A Contribution to the Efficient Solution

of Extensive Symbolic Computations

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Abstract

The solving of extensive algebraic problems with computers is difficult owing to the large amount of storage and computing time needed. This paper describes a method that obviates these difficulties in many cases by using appropriate operators instead of extensive polynomials.
This calculation is prompted by the question whether a class of algebraic equilibria in plasma physics governed by MHD theory is stable. Great accuracy is called for in treating this problem.

We describe the calculation of a three-dimensional integral containing Fourier, Taylor and trigonometric series.

Introduction

The ability to perform extensive algebraic calculations has been made possible by introducing techniques for handling strings by computer. The structure of program systems for operations on polynomials has been discussed by, for example, Knuth [1]. For this purpose there are several well-known program systems, each restricted to certain classes of problems.

Details of the REDUCE 2 program system used by the authors to solve the problem described in this paper are given in [2], [3]. In general, no satisfactory information is available on the computing time and storage required, these being decisive in solving such problems.

This paper represents a continuation of the MHD stability calculations of Tasso [4], [5], for which the large amount of storage space required made it necessary to partition the algebraic expressions. In the treatment of the general class of equilibria mentioned in [5] the partitioning was expected to be such that the amount of storage and computing time would have been ten times larger than in [5].

Here we have to calculate a three-dimensional integral $\delta W = \int d^3x \; f(\vec{x})$, the integrand of which is a complicated function containing essentially trigonometric functions and polynomials which can be integrated analytically.

An important feature of these problems is that the algebraic output extends over many pages. This implies a numerical evaluation of the algebraic formulae.

For such problems the question where to set the partition between algebraic and numerical calculations is important. For example, some operations on polynomials should be saved in the algebraic processing and done numerically in the output. This might help in the case where the required numerical accuracy implies that some series have to be truncated at a very high order. In this way the results of symbolic manipulation are algebraic formulae together with an operation prescription for the numerical program.

In Section 1 an introduction to the physics underlying our task is given and the algebraic problems are formulated.

Section 2 shows a way of minimizing storage and computing time by using appropriate operators. Though we are not presenting a general method or even an algorithm, we hope to provide some new ideas that are not restricted solely to our special case.

Section 3 shows the saving in storage and computing time.

1. Problem

The problem treated here arises in plasma physics. For details the reader is referred to [6], [5]. The plasma is described by magneto-hydrodynamic (MHD) theory.

We are given a class of axisymmetric toroidal plasma equilibria. According to the energy principle [7], which we write in covariant form, stability or instability can be deduced from the sign of the minimum energy variation δW due to perturbations around the equilibrium

1.1. $$\delta W = \frac{1}{2} \int_{\substack{\text{Plasma-} \\ \text{Volume}}} d^3\tau \left\{ g_{ij} Q^i Q^j + \xi^i \epsilon_{ipq} Q^p J^q \right\}$$

We consider an incompressible plasma, this is $\nabla \cdot \vec{\xi} = 0$.

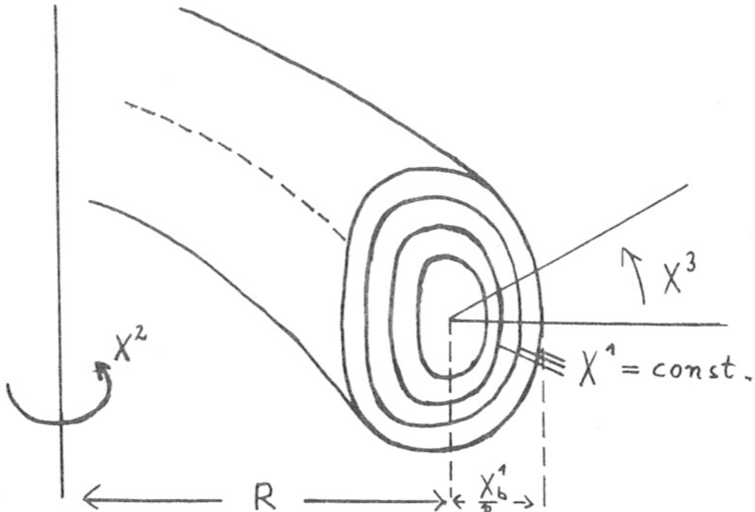
Repeated indices in the lower and upper positions are summed from 1 to 3. B^i, J^i are the contravariant components of the magnetic field and current density in equilibrium, and Q^i the magnetic field perturbation caused by the displacement ξ^i of the plasma; ϵ_{ipq}, ϵ^{ipq} denote the covariant and contravariant components of the unit tensor, g_{ij} the covariant components of the metric tensor and $g = \det(g_{ij})$. Furthermore, with $\partial_p = \frac{\partial}{\partial X^p}$ one has from $\vec{Q} = \nabla \times (\vec{\xi} \times \vec{B})$

