 2

FOR SMALL MASS RATIO: MS = ME << MI = MT

```
                                                       3
A( 1,1) = (3*ALFA*( - U*EXPU + ERFU))/(2*U )

                        3                                          4
A( 2,1) = (3*ALFA*(2*U *EXPU + 3*U*EXPU - 3*ERFU))/(4*U )

                              3                                       4
A( 2,2) = (3*ME*ALFA*TI*(2*U *EXPU + 3*U*EXPU - 3*ERFU))/(4*U *MI*

           TE)

                           5            3
A( 2,3) = (3*ALFA*( - 4*U *EXPU - 10*U *EXPU - 15*U*EXPU + 15*ERFU)

                5
           )/(8*U )

                                 5            3
A( 2,4) = (3*ME*ALFA*TI*( - 4*U *EXPU - 10*U *EXPU - 15*U*EXPU + 15

                      5
           *ERFU))/(8*U *MI*TE)

A( 3,1) = (3*EXPU)/5

A( 3,2) = ( - 3*ME*EXPU*TI)/(5*MI*TE)

A( 3,3) = ( - 6*U*EXPU)/5

A( 3,4) = (6*U*ME*EXPU*TI)/(5*MI*TE)

A( 4,1) = (3*K*( - U*EXPU*MI*TE + ME*ERFU*TI - ME*ERFU*TE + ERFU*MI

           *TE))/(U*ME*MI)

                           5           3
A( 5,1) = (3*K*TE*( - 4*U *EXPU - 6*U *EXPU - 9*U*EXPU + 9*ERFU))/(

             3
           4*U *ME)

                           5           3
A( 5,2) = (3*K*TI*( - 4*U *EXPU - 2*U *EXPU - 3*U*EXPU + 3*ERFU))/(

             3
           4*U *MI)

                                      2
A( 6,1) = (3*U*EXPU*ALFA*( - 2*U  + 1))/5

                                         2
A( 6,2) = (6*U*ME*EXPU*ALFA*TI*(U  - 2))/(5*MI*TE)
```

TABLE 2

$$A(7,1) = (K*TE*(4*U^{3}*EXPU + 2*U^{2}*ERFU + 9*U*EXPU - 9*ERFU))/(2*U^{3}*ME)$$

$$A(8,1) = (9*K*TE*(- 2*U^{2}*ERFU - 3*U*EXPU + 3*ERFU))/(4*U^{5}*ME)$$

$$A(8,2) = (3*K*TI*(- 4*U^{3}*EXPU - 2*U^{2}*ERFU - 9*U*EXPU + 9*ERFU))/(4*U^{5}*MI)$$

$$A(8,3) = (K*TE*(16*U^{7}*EXPU + 48*U^{5}*EXPU + 108*U^{3}*EXPU - 18*U^{2}*ERFU + 135*U*EXPU - 135*ERFU))/(8*U^{5}*ME)$$

$$A(8,4) = (K*TI*(16*U^{7}*EXPU + 32*U^{5}*EXPU + 84*U^{3}*EXPU + 6*U^{2}*ERFU + 135*U*EXPU - 135*ERFU))/(8*U^{5}*MI)$$

$$A(8,5) = (3*K*TE*(4*U^{5}*EXPU + 18*U^{3}*EXPU + 12*U^{2}*ERFU + 45*U*EXPU - 45*ERFU))/(2*U^{5}*ME)$$

$$A(8,6) = (3*K*TI*(8*U^{5}*EXPU + 24*U^{3}*EXPU + 6*J^{2}*ERFU + 45*U*EXPU - 45*ERFU))/(2*U^{5}*MI)$$

$$A(9,1) = (3*ALFA*(2*U^{5}*EXPU + 6*U^{3}*EXPU + 9*U*EXPU - 9*ERFU))/(5*U^{4})$$

$$A(9,2) = (3*ME*ALFA*TI*(- 4*U^{5}*EXPU - 2*U^{3}*EXPU - 3*J*EXPU + 3*ERFU))/(5*U^{4}*MI*TE)$$

$$A(9,3) = (ALFA*(8*U^{7}*EXPU + 12*U^{5}*EXPU + 30*J^{3}*EXPU + 45*U*EXPU - 45*ERFU))/(10*U^{4})$$

TABLE 2

$$A(9,4) = (ME*ALFA*TI*(-8*U^{7}*EXPU - 4*U^{5}*EXPU - 10*U^{3}*EXPU - 15*U*EXPU + 15*ERFU))/(10*U^{4}*MI*TE)$$

$$A(10,1) = (9*K*ALFA*TE*(-U^{3}*EXPU - U*EXPU + ERFU))/(2*U^{3}*ME)$$

$$A(11,1) = (3*K*ALFA*TE*(2*U^{5}*EXPU + 10*U^{3}*EXPU + 8*U^{2}*ERFU + 27*U*EXPU - 27*ERFU))/(4*U^{4}*ME)$$

$$A(11,2) = (9*K*ALFA*TI*(2*U^{5}*EXPU + 2*U^{3}*EXPU + 3*U*EXPU - 3*ERFU))/(4*U^{4}*MI)$$

$$A(11,3) = (3*K*ALFA*TE*(-12*U^{7}*EXPU - 28*U^{5}*EXPU - 78*U^{3}*EXPU - 12*U^{2}*ERFU - 135*U*EXPU + 135*ERFU))/(8*U^{5}*ME)$$

