On the Holding Power of an
Relativistic Electron Ring

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Abstract

The maximum acceleration of ions possible in an electron ring accelerator is studied. The case of slender rings without external focusing is treated in beam approximation using macroscopic fluid equations. The acceleration forces are taken as a perturbation acting on an unaccelerated equilibrium. The dependence of the maximum acceleration on the ion loading and electron energy is discussed. At the Budker limits no acceleration is possible, while over a wide range in between the maximum acceleration is only weakly dependent on the ion loading.

Introduction

In the electron ring accelerator an upper limit is imposed on ion acceleration by the strength of ion binding, the so-called holding power, in the electron ring. The ions are trapped in the electric potential well of the electrons. During acceleration the inertial force acting on the ions polarizes the ring by displacing the ions against the electrons. If the inertial force exceeds the binding force of the electron ring the ring equilibrium will be destroyed and the ions lost. For an electron ring with a maximum electric field strength E_{max}, the rate of ion acceleration will be less than $e \cdot E_{max}/M$ (M = ion mass), where E_{max} is the maximum electric field strength produced by the electron ring. This paper will consider the question of how much smaller values of possible acceleration are to be expected from a more detailed examination.

It is assumed that for slender rings $r/R \ll 1$ (r = minor, R = major ring radius) toroidal effects on the holding power can be neglected; the problem then reduces to treating straight electron ion beam equilibria when accelerated perpendicularly to the beam direction.

Unaccelerated relativistic electron-ion beam equilibria without external focusing have been studied by many authors [e.g. BENNETT, 1934, 1955]. Bennett equilibria exist in the range of f: $1/\gamma^2 < f < 1$, where f is the ratio of ion to electron density and γ is the relativistic factor of the electrons. The particle ratio f is a function of the electron and ion temperatures T_e, T_i. The temperatures can be regarded as a measure of the transverse energies of electrons and ions, taken in the rest frame of the particles.

Accelerated slender-ring equilibria without external focusing have been studied by PERELSHTEIN et al. (1971); BARKHUDARYAN et al. (1972); KAZARINOV (1972). BARKHUDARYAN et al. got an approximate solution in beam geometry for the special case $T_e = T_i$, corresponding to a particle ratio $f = 1/\gamma$. They found a maximum ion acceleration $g_{max} = 0.4 \frac{e}{M} E_{max}$.

This paper attempts to find the dependence of the maximum acceleration on the ion loading f or, in other words, on the temperatures of the electrons and ions. In Section 1 the basic equations are given. The electrons and ions are described by the macroscopic relativistic magneto-hydrodynamic equations, and the electromagnetic field produced by the charges and currents by Maxwell's equations.

The electron-ion beam will be accelerated in an external constant electric field, and it will be assumed that the beam is stationary in the coordinate system accelerated with the beam, this being known also as the Möller system [MÖLLER (1943), PERELSHTEIN (1971)]. For small acceleration g and a not too large beam dimension in the acceleration direction the Möller system can be approximated by a system with a uniform gravitation force g, which is equal to the acceleration in the laboratory system.

Because of the difficulty in finding the general solution of the equation for a beam equilibrium with external electric field and a gravitation force, we discuss an approximation linearizing the equation about the unaccelerated Bennett equilibrium. This is performed in Section 2, where the solution is discussed also.

1. Basic equations

Considering a relativistic electron-ion beam accelerated perpendicularly to the beam direction in a constant electric field, it is advantageous to use the coordinate system accelerated with the beam. In this system the external forces are the accelerating electric field E, which has the same value as in the laboratory system, and a gravitational force g equal to the acceleration in the laboratory system. Of interest are solutions that are stationary in the accelerated coordinate system. To find such a stationary equilibrium of ions at rest and relativistic electrons, the following equations will be used:

$$-\nabla(n_e T_e) - (\nabla\Phi + \vec{v}_e \times \vec{B})\, e n_e \gamma + \vec{K}_e n_e \gamma = 0 \; ,$$

$$-\nabla(n_i T_i) + \nabla\Phi \cdot e n_i + \vec{K}_i n_i = 0 \; ,$$

(1)

$$\Delta\Phi = \Delta\Phi_e + \Delta\Phi_i = -4\pi e n_e \gamma + 4\pi e n_i \; ,$$

$$\nabla \times \vec{B} = -4\pi e n_e \gamma \vec{v}_e \; ,$$

$$\nabla \vec{B} = 0 \; .$$

The indices e and i denote electrons and ions resp.; T_e, T_i are the temperatures, and n_e, n_i are the particle densities in the rest systems. In the system in which the ions at rest, the electron and ion densities are $n_e \gamma$ and n_i resp.. Φ_e, Φ_i are the electric potentials of the particles and $\vec{B}$ is the magnetic field produced by the electrons.

The electron beam has the velocity $\vec{v}_e = (0, 0, v)$ parallel to the z-axis and all quantities are independent of z. In such a geometry it holds that $\vec{B} = \vec{v}_e \times \nabla\Phi_e$, which will be used to eliminate

$\vec{B}$ in eqs. (1). The external forces $\vec{K}_e$, $\vec{K}_i$ have x-components k_e, k_i only:

$$\vec{K}_e = (k_e, 0, 0) = (-eE + m\gamma g, 0, 0),$$
$$(2) \qquad \vec{K}_i = (k_i, 0, 0) = (+eE + Mg, 0, 0),$$

where m, M are the rest masses of electrons and ions, resp. The electric field E and the gravitation g are not independent parameters. Because the total force on electrons and ions has to vanish for a stationary equilibirum, integration of the first two equations of (1) over the (x-y) plane gives the following relation between E and g:

$$(3) \qquad g = \frac{1 - N_i/N_e}{m\gamma + M N_i/N_e}\, eE \;, \qquad N_e = \int n_e \gamma \, dx\, dy \,, \quad N_i = \int n_i \, dx\, dy \,.$$

N_e, N_i are the line densities of electron and ions in the laboratory system.