1.2. $$Q^i = \epsilon^{ipq} \partial_p \left(\epsilon_{qst} \xi^s B^t \right)$$

Here it is only the energy variation of the plasma that is treated. It is convenient [5] to introduce curvilinear coordinates X^1, X^2, X^3 which take into account axisymmetry and which contain the flux surfaces as one coordinate.

X_b^1/R^2 is the inverse aspect ratio and is assumed to be less then 1/2. X^2, $X^3 \in [0,2\pi]$ are angles the long and short ways round the torus, respectively. The situation is outlined in Fig. 1.

Fig. 1



We decompose ξ^i into Fourier series with respect to both angles X^2, X^3.

$$\xi^i = \sum_{k,m} \xi^i_{m,k}(X^1)\, e^{ikX^2}\, e^{imX^3}$$

Because of axisymmetry we can treat δW for each k separately. In addition, we multiply ξ^i by a polynomial $K_0 = K_0(X^1, X^3)$ and consider the real or imaginary part of the exponential functions $e^{i(kX^2 + mX^3)}$.

The description of the $\xi^i = \xi^i(m, X^1, X^2, X^3)$ is abbreviated as follows:

1.3 $$\rho_z = kX^2 + m_z X^3$$

with $k \in \mathbb{N}$; $m_z \in [M_0, M_1]$; $z = 1, 2$; $M_0, M_1 \in \mathbb{Z}$ +)

1.4 $$K_0 = \sum_{\ell=0}^{L} d_\ell \left(\frac{2X^1}{R^2}\cos X^3\right)^\ell$$

with $d_0 = 1$, $d_\ell = const.$

1.5 $$\xi^1_z = f^1_{m_z}(X^1)\, K_0 \cos\rho_z$$
$$\xi^3_z = f^3_{m_z}(X^1)\, K_0 \cdot \sin\rho_z$$

ξ^2_z is calculated from $\nabla\cdot\vec{\xi} = \partial_i(\sqrt{g}\,\xi^i)/\sqrt{g} = 0$ on the assumption that

$$\partial_2\, \xi^2_z \neq 0 \qquad \text{i.e. } k \neq 0.$$

+) $\mathbb{N}$ and $\mathbb{Z}$ denote the set of ordinary numbers and integers respectively.

$f^1_{m_z}$, $f^3_{m_z}$ are chosen such as to minimize δW. We discuss for

$f^s_{m_z}(X^1)$:

1.6.1 Polynomials in X^1

1.6.2 Representation by Bessel functions of integer order.

In the former case the X^1-integration is performed analytically, in the latter case numerically.

After some operations the coordinates chosen allow analytic integration of δW with respect to X^2, X^3, and with 1.6.1 even with respect to X^1. With the definitions 1.2 to 1.6 obtain

1.7 $$\delta W = \frac{1}{2} \sum_{m_1=M_0}^{M_1} \sum_{m_2=M_1}^{M_1} \int_0^{X_b^1} dX^1 \int_0^{2\pi} dX^2 \int_0^{2\pi} dX^3 \sqrt{g} \left\{ g_{ij} Q_1^i Q_2^j + \xi_1^i \epsilon_{ipq} Q_2^p J^q \right\}$$

The equilibrium is characterized as follows:

1.8
$$B^1 = 0 \qquad B^2 = \frac{T(X^1)}{r^2} \qquad B^3 = c_1 \tilde{r}$$
$$J^1 = 0 \qquad J^2 = c_2(1 + c_3/r^2) \qquad J^3 = c_4 \tilde{r}/T$$

with $T = \left[c_5{}^2 + c_6 (X^1)^2 \right]^{1/2}$ $\quad c_i \in \mathbb{R}$ are constants +)

1.9 $$r = (R^2 + 2X^1 \cos X^3)^{1/2} \qquad \tilde{r} = (r^2 - \gamma)^{1/2}$$

$$\gamma = \text{const. so that } 0 \le \frac{X^1_b}{R^2 - \gamma} < 1/2$$

1.10 $g_{12} = g_{21} = g_{23} = g_{32} = 0,\qquad g_{22} = r^2.$

The g_{ij} are of the form $g_{ij} = g_{ij}(\sin X^3, \cos X^3, r, \tilde{r})$; for example

+) $\mathbb{R}$ denotes the set of real numbers.

1.11 $$g_{33} = \frac{(X^1)^2}{\tilde{r}^2}\left[\alpha^2\cos^2X^3 + \frac{\tilde{r}^2}{r^2}\sin^2X^3 + \frac{\alpha^2X^1\sin^2X^3}{\tilde{r}^2}\left(2\cos X^3 + \frac{X^1\sin^2X^3}{\tilde{r}^2}\right)\right]$$

$$\alpha \in \mathbb{R}$$

1.12 $$\sqrt{g} = \frac{\alpha X^1}{\tilde{r}}$$

2. Symbolic manipulation

We have to evaluate δW from eq. 1.7. The integration with respect to X^3 yields elliptic integrals due to r and $\tilde{r}$ and powers of these but the integration with respect to X^1 cannot be solved by symbolic manipulation. In order to avoid these elliptic integrals, we expand r and $\tilde{r}$ into Taylor series. With the given restrictions on X_b^1/R^2 and γ this expansion is always possible. First we notice that we have to handle Fourier series with a variable number of terms in the double sum with respect to m_1, m_2 in 1.7. Secondly, the necessary Taylor expansion involves the following difficulties: There is no general truncation criterion because of the large number and wide range of parameters to be investigated. Owing to the high accuracy required and with $2\,X_b^1/(R^2-\gamma)$ close to unity we have to truncate beyond the 21^{st} term in some cases. This greatly exeeds the storage capacity.

Furthermore, the splitting into partial expressions should be avoided since it does not make full use of algebraic simplification such as cancellation and factorization.

The necessary optimization with respect to storage and computing time may be discussed from another point of view. The evaluation of δW is completely determined by the equilibrium, by the ansatz for the test functions ξ^i and by the details of solution, e.g. the Taylor expansion of special terms or the sequence of integrations. We have therefore all information necessary for solution. However, inserting the algebraic values of all variables in δW yields expressions of such extent that evaluation is practically impossible even by computer. It is concluded that our information is represented in a disadvantageous form. We have to look for a compact, but sufficient representation. This problem of symbolic manipulation may be characterized as the search for the optimal representation of a given item of information. Some polynomials like Taylor series are completely described by an initial term together with a recursive formula.

The desired optimization is achieved by representing every operation in the most compact form. This is described by the following rule:

2.1 rule An operation is applied to a variable in such a way that instead of the algebraic value of the variable that information is inserted and manipulated which gives a result of minimal extent.

This is illustrated by the following examples.

1) $g_{12}=0$ The algebraic value is inserted because it is of minimal extent and shortens the expressions.

$g_{33}=g_{33}(X^1, X^3)$ The variable name is kept at the beginning.