$$A(11,4) = (9*K*ALFA*TI*(-4*U^{7}*EXPU - 4*U^{5}*EXPU - 10*U^{3}*EXPU - 15*U*EXPU + 15*ERFU))/(8*U^{5}*MI)$$

$$A(12,1) = (3*K*TE*(2*U^{5}*EXPU + 7*U^{3}*EXPU + 17*U*EXPU - 17*ERFU))/(10*U^{3}*ME)$$

$$A(12,2) = (9*K*EXPU*TI*(-2*U^{2} + 3))/(10*MI)$$

$$A(12,3) = (3*K*TE*(-12*U^{7}*EXPU - 2*U^{5}*EXPU - 34*U^{3}*EXPU - 51*U*EXPU + 51*ERFU))/(10*U^{4}*ME)$$

$$A(12,4) = (9*K*U*EXPU*TI*(2*U^{2} - 5))/(5*MI)$$

TABLE 3

FOR SMALL MASS RATIO: MS = MI >> ME = MT

```
                                                   3
A( 1,1) = (3*ALFA*( - U*EXPU + ERFU))/(2*U )

                                 3                                   4
A( 2,1) = (3*ME*ALFA*TI*(2*U *EXPU + 3*U*EXPU - 3*ERFU))/(4*U *MI*
          TE)

                          3                                  4
A( 2,2) = (3*ALFA*(2*U *EXPU + 3*U*EXPU - 3*ERFU))/(4*U )

                                        5            3
A( 2,3) = (3*ME*ALFA*TI*( - 4*U *EXPU - 10*U *EXPU - 15*U*EXPU + 15

                5
          *ERFU))/(8*U *MI*TE)

                              5             3
A( 2,4) = (3*ALFA*( - 4*U *EXPU - 10*U *EXPU - 15*U*EXPU + 15*ERFU)

             5
          )/(8*U )
A( 3,1) = (3*ME*EXPU*TI)/(5*MI*TE)
A( 3,2) = ( - 3*EXPU)/5
A( 3,3) = ( - 6*U*ME*EXPU*TI)/(5*MI*TE)
A( 3,4) = (6*U*EXPU)/5
A( 4,1) = (3*K*( - U*EXPU*TI + ERFU*TE))/(U*MI)

                              3                                    3
A( 5,1) = (3*K*TI*( - 2*U *EXPU - 3*U*EXPU + 3*ERFU))/(2*U *MI)

                          5                3
A( 5,2) = (3*K*( - 4*U *EXPU*TI - 2*U *EXPU*TE - 3*U*EXPU*TE + 3*

                              3
          ERFU*TE))/(4*U *MI)
A( 6,1) = ( - 9*U*ME*EXPU*ALFA*TI)/(5*MI*TE)

                                         2
A( 6,2) = (3*U*ME*EXPU*ALFA*(2*U *TI + TE - 5*TI))/(5*MI*TE)
```

TABLE 3

$$A(7,1) = (K*(4*U^{3}*EXPU*TI + 2*U^{2}*ERFU*TE + 3*U*EXPU*TE + 6*U*EXPU*TI - 3*ERFU*TE - 6*ERFU*TI))/(2*U^{3}*MI)$$

$$A(8,1) = (3*K*TI*(U*EXPU - ERFU))/(U^{3}*MI)$$

$$A(8,2) = (3*K*(- 4*U^{3}*EXPU*TI - 2*U^{2}*ERFU*TE - 3*U*EXPU*TE - 6*U*EXPU*TI + 3*ERFU*TE + 6*ERFU*TI))/(4*U^{5}*MI)$$

$$A(8,3) = (K*TI*(2*U^{3}*EXPU + 3*U*EXPU - 3*ERFU))/(U^{3}*MI)$$

$$A(8,4) = (K*(16*U^{7}*EXPU*TI + 8*U^{5}*EXPU*TE + 24*U^{5}*EXPU*TI + 24*U^{3}*EXPU*TE + 60*U^{3}*EXPU*TI + 6*U^{2}*ERFU*TE + 45*U*EXPU*TE + 90*U*EXPU*TI - 45*ERFU*TE - 90*ERFU*TI))/(8*U^{5}*MI)$$

$$A(8,5) = (3*K*TI*(- 2*U^{3}*EXPU - 3*U*EXPU + 3*ERFU))/(U^{3}*MI)$$

$$A(8,6) = (3*K*(8*U^{5}*EXPU*TI + 4*U^{3}*EXPU*TE + 20*U^{3}*EXPU*TI + 6*U^{2}*ERFU*TE + 15*U*EXPU*TE + 30*U*EXPU*TI - 15*ERFU*TE - 30*ERFU*TI))/(2*U^{5}*MI)$$

$$A(9,1) = (6*ME*ALFA*TI*(- U^{5}*EXPU + 2*U^{3}*EXPU + 3*U*EXPU - 3*ERFU))/(5*U^{4}*MI*TE)$$

$$A(9,2) = (3*ME*ALFA*(- 4*U^{5}*EXPU*TI - 2*U^{3}*EXPU*TE - 3*U*EXPU*TE + 3*ERFU*TE))/(5*U^{4}*MI*TE)$$

$$A(9,3) = (ME*ALFA*TI*(4*U^{5}*EXPU + 10*U^{3}*EXPU + 15*U*EXPU - 15*ERFU))/(5*U^{4}*MI*TE)$$

TABLE 3

$$A(9,4) = (ME*ALFA*(- 8*U^{7}*EXPU*TI - 4*U^{5}*EXPU*TE - 10*U^{3}*EXPU*TE - 15*U*EXPU*TE + 15*ERFU*TE))/(10*J^{4}*MI*TE)$$

$$A(10,1) = (9*K*ME*ALFA*TI*(- U^{3}*EXPU*TI - U*EXPU*TE + ERFU*TE))/(2*U^{3}*MI^{2}*TE)$$

$$A(11,1) = (3*K*ALFA*TI*(- U*EXPU + ERFU))/(2*U^{2}*MI)$$

$$A(11,2) = (9*K*ME*ALFA*TI*(2*U^{5}*EXPU*TI + 2*U^{3}*EXPU*TE + 3*U*EXPU*TE - 3*ERFU*TE))/(4*U^{4}*MI^{2}*TE)$$

$$A(11,3) = (3*K*ME*ALFA*TI*(- 8*U^{5}*EXPU*TI - 4*U^{3}*EXPU*TE - 20*U^{3}*EXPU*TI - 6*U^{2}*ERFU*TE - 15*U*EXPU*TE - 30*U*EXPU*TI + 15*ERFU*TE + 30*ERFU*TI))/(4*U^{5}*MI^{2}*TE)$$

$$A(11,4) = (9*K*ME*ALFA*TI*(- 4*U^{7}*EXPU*TI - 4*U^{5}*EXPU*TE - 10*U^{3}*EXPU*TE - 15*U*EXPU*TE + 15*ERFU*TE))/(8*U^{5}*MI^{2}*TE)$$

$$A(12,1) = (3*K*TI*(- 4*U^{3}*EXPU + 9*U*EXPU - 9*ERFU))/(10*U^{3}*MI)$$

$$A(12,2) = (9*K*ME*EXPU*TI*(- 2*U^{2}*TI - 2*TE + 5*TI))/(10*MI^{2}*TE)$$

$$A(12,3) = (27*K*TI*(- 2*U^{3}*EXPU - 3*U*EXPU + 3*ERFU))/(10*U^{4}*MI)$$

$$A(12,4) = (9*K*U*ME*EXPU*TI*(2*U^{2}*TI + 2*TE - 7*TI))/(5*MI^{2}*TE)$$

TABLE 4

FOR EQUAL MASSES: MS = MT

```
                                                 3
A( 1,1)  =  (3*ALFA*( - U*EXPU + ERFU))/(2*U )

                         3                                          4
A( 2,1)  =  (3*TS*ALFA*(2*U *EXPU + 3*U*EXPU - 3*ERFU))/(4*U *ST)

                         3                                          4
A( 2,2)  =  (3*TT*ALFA*(2*U *EXPU + 3*U*EXPU - 3*ERFU))/(4*U *ST)

                               5            3
A( 2,3)  =  (3*TS*ALFA*( - 4*U *EXPU - 10*U *EXPU - 15*U*EXPU + 15*

                         5
             ERFU))/(8*U *ST)

                               5            3
A( 2,4)  =  (3*TT*ALFA*( - 4*U *EXPU - 10*U *EXPU - 15*U*EXPU + 15*

                         5
             ERFU))/(8*U *ST)

A( 3,1)  =  (3*TS*EXPU)/(5*ST)

A( 3,2)  =  ( - 3*TT*EXPU)/(5*ST)

A( 3,3)  =  ( - 6*U*TS*EXPU)/(5*ST)

A( 3,4)  =  (6*U*TT*EXPU)/(5*ST)

A( 4,1)  =  (3*K*( - 2*U*TS*EXPU + TT*ERFU + TS*ERFU))/(2*U*MS)

                          5                3                    3
A( 5,1)  =  (3*K*TS*( - 8*U *TS*EXPU - 10*U *TT*EXPU - 10*U *TS*EXPU

              - 15*U*TT*EXPU - 15*U*TS*EXPU + 15*TT*ERFU + 15*TS*ERFU

                      3
             ))/(8*U *ST*MS)

                          5               3                 3
A( 5,2)  =  (3*K*TT*( - 8*U *TS*EXPU - 2*U *TT*EXPU - 2*U *TS*EXPU -

             3*U*TT*EXPU - 3*U*TS*EXPU + 3*TT*ERFU + 3*TS*ERFU))/(8*U

             3
             *ST*MS)
```