2) We do not perform the differentiations of the polynomial K_o of eq. 1.4 explicitly but define:

2.2. $$\frac{\partial K_s}{\partial X^1} = \frac{2\cos X^3}{R^2} K_{s+1} \; ; \quad \frac{\partial K_s}{\partial X^2} = 0 \; ; \quad \frac{\partial K_s}{\partial X^3} = -\frac{2X^1 \sin X^3}{R^2} K_{s+1}$$

for $0 \leq s \leq 2$

K_{s+1} denotes the s th differentation of K_o with respect to the argument $u = \frac{2X^1}{R^2} \cos X^3$

$$K_s = s! \sum_{\ell=s}^{L} \binom{\ell}{s} d_\ell \, u^\ell$$

In eq. 2.2. K_{s+1} is kept as variable name. With rule 2.1 as a side relation we find the following flow-chart:

1) Compute Q^i and then the integrand of 1.7
2) Perform the integration with respect to X^2
3) Prepare the integration with respect to X^3
   a) Take into account the variable range of m_1, m_2
   b) Expand, if necessary, into Taylor series
   c) Include K_o and its derivatives
4) Perform the integration with respect to X^3
5) a) For numerical integration with respect to X^1 rearrange the result
   b) For analytical integration with respect to X^1 expand T and integrate
6) Output of symbolic manipulation.

Some details are given in Appendix A - D.

Step 1) We define the differentiation of r (or $\tilde{r}$) and T with $r, \tilde{r}$ and T as variables:

2.4 $$\frac{\partial r}{\partial X^1} = \frac{\cos X^3}{r} \qquad \frac{\partial r}{\partial X^2} = 0 \qquad \frac{\partial r}{\partial X^3} = -\frac{X^1 \sin X^3}{r}$$

2.5 $$\frac{\partial T}{\partial X^1} = \frac{c_7 X^1}{T} \qquad \frac{\partial T}{\partial X^2} = 0 \qquad \frac{\partial T}{\partial X^3} = 0 \qquad c_7 = \text{const.}$$

We eliminate ξ^2_z with $\nabla \cdot \vec{\xi} = 0$, and with decomposition A2 we compute the Q^i_z from eq. 1.2 and then, with $g_{ij} \neq 0$ as variables, the integrand of δW.

Step 2) Since X^2 occurs only in ρ_z (eq. 1.4), we decompose ξ^i_z, Q^i_z with respect to $\sin \rho_z$ and $\cos \rho_z$. The X^2 - integration yields (A1 - A3):

$$2.6 \qquad \delta W = \frac{\pi}{2} \int_0^{X^1_b} dX^1 \int_0^{2\pi} dX^3 \sum_{\substack{m_1, m_2 \\ m_2 \le m_1}} \sqrt{g} \left\{ B_0 \cos (m_1 - m_2) X^3 + B_1 \sin (m_1 - m_2) X^3 \right\}$$

Step 3a) The double sum resulting from the Fourier series is taken into account by the index z of ξ^i_z and Q^i_z in the form of two different variables m_1, m_2 throughout the definitions in Section 1. We expand $\cos \mu X^3$ and $\sin \mu X^3$ with $\mu = m_1 - m_2$ and distinguish between even and odd μ with an index p (B1 - B4).

We define: $A_0 = B_0 \cdot \sqrt{g}$

$A_1 = B_1 \cdot \sqrt{g} \;\sin X^3$

We rearrange and get:

$$2.7 \qquad \delta W = \frac{\pi}{2} \int_0^{X^1_b} dX^1 \int_0^{2\pi} dX^3 \sum_{t=0,1} \sum_{\mu=0}^{M_1 - M_0} \sum_{\kappa = t(p)}^{\hat{\mu}(p)} \cos^{\lambda(p)} X^3 \, \alpha^{(p)}_{t\mu\kappa} \cos^{2\hat{\mu} - 2\kappa} \sum_{\substack{m_1, m_2 = M_0 - M_1 \\ m_2 \le m_1 \\ m_1 - m_2 = \mu}} A_t$$

At this point it is quite sufficient to compute only the A_t by symbolic manipulation. According to rule 2.1 we have found a representation of δW containing an algorithm for expanding sin and cos with multiple argument.