TABLE 4

$$A(6,1) = (3*U*TS*EXPU*ALFA*(-4*U^{2}*TS - 7*TT + 3*TS))/(10*ST^{2})$$

$$A(6,2) = (3*U*TT*EXPU*ALFA*(4*U^{2}*TS + TT - 9*TS))/(10*ST^{2})$$

$$A(7,1) = (K*(8*U^{3}*TS*EXPU + 2*U^{2}*ERFU*TT + 2*U^{2}*ERFU*TS + 3*U*TT*EXPU + 15*U*TS*EXPU - 3*ERFU*TT - 15*ERFU*TS))/(4*U^{3}*MS)$$

$$A(8,1) = (3*K*TS*(8*U^{3}*TT*EXPU - 10*U^{2}*ERFU*TT - 10*U^{2}*ERFU*TS - 3*U*TT*EXPU - 15*U*TS*EXPU + 3*ERFU*TT + 15*ERFU*TS))/(8*U^{5}*ST*MS)$$

$$A(8,2) = (3*K*TT*(-8*U^{3}*TS*EXPU - 2*U^{2}*ERFU*TT - 2*U^{2}*ERFU*TS - 3*U*TT*EXPU - 15*U*TS*EXPU + 3*ERFU*TT + 15*ERFU*TS))/(8*U^{5}*ST*MS)$$

$$A(8,3) = (K*TS*(32*U^{7}*TS*EXPU + 40*U^{5}*TT*EXPU + 88*U^{5}*TS*EXPU + 72*U^{3}*TT*EXPU + 192*U^{3}*TS*EXPU - 42*U^{2}*ERFU*TT - 42*U^{2}*ERFU*TS + 45*U*TT*EXPU + 225*U*TS*EXPU - 45*ERFU*TT - 225*ERFU*TS))/(16*U^{5}*ST*MS)$$

$$A(8,4) = (K*TT*(32*U^{7}*TS*EXPU + 8*U^{5}*TT*EXPU + 56*U^{5}*TS*EXPU + 24*U^{3}*TT*EXPU + 144*U^{3}*TS*EXPU + 6*U^{2}*ERFU*TT + 6*U^{2}*ERFU*TS + 45*U*TT*EXPU + 225*U*TS*EXPU - 45*ERFU*TT - 225*ERFU*TS))/(16*U^{5}*ST*MS)$$

TABLE 4

$$A(8,5) = (3*K*TS*(- 8*U^{5}*EXPU*TT + 8*U^{5}*EXPU*TS - 8*U^{3}*EXPU*TT + 32*U^{3}*EXPU*TS + 18*U^{2}*ERFU*TT + 18*U^{2}*ERFU*TS + 15*U*EXPU*TT + 75*U*EXPU*TS - 15*ERFU*TT - 75*ERFU*TS))/(4*U^{5}*ST*MS)$$

$$A(8,6) = (3*K*TT*(16*U^{5}*EXPU*TS + 4*U^{3}*EXPU*TT + 44*U^{3}*EXPU*TS + 6*U^{2}*ERFU*TT + 6*U^{2}*ERFU*TS + 15*U*EXPU*TT + 75*U*EXPU*TS - 15*ERFU*TT - 75*ERFU*TS))/(4*U^{5}*ST*MS)$$

$$A(9,1) = (3*ALFA*TS*(- 4*U^{5}*EXPU*TT + 4*U^{5}*EXPU*TS + 10*U^{3}*EXPU*TT + 10*U^{3}*EXPU*TS + 15*U*EXPU*TT + 15*U*EXPU*TS - 15*ERFU*TT - 15*ERFU*TS))/(10*U^{4}*ST^{2})$$

$$A(9,2) = (3*ALFA*TT*(- 8*U^{5}*EXPU*TS - 2*U^{3}*EXPU*TT - 2*U^{3}*EXPU*TS - 3*U*EXPU*TT - 3*U*EXPU*TS + 3*ERFU*TT + 3*ERFU*TS))/(10*U^{4}*ST^{2})$$

$$A(9,3) = (ALFA*TS*(16*U^{7}*EXPU*TS + 20*U^{5}*EXPU*TT + 20*U^{5}*EXPU*TS + 50*U^{3}*EXPU*TT + 50*U^{3}*EXPU*TS + 75*U*EXPU*TT + 75*U*EXPU*TS - 75*ERFU*TT - 75*ERFU*TS))/(20*U^{4}*ST^{2})$$

$$A(9,4) = (ALFA*TT*(- 16*U^{7}*EXPU*TS - 4*U^{5}*EXPU*TT - 4*U^{5}*EXPU*TS - 10*U^{3}*EXPU*TT - 10*U^{3}*EXPU*TS - 15*U*EXPU*TT - 15*U*EXPU*TS + 15*ERFU*TT + 15*ERFU*TS))/(20*U^{4}*ST^{2})$$

TABLE 4

$$A(10,1) = (9*K*ALFA*TS*(- 2*U^{3}*EXPU*TS - U*TT*EXPU - U*EXPU*TS + ERFU*TT + ERFU*TS))/(4*U^{3}*ST*MS)$$

$$A(11,1) = (3*K*ALFA*TS*(- 8*U^{5}*TT*EXPU*TS + 4*U^{5}*EXPU*TS^{2} - 6*U^{3}*TT^{2}*EXPU + 10*U^{3}*TT*EXPU*TS + 16*U^{3}*EXPU*TS^{2} + 10*U^{2}*ERFU*TT^{2} + 20*U^{2}*ERFU*TT*TS + 10*U^{2}*ERFU*TS^{2} + 6*U*TT^{2}*EXPU + 45*U*TT*EXPU*TS + 39*U*EXPU*TS^{2} - 6*ERFU*TT^{2} - 45*ERFU*TT*TS - 39*ERFU*TS^{2}))/(8*U^{4}*ST^{2}*MS)$$

$$A(11,2) = (9*K*TT*ALFA*TS*(4*U^{5}*EXPU*TS + 2*U^{3}*TT*EXPU + 2*U^{3}*EXPU*TS + 3*U*TT*EXPU + 3*U*EXPU*TS - 3*ERFU*TT - 3*ERFU*TS))/(8*U^{4}*ST^{2}*MS)$$

$$A(11,3) = (3*K*ALFA*TS*(- 24*U^{7}*EXPU*TS^{2} - 44*U^{5}*TT*EXPU*TS - 44*U^{5}*EXPU*TS^{2} - 8*U^{3}*TT^{2}*EXPU - 126*U^{3}*TT*EXPU*TS - 118*U^{3}*EXPU*TS^{2} - 12*U^{2}*ERFU*TT^{2} - 24*U^{2}*ERFU*TT*TS - 12*U^{2}*ERFU*TS^{2} - 30*U*TT^{2}*EXPU - 225*U*TT*EXPU*TS - 195*U*EXPU*TS^{2} + 30*ERFU*TT^{2} + 225*ERFU*TT*TS + 195*ERFU*TS^{2}))/(16*U^{5}*ST^{2}*MS)$$

$$A(11,4) = (9*K*TT*ALFA*TS*(- 8*U^{7}*EXPU*TS - 4*U^{5}*TT*EXPU - 4*U^{5}*EXPU*TS - 10*U^{3}*TT*EXPU - 10*U^{3}*EXPU*TS - 15*U*TT*EXPU - 15*U*EXPU*TS + 15*ERFU*TT + 15*ERFU*TS))/(16*U^{5}*ST^{2}*MS)$$

TABLE 4

$$A(12,1) = (3*K*TS*(- 4*U^{5}*EXPU*TT*TS + 2*U^{5}*EXPU*TS^{2} - 5*U^{3}*EXPU*TT^{2} + 15*U^{3}*EXPU*TT*TS + 5*U^{3}*EXPU*TS^{2} + 13*U*EXPU*TT^{2} + 26*U*EXPU*TT*TS + 13*U*EXPU*TS^{2} - 13*ERFU*TT^{2} - 26*ERFU*TT*TS - 13*ERFU*TS^{2}))/(10*U^{3}*ST^{2}*MS)$$

$$A(12,2) = (9*K*EXPU*TT*TS*(- 2*U^{2}*TS - TT + 4*TS))/(10*ST^{2}*MS)$$

$$A(12,3) = (3*K*TS*(- 12*U^{7}*EXPU*TS^{2} - 38*U^{5}*EXPU*TT*TS + 4*U^{5}*EXPU*TS^{2} - 26*U^{3}*EXPU*TT^{2} - 52*U^{3}*EXPU*TT*TS - 26*U^{3}*EXPU*TS^{2} - 39*U*EXPU*TT^{2} - 78*U*EXPU*TT*TS - 39*U*EXPU*TS^{2} + 39*ERFU*TT^{2} + 78*ERFU*TT*TS + 39*ERFU*TS^{2}))/(10*U^{4}*ST^{2}*MS)$$