Step 3b) Now we give the $g_{ij} \neq 0$ their algebraic values according to eq. 1.11. The A_t can be represented as a double sum:

$$A_t = \sum_i \sum_j \beta_{tij} \tilde{r}^i r^j \qquad \text{with } i,j \in \mathbb{Z} \text{ and almost all } i,j < 0.$$

The necessary Taylor expansion of the $\tilde{r}^i r^j$ is abbreviated by a summation operator P (C1 - C4) :

$$2.8 \qquad A_t = P_{x^1 \cos \chi^3 (v)} \sum_{i,j} \beta_{tij}$$

This is a representation with information about the function to be expanded together with an algorithm of the expansion.

S_N as remainder of the Taylor series and $N = N(i,j)$are determined in such a way that $|S_N| < \varepsilon$ for any given $\varepsilon > 0$. It is now possible to manipulate the Taylor expansion symbolically in a simple manner. At the same time the unique application of the operator P with variable N in the numerical solution of the problem is formulated. It is thus possible to decide when to truncate the series and hence the zero of δW can be approximated with an arbitrary degree of accuracy.

Step 3c) With K_s defined by 2.3 only expressions of the form

$$\varphi_s = K_s^2 \quad ; \quad \varphi_3 = K_0 K_1 \quad ; \quad \varphi_4 = K_0 K_1 \quad ; \quad \varphi_5 = K_1 K_2$$

appear in A_0, A_1. With $0 \leq \tau \leq 5$ the general representation is polynomial:

$$2.9 \qquad \varphi_\tau = \sum_{\ell=0}^{24} \chi_{\tau\ell} \left(\frac{2x^1}{R^2} \cos \chi^3 \right)^\ell$$, where the $\chi_{\tau\ell}$ are uniquely determined.

Step 4) It is found that the β_{tij} can be reduced for all t,i,j in the following way:

$$\beta_{tij} = \sum_\varrho \sum_\sigma \sum_\tau \beta_{tij\varrho\sigma\tau} \cos^\varrho \chi^3 \sin^\sigma \chi^3 \ \varphi_\tau$$

With $\omega = \rho + \ell + \nu + 2\hat{\mu}(P) - 2\kappa + \lambda(P)$ regarding all powers of $\cos X^3$

and $\vartheta_2 = \frac{2}{R^2}$ we get (D1, D2)

$$\delta W = \pi^2 \int_0^{X_b^1} dX^1 \sum_{t=0,1} \sum_{\mu=0}^{M_1-M_0} \sum_{\kappa=t(P)}^{\hat{\mu}(P)} \alpha_{t\mu\kappa}^{(P)} \sum_{i,j} P_{X^1\cos X^3}(\nu) \sum_{\tau=0}^{5} \sum_{\ell=0}^{2L} \chi_{\tau\ell} (\vartheta_2 X^1)^\ell \cdot$$

$$\cdot \sum_{\rho,\sigma} C_{\omega\sigma} \sum_{\substack{m_1-m_2=\mu \\ m_2 \le m_1}} \beta_{tij\rho\sigma\tau}$$

With the operator

$$2.10 \qquad S_{X^1}(\omega) = \sum_{\kappa=t(P)}^{\hat{\mu}(P)} \alpha_{t\mu\kappa}^{(P)} \sum_{i,j} P_{X^1\cos X^3}(\nu) \sum_{\tau=0}^{5} \sum_{\ell=0}^{2L} \chi_{\tau\ell}(\vartheta_2 X^1) \sum_{\rho} \sum_{\sigma} C_{\omega\sigma} \sum_{m_1-m_2=\mu}$$

this result can be simplified to yield

$$2.11 \qquad \delta W = \pi^2 \int_0^{X_b^1} dX^1 \sum_{t=0,1} \sum_{\mu=0}^{M_1-M_0} S_{X^1}(\omega)\, \beta_{tij\rho\sigma\tau}(m_1, m_2, X^1)$$

It is sufficient to determine the $\beta_{tij\rho\sigma\tau}$ by symbolic manipulation - only for even μ since those with odd μ vanish according to D2 - and then interpret the operator $S_{X^1}(\omega)$ later on.

<u>Step 5)</u> We separate the function $f_z^i(X^1)$ from the expressions. They occur in bilinear terms with nine combinations:

$$2.12 \qquad \phi_\eta = f_1^i(X^1) \cdot f_2^j(X^1) \qquad i,j = 1,2,3 \qquad f_z^2 = \partial_1 f_z^1$$

$$\eta = 1, 2 \dots 9$$

$$\beta_{tij\rho\sigma\tau} = \sum_{\eta=1}^{9} \beta_{tij\rho\sigma\tau\eta}\, \phi_\eta$$

a) The separation 2.12 shortens the expressions and is convenient for numerical evaluation. It is advisable to rearrange the operator $S_{X^1}(\omega)$

$$2.13 \qquad S_{X^1}(\omega) = \sum_{i,j} \sum_{\rho,\sigma} \sum_{\tau=0}^{5} \sum_{\eta=1}^{q} \sum_{m_1,m_2,\mu} (S_t)_{ij\rho\sigma\tau}(m_1,m_2,X^1)\cdot\phi_\eta(X^1)$$

2.14 The $(S_t)_{ij\rho\sigma\tau} = (R^2-1)^{i/2} R^j \sum_{\kappa=t(\rho)}^{\hat\mu(\rho)} \alpha^{(\rho)}_{t\mu\kappa} \sum_{\nu=0}^{N(i,j)} t_{ij\nu}(\vartheta_1 X^1)^\nu \sum_{l=0}^{2L} \chi_{\tau l}(\vartheta_2 X^1)^l c_{\omega\tau}$ can be regarded as elements of a fifth rank tensor. Because most of its elements are zero, we introduce a label n_I by $I : (i,j,\rho,\sigma,\tau) \to n_I \in \mathbb{N}$.
In this manner the problem of Kerner [7] has been solved.

b) The function ϕ_η are now polynomials with respect to X^1 with given constants. The expressions $\beta_{tij\rho\sigma\tau\eta}$ are reduced according to powers k_T of $T^{k_T/2}$, $k_T \in \mathbb{Z}$, to be expanded into Taylor series and then according to powers of X^1. Introducing an operator $\hat{S}(\omega)$ now at points $X^1 = 0, X^1_b$ containing the constants of the news Taylor expansion and of the X^1 integration, we continue as in 5a.

<u>Step 6)</u> The output of the symbolic manipulation contains algebraic expressions, operators $O(n_I)$ labeled by an index n_I and the 5-tuples (i,j,ρ,σ,τ) together with the respective labels n_I.

The algebraic results are general with respect to the number and choise of Fourier components of $\mathcal{J}^i$, to truncation of Taylor expansions and to polynomial K_0, i.e. for every allowed value of the input data for δW. The numerical evaluation of these expressions for any set of parameters is straight forward with the given definitions and algorithms for the operators $O(n_I)$.

3. Computing time

In the following table we present the effects of the optimization shown above. It is difficult to give the exact storage space required since only versions of REDUCE 2 with 480K, 720 K or 1440 K were available at the time.

Column 1: problem treated in [5], [4] :

Column 2: our problem according to step 5a with fourth rank tensor, not including index τ

Column 3: like column 2, but with array n_I instead of fifth rank tensor including index τ

In all three cases the X^1-integration is done numerically with 1024 points in the interval $[0,X_b^1]$. Columns 2 and 3 include the normalization integral .

		1	2	3
Symbolic manipulation with REDUCE 2	Storage	1440 K	480 K	720 K
	Comp.time 1) min.	92	78 3)	26 3)
Numerical calculation with FORTRAN	Storage	720 K	180 K	120 K
	Comp.time 2) sec.	36	36 4)	20 4)

1) For the entire symbolic manipulation

2) For the numerical evaluation of the result with one set of parameters

3) Owing to the type of problem involved the time required for columns 2 and 3 was estimated to be ten times that for column 1

4) Owing to the type of problem involved the time requirement for 2 and 3 was estimated to be two times that for 1.