$$A(12,4) = (9*K*U*EXPU*TT*TS*(2*U^{2}*TS + TT - 6*TS))/(5*ST^{2}*MS)$$

TABLE 5

```
A( 1,1) = ALFA

A( 2,1) = ( - 3*U*MT*TS*ALFA)/(5*S)

A( 2,2) = ( - 3*U*MS*TT*ALFA)/(5*S)

              2
A( 2,3) = (3*U *MT*TS*ALFA)/(7*S)

              2
A( 2,4) = (3*U *MS*TT*ALFA)/(7*S)

A( 3,1) = (3*MT*TS)/(5*S)

A( 3,2) = ( - 3*MS*TT)/(5*S)

A( 3,3) = ( - 6*U*MT*TS)/(5*S)

A( 3,4) = (6*U*MS*TT)/(5*S)

                   2            2            2
A( 4,1) = (K*( - U *MS*TT + 3*U *MS*TS + 2*U *MT*TS + 3*MS*TT - 3*

              MS*TS))/(MO*MS)

                 2         2                                     2
A( 5,1) = (3*K*U *TS*(2*MS *TT + 3*MS*TT*MT - 3*MS*MT*TS - 2*MT *TS

          ))/(5*S*MO*MS)

                 2
A( 5,2) = (3*K*U *TT*(MS*TT - 5*MS*TS - 4*MT*TS))/(5*S*MO)

                                     2
A( 6,1) = (3*U*MT*TS*ALFA*( - 3*MS *TT - 4*MS*TT*MT + 2*MS*MT*TS +

             2        2
          MT *TS))/(5*S *MO)

                                                                2
A( 6,2) = (3*U*MS*TT*MT*ALFA*(MS*TT - 5*MS*TS - 4*MT*TS))/(5*S *MO)

                 2
A( 7,1) = (4*K*U *(MS*TT - 3*MS*TS - 2*MT*TS))/(15*MO*MS)

                              2                                  2
A( 8,1) = (2*K*TS*( - 5*MS *TT - 6*MS*TT*MT - 2*MS*MT*TS - 3*MT *TS

          ))/(5*S*MO*MS)

A( 8,2) = (2*K*TT*( - MS*TT + 3*MS*TS + 2*MT*TS))/(5*S*MO)
```

TABLE 5

$$A(8,3) = (4*K*U^{2}*TS*(MT^{2}*TS - 9*MT*MS*TT + 3*MT*MS*TS - 7*MS^{2}*TT))/(35*S*MO*MS)$$

$$A(8,4) = (8*K*U^{2}*TT*(4*MT*TS - MS*TT + 5*MS*TS))/(35*S*MO)$$

$$A(8,5) = (12*K*U^{2}*TS*(2*S*MT - 3*MO*MT*TS + 7*MO*MS*TT))/(35*S*MO*MS)$$

$$A(8,6) = (24*K*U^{2}*TT*(S - 5*MO*TS))/(35*S*MO)$$

$$A(9,1) = (6*U*ALFA*MT*TS*(- MT^{2}*TS - 11*MT*MS*TT + MT*MS*TS - 9*MS^{2}*TT))/(25*S^{2}*MC)$$

$$A(9,2) = (12*U*ALFA*MT*MS*TT*(- 4*MT*TS + MS*TT - 5*MS*TS))/(25*S^{2}*MO)$$

$$A(9,3) = (4*U^{3}*ALFA*MT*TS*(4*MT^{2}*TS - 3*MT*MS*TT + 5*MT*MS*TS - 2*MS^{2}*TT))/(35*S^{2}*MO)$$

$$A(9,4) = (4*U^{3}*ALFA*MT*MS*TT*(- 6*MT*TS + MS*TT - 7*MS*TS))/(35*S^{2}*MO)$$

$$A(10,1) = (3*K*ALFA*MT*TS*(- MT*TS + 2*MS*TT - 3*MS*TS))/(2*S*MO*MS)$$

$$A(11,1) = (K*U*ALFA*TS*(MT^{3}*TS^{2} - 16*MT^{2}*MS*TT*TS + MT^{2}*MS*TS^{2} + 28*MT*MS^{2}*TT^{2} - 34*MT*MS^{2}*TT*TS + 10*MS^{3}*TT^{2}))/(10*S^{2}*MO*MS)$$

$$A(11,2) = (9*K*U*ALFA*MT*TT*TS*(3*MT*TS - 2*MS*TT + 5*MS*TS))/(10*S^{2}*MC)$$

TABLE 5

$$A(11,3) = (3*K*U^{2}*ALFA*TS*MT*(- 8*MS^{2}*TT^{2} + 40*MS^{2}*TT*TS + 54*MS*TT*TS*MT - 65*MS*TS^{2}*MT - 43*TS^{2}*MT^{2}))/(70*S^{2}*MO*MS)$$

$$A(11,4) = (9*K*U^{2}*TT*ALFA*TS*MT*(2*MS*TT - 7*MS*TS - 5*TS*MT))/(14*S^{2}*MO)$$

$$A(12,1) = (K*TS*(- 30*MS^{3}*TT^{2} - 52*MS^{2}*TT^{2}*MT + 6*MS^{2}*TT*TS*MT - 20*MS*TT*TS*MT^{2} - 9*MS*TS^{2}*MT^{2} - 13*TS^{2}*MT^{3}))/(10*S^{2}*MO*MS)$$