Summary

We describe the three-dimensional analytic integration of a functional δW which occurs in plasma physics. Since the zeros of δW are of special interest, this implies a high degree of accuracy at any stage of the calculation.
The calculation of the integrand and the performing of the integrations, which are possible after Taylor expansion of certain functions, can be done by computer using the language REDUCE.

At first glance this procedure seemes to be limited by the large amount of computing time and storage needed. But introducing appropriate operators instead of inserting polynomials, trigonometric series and Taylor series we shift the large extent of the algebraic expressions to unique operating rules in the last step of the program, the numerical part. We thus get a compact description of the problem. Here we have made use of the fact that the numerical evaluation of the algebraic formulae is necessary because of their large extent. By suitable partitioning into algebraic and numerical calculations we can optimize the storage and computing time.

The truncation of Taylor series or polynomials is given by the final numerical result, so that we reach an arbitrarily high accuracy for every set of allowed input data, this being limited only by the numerical accuracy of the computer.

We believe that this compact manipulation need not be restricted to the special problem or to the language REDUCE used here.

Appendix

A) X^2 - integration

We have the integrals

A1

$$\int_0^{2\pi} \begin{pmatrix} \cos\varrho_1 \; \cos\varrho_2 \\ \sin\varrho_1 \; \sin\varrho_2 \end{pmatrix} dX^2 = \pi \cos(m_1 - m_2)\, X^3$$

and

$$\int_0^{2\pi} \sin\varrho_1 \cos\varrho_2 \; dX^2 = \pi \sin(m_1 - m_2)\, X^3$$

We therefore decompose ξ^i_z and Q^i_z

A2

$$\xi^i_z = \xi^{i,c}_z \cos\varrho_z + \xi^{i,s}_z \sin\varrho_z$$

$$Q^i_z = Q^{i,c}_z \cos\varrho_z + Q^{i,s}_z \sin\varrho_z$$

In order to get $\mu = m_1 - m_2$ with $\mu \geqslant 0$ we define

$$B_0 = \left\{ g_{ij} \left[\left(Q^{i,s}_1 Q^{j,s}_2 + Q^{i,c}_1 Q^{j,c}_2 \right) + (1-\delta_{ij}) \left(Q^{i,s}_2 Q^{j,s}_1 + Q^{i,c}_2 Q^{j,c} \right) \right] \right\} \cdot \Delta$$

$$+ \epsilon_{ipq} J^q \left[\xi^{i,s}_1 Q^{p,s}_2 + \xi^{i,c}_1 Q^{p,c}_2 + \xi^{i,s}_2 Q^{p,s}_1 + \xi^{i,c}_2 Q^{p,c}_1 \right] \cdot \frac{\Delta}{2}$$

A3 $$B_1 = \left\{ \partial_{ij} \left[\left(Q_2^{i,c} Q_1^{j,s} - Q_2^{i,s} Q_1^{j,c} \right) - (1-\delta_{ij}) \left(Q_1^{i,c} Q_2^{j,s} - Q_1^{i,s} Q_2^{j,c} \right) \right] \right\} \cdot \Delta$$
$$+ \epsilon_{ipq} \partial^q \left[\xi_2^{i,c} Q_1^{p,s} - \xi_2^{i,s} Q_1^{p,c} - \xi_1^{i,c} Q_2^{p,s} + \xi_1^{i,s} Q_2^{p,c} \right] \cdot \frac{\Delta}{2}$$

with $\Delta = 2 - \delta_{m_1 m_2}$ where $\delta_{m_1 m_2}$ denotes the Kronecker delta.

B) Trigonometric expansion

We expand $\cos \mu X^3$ and $\sin \mu X^3$ with $\mu \geq 0$ according to trigonometric relations:

B1) $$\cos \mu X^3 = \cos^{\lambda_0(P)} X^3 \sum_{K=t(P)}^{\hat{\mu}(P)} \alpha_{0\mu K}^{(P)} \cos^{2\hat{\mu}-2K} X^3$$

B2) $$\sin \mu X^3 = \sin X^3 \cdot \cos^{\lambda_1(P)} X^3 \sum_{K=t(P)}^{\hat{\mu}(P)} \alpha_{1\mu K}^{(P)} \cos^{2\hat{\mu}-2K} X^3$$

B3) μ even $\quad \lambda_0(P) = 0 \quad \hat{\mu}(P) = \frac{\mu}{2} \quad t(P) = t = 0, 1$

$\lambda_1(P) = 1$

B4) μ odd $\quad \lambda_0(P) = 1 \quad \hat{\mu}(P) = \frac{\mu - 1}{2} \quad t(P) = 0$

$\lambda_1(P) = 0$

It is easy to decribe the $\alpha_{t\mu K}^{(P)}$ by an algorithm.