$$A(12,2) = (9*K*TT*TS*MT*(- 2*MS*TT + 5*MS*TS + 3*TS*MT))/(10*S^{2}*MO)$$

$$A(12,3) = (3*K*U*TS*(18*MS^{3}*TT^{2} + 34*MS^{2}*TT^{2}*MT - 44*MS^{2}*TT*TS*MT - 42*MS*TT*TS*MT^{2} + 43*MS*TS^{2}*MT^{2} + 29*TS^{2}*MT^{3}))/(25*S^{2}*MO*MS)$$

$$A(12,4) = (9*K*U*TT*TS*MT*(2*MS*TT - 7*MS*TS - 5*TS*MT))/(5*S^{2}*MO)$$

TABLE 6

FOR SMALL MASS RATIO: MS = ME << MI = MT

$A(1,1) = ALFA$

$A(2,1) = (-3*U*ALFA)/5$

$A(2,2) = (-3*U*ME*TI*ALFA)/(5*MI*TE)$

$A(2,3) = (3*U^{2}*ALFA)/7$

$A(2,4) = (3*U^{2}*ME*TI*ALFA)/(7*MI*TE)$

$A(3,1) = 3/5$

$A(3,2) = (-3*ME*TI)/(5*MI*TE)$

$A(3,3) = (-6*U)/5$

$A(3,4) = (6*U*ME*TI)/(5*MI*TE)$

$A(4,1) = (K*(2*U^{2}*MI*TE + 3*ME*TI - 3*ME*TE))/(ME*MI)$

$A(5,1) = (-6*K*U^{2}*TE)/(5*ME)$

$A(5,2) = (-12*K*U^{2}*TI)/(5*MI)$

$A(6,1) = (3*U*ALFA)/5$

$A(6,2) = (-12*U*ME*TI*ALFA)/(5*MI*TE)$

$A(7,1) = (-8*K*U^{2}*TE)/(15*ME)$

$A(8,1) = (-6*K*TE)/(5*ME)$

$A(8,2) = (4*K*TI)/(5*MI)$

TABLE 6

$A(8,3) = (4*K*U^{2}*TE)/(35*ME)$

$A(8,4) = (32*K*U^{2}*TI)/(35*MI)$

$A(8,5) = (- 12*K*U^{2}*TE)/(35*ME)$

$A(8,6) = (- 96*K*U^{2}*TI)/(35*MI)$

$A(9,1) = (- 6*U*ALFA)/25$

$A(9,2) = (- 48*U*TI*ME*ALFA)/(25*TE*MI)$

$A(9,3) = (16*U^{3}*ALFA)/35$

$A(9,4) = (- 24*U^{3}*TI*ME*ALFA)/(35*TE*MI)$

$A(10,1) = (- 3*K*TE*ALFA)/(2*ME)$

$A(11,1) = (K*U*TE*ALFA)/(10*ME)$

$A(11,2) = (27*K*U*TI*ALFA)/(10*MI)$

$A(11,3) = (- 129*K*U^{2}*TE*ALFA)/(70*ME)$

$A(11,4) = (- 45*K*U^{2}*TI*ALFA)/(14*MI)$

$A(12,1) = (- 13*K*TE)/(10*ME)$

$A(12,2) = (27*K*TI)/(10*MI)$

$A(12,3) = (87*K*U*TE)/(25*ME)$

$A(12,4) = (- 9*K*U*TI)/MI$

TABLE 7

FOR SMALL MASS RATIO: MS = MI >> ME = MT

A(1,1) = ALFA

A(2,1) = (- 3*U*ME*TI*ALFA)/(5*MI*TE)

A(2,2) = (- 3*U*ALFA)/5

A(2,3) = (3*U^2*ME*TI*ALFA)/(7*MI*TE)

A(2,4) = (3*U^2*ALFA)/7

A(3,1) = (3*ME*TI)/(5*MI*TE)

A(3,2) = (- 3)/5

A(3,3) = (- 6*U*ME*TI)/(5*MI*TE)

A(3,4) = (6*U)/5

A(4,1) = (3*K*(TE - TI))/MI

A(5,1) = (6*K*U^2*TI)/(5*MI)

A(5,2) = (3*K*U^2*(TE - 5*TI))/(5*MI)

A(6,1) = (- 9*U*ME*TI*ALFA)/(5*MI*TE)

A(6,2) = (3*U*ME*ALFA*(TE - 5*TI))/(5*MI*TE)

A(7,1) = (4*K*U^2*(TE - 3*TI))/(15*MI)

A(8,1) = (- 2*K*TI)/MI

A(8,2) = (2*K*(- TE + 3*TI))/(5*MI)

TABLE 7

$$A(8,3) = (-4*K*U^{2}*TI)/(5*MI)$$

$$A(8,4) = (8*K*U^{2}*(-TE+5*TI))/(35*MI)$$

$$A(8,5) = (12*K*U^{2}*TI)/(5*MI)$$

$$A(8,6) = (24*K*U^{2}*(TE-5*TI))/(35*MI)$$

$$A(9,1) = (-54*U*ME*ALFA*TI)/(25*MI*TE)$$

$$A(9,2) = (12*U*ME*ALFA*(TE-5*TI))/(25*MI*TE)$$

$$A(9,3) = (-8*U^{3}*ME*ALFA*TI)/(35*MI*TE)$$

$$A(9,4) = (4*U^{3}*ME*ALFA*(TE-7*TI))/(35*MI*TE)$$

$$A(10,1) = (3*K*ME*ALFA*TI*(2*TE-3*TI))/(2*MI^{2}*TE)$$

$$A(11,1) = (K*U*ALFA*TI)/MI$$

$$A(11,2) = (9*K*U*ME*ALFA*TI*(-2*TE+5*TI))/(10*MI^{2}*TE)$$

$$A(11,3) = (12*K*U^{2}*ME*ALFA*TI*(-TE+5*TI))/(35*MI^{2}*TE)$$

$$A(11,4) = (9*K*U^{2}*ME*ALFA*TI*(2*TE-7*TI))/(14*MI^{2}*TE)$$

$$A(12,1) = (-3*K*TI)/MI$$

$$A(12,2) = (9*K*ME*TI*(-2*TE+5*TI))/(10*MI^{2}*TE)$$

$$A(12,3) = (54*K*U*TI)/(25*MI)$$

$$A(12,4) = (9*K*U*ME*TI*(2*TE-7*TI))/(5*MI^{2}*TE)$$

TABLE 8

FOR EQUAL MASSES: MS = MT

$A(1,1) = ALFA$

$A(2,1) = (-3*U*TS*ALFA)/(5*ST)$

$A(2,2) = (-3*U*TT*ALFA)/(5*ST)$

$A(2,3) = (3*U^{2}*TS*ALFA)/(7*ST)$

$A(2,4) = (3*U^{2}*TT*ALFA)/(7*ST)$

$A(3,1) = (3*TS)/(5*ST)$

$A(3,2) = (-3*TT)/(5*ST)$

$A(3,3) = (-6*U*TS)/(5*ST)$

$A(3,4) = (6*U*TT)/(5*ST)$

$A(4,1) = (3*K*(TT-TS))/(2*MS)$

$A(5,1) = (3*K*U^{2}*TS*(TT-TS))/(2*ST*MS)$

$A(5,2) = (3*K*U^{2}*TT*(TT-9*TS))/(10*ST*MS)$

$A(6,1) = (3*U*TS*ALFA*(-7*TT+3*TS))/(10*ST^{2})$

$A(6,2) = (3*U*TT*ALFA*(TT-9*TS))/(10*ST^{2})$

$A(7,1) = (2*K*U^{2}*(TT-5*TS))/(15*MS)$

$A(8,1) = (K*TS*(-11*TT-5*TS))/(5*ST*MS)$

$A(8,2) = (K*TT*(-TT+5*TS))/(5*ST*MS)$

TABLE 8

$$A(8,3) = (8*K*U^{2}*TS*(- 4*TT + TS))/(35*ST*MS)$$

$$A(8,4) = (4*K*U^{2}*TT*(- TT + 9*TS))/(35*ST*MS)$$

$$A(8,5) = (24*K*U^{2}*TS*(4*TT - TS))/(35*ST*MS)$$

$$A(8,6) = (12*K*U^{2}*TT*(TT - 9*TS))/(35*ST*MS)$$

$$A(9,1) = (- 12*U*ALFA*TT*TS)/(5*ST^{2})$$

$$A(9,2) = (6*U*ALFA*TT*(TT - 9*TS))/(25*ST^{2})$$

$$A(9,3) = (2*U^{3}*ALFA*TS*(- 5*TT + 9*TS))/(35*ST^{2})$$

$$A(9,4) = (2*U^{3}*ALFA*TT*(TT - 13*TS))/(35*ST^{2})$$

$$A(10,1) = (3*K*ALFA*TS*(TT - 2*TS))/(2*ST*MS)$$

$$A(11,1) = (K*U*ALFA*TS*(19*TT^{2} - 25*TT*TS + TS^{2}))/(10*ST^{2}*MS)$$

$$A(11,2) = (9*K*U*ALFA*TT*TS*(- TT + 4*TS))/(10*ST^{2}*MS)$$

$$A(11,3) = (3*K*U^{2}*ALFA*TS*(- 4*TT^{2} + 47*TT*TS - 54*TS^{2}))/(70*ST^{2}*MS)$$

$$A(11,4) = (9*K*U^{2}*ALFA*TT*TS*(TT - 6*TS))/(14*ST^{2}*MS)$$

$$A(12,1) = (K*TS*(- 41*TT^{2} - 7*TT*TS - 11*TS^{2}))/(10*ST^{2}*MS)$$

$$A(12,2) = (9*K*TT*TS*(- TT + 4*TS))/(10*ST^{2}*MS)$$

$$A(12,3) = (3*K*U*TS*(26*TT^{2} - 43*TT*TS + 36*TS^{2}))/(25*ST^{2}*MS)$$

$$A(12,4) = (9*K*U*TT*TS*(TT - 6*TS))/(5*ST^{2}*MS)$$