C Taylor expansion

We expand $r, \tilde{r}$ around the point $X^1 \cos X^3 = 0$ with

$$\vartheta_1 = \frac{2}{R^2 - f} \quad ; \quad w = \frac{R^2 - f}{R^2} \quad ; \quad \frac{2X^1}{R^2 - f} < 1$$

C1 $$\tilde{r}^i r^j = (R^2 - f)^{i/2} R^j \sum_{\nu=0}^{N(i,j)} t_{ij\nu} (\vartheta_1 X^1 \cos X^3)^\nu + S_N$$

C2 $\quad t_{ij0} = 1 \qquad t_{ij\nu} = \frac{1}{2^\nu \nu!}\left[\prod_{\varrho=0}^{\nu-1}(i-2\varrho) + w\binom{\nu}{1}\prod_{\varrho=0}^{\nu-2}(i-2\varrho)\prod_{\sigma=\nu-1}^{\nu-1}(j-2\sigma)\cdots\right.$

$$\left. + w^\nu \prod_{\sigma=0}^{\nu-1}(j-2\sigma)\right] \qquad \nu > 0$$

Since the series for $(1+x)^\alpha$ with real α and x converges absolutely for $|x| < 1$; the Taylor series for $\hat{r}^i r^j$ converges absolutely, too.

We define a summation operator containing two variables

C3 $$P_{XY}(\nu) = (R^2-j)^{i/2} R^j \sum_{\nu=0}^{N(i,j)} t_{ij\nu} (\vartheta_1 XY)^\nu$$

and get

C4 $$\hat{r}^i r^j = P_{X^1 \cos X^3}(\nu) + S_N$$

D X^3-integration

D1 $$\int_0^{2\pi} \sin^\mu x \cos^\nu x \, dx = 2\pi c_{\mu\nu}$$ [8] with

D2 $$c_{\mu\nu} = \begin{cases} \dfrac{(\mu-1)!!\,(\nu-1)!!}{2^{\frac{\mu+\nu}{2}}\left(\frac{\mu+\nu}{2}\right)!} & \text{if } \mu \in 2\mathbb{N}_0 \wedge \nu \in 2\mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases}$$

From this it is easy to evaluate the coefficients $\alpha^{(p)}_{t\mu\kappa}$, $t_{ij\nu}$, $c_{\mu\nu}$ numerically.

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References

[1] Knuth, D.E. — The Art of Computer Programming, Addison Wesley Publishing Comp., Reading (Mass.) (1969)

[2] Hearn, A.C. — REDUCE 2 User's Manual, Stanford Artificial Intelligence Project Memo-No AIM-133 (1970)

[3] Hearn, A.C. — Applications of Symbol Manipulation in Theoretical Physics, University of Utah, Contract-No. F 30602-70-C-0300

[4] Steuerwald, J. Tasso, H. — Mehrfache Integration durch Formelmanipulation, to be published

[5] G. Küppers, D. Pfirsch, H. Tasso — MHD stability of Axisymmetric Plasmas Proc. of the I.A.E.A. Conference on "Plasma Physics and Controlled Nuclear Fusion Research" Madison 1971

[6] Kerner, W. — Doctoral Thesis, to be published.

[7] Bernstein, Frieman, Kruskal, Kulsrud, — An Energy Principle for Hydromagnetic Stability Proc. Roy. Soc. (London) A 244, 17 (1958)

[8] Gröbner, W. Hofreiter, N. — Integrationstafel, Springer-Verlag Berlin (1965)