If $\vec{B}$ is eliminated and uniform temperatures T_e, T_i are assumed, the first two equations (1) can be integrated to yield in cylindrical coordinates (r, y, z)

$$T_e \ln n_e + e\left(\tfrac{1}{\gamma}\Phi_e + \gamma \Phi_i\right) - \gamma k_e r \cos y + c_e = 0,$$
$$(4) \qquad T_i \ln n_i - e(\Phi_e + \Phi_i) - k_i r \cos y + c_i = 0,$$
$$\Delta \Phi_e + 4\pi e n_e \gamma = 0,$$
$$\Delta \Phi_i - 4\pi e n_i = 0,$$

where $k_e = -eE + mg$, $k_i = eE + mg$ and c_e, c_i are the constants of integration.

2. The accelerated beam

A general analytic treatment of eqs. (4) is difficult. To get an approximate solution, one begins with the known Bennett solution [BENNETT (1934)] for a beam without external fields ($k_e = k_i = o$) and then takes the external forces as a perturbation. The particle densities of the unaccelerated beam equilibrium are given by

$$(5)\qquad n_e^{(0)} = \frac{A^2}{\{r^2A^2 + a^2\}^2}\,, \qquad n_i^{(0)} = f\gamma n_e^{(0)}\,,$$

and the potentials are apart from an unimportant constant

$$(6)\qquad \Phi_e^{(0)} = -\frac{2\gamma(T_e + \gamma T_i)}{e(\gamma^2-1)} \ln\{r^2A^2 + a^2\}\,, \qquad \Phi_i^{(0)} = -f\,\Phi_e^{(0)}\,,$$

where

$$(7)\qquad a^2 = \frac{\pi e^2 v^2 \gamma^2}{2(T_e + T_i\gamma)}\,, \qquad f = \frac{\gamma T_e + T_i}{\gamma(T_e + \gamma T_i)}$$

and where A is a free parameter. As can be seen, the particle density ratio f is a function of the temperature ratio T_i/T_e only.

Now, the equations (4) will be linearized about the Bennett equilibrium. With the substitution

$$n_e = n_e^{(0)} + n_e^{(1)}\,, \qquad \Phi_e = \Phi_e^{(0)} + \Phi_e^{(1)}\,,$$
$$n_i = n_i^{(0)} + n_i^{(1)}\,, \qquad \Phi_i = \Phi_i^{(0)} + \Phi_i^{(1)}\,,$$

one obtains the following system of linear equations for the particle density and potential perturbation:

$$T_e \frac{n_e^{(1)}}{n_e^{(0)}} + e\left(\frac{1}{\gamma}\Phi_e^{(1)} + \gamma\Phi_i^{(1)}\right) - \gamma k_e r\cos\varphi + d_e = 0 ,$$

$$T_i \frac{n_i^{(1)}}{n_i^{(0)}} - e\left(\Phi_e^{(1)} + \Phi_i^{(1)}\right) - k_i r\cos\varphi + d_i = 0 ,$$

(8)
$$\Delta\Phi_e^{(1)} + 4\pi e n_e^{(1)}\gamma = 0 ,$$

$$\Delta\Phi_i^{(1)} - 4\pi e n_i^{(1)} = 0 ,$$

where d_i, d_e are constants of integration. Confining attention to solutions in which the total numbers of particles are unchanged compared with the unperturbed solution, it follows that the potentials $\Phi_e^{(1)}$, $\Phi_i^{(1)}$ will approach constants, as $r \rightarrow \infty$, while the densities behave like: $n_e^{(1)} \propto n_e^{(0)} r \cos\varphi$, $n_i^{(1)} \propto n_i^{(0)} r \cos\varphi$. This leads to negative densities $n_e^{(0)} + n_e^{(1)}$, $n_i^{(0)} + n_i^{(1)}$. This point will be discussed later in greater detail.

It is advantageous to introduce functions u and v, defined by

(9)
$$\frac{n_e^{(1)}}{n_e^{(0)}} =: u - \gamma^2 f(1-f) v ,$$

$$\frac{n_i^{(1)}}{n_i^{(0)}} =: u + (\gamma^2 f - 1) v ,$$

and the variable $x = \frac{r^2 A^2}{a^2}$. Inserting these in eqs. (8) and eliminating $\Phi_e^{(1)}$, $\Phi_i^{(1)}$ yields the following uncoupled set of equations:

(10)
$$\left\{\frac{\partial}{\partial x} x \frac{\partial}{\partial x} + \frac{1}{4x}\frac{\partial}{\partial x} + \frac{2}{(1+x)^2}\right\} u(x,\varphi) = 0$$

$$\left\{\frac{\partial}{\partial x} x \frac{\partial}{\partial x} + \frac{1}{4x}\frac{\partial}{\partial x} - \frac{2\lambda}{(1+x)^2}\right\} v(x,\varphi) = 0 , \quad \lambda = \frac{(\gamma^2-1) f}{(\gamma^2 f - 1)(1-f)} .$$

Because of the fact that the acceleration forces are parallel to the $\varphi = o$ direction, attention can be confined to solutions symmetric about $\varphi = o$.

The ansatz

$$
(11)\qquad
\begin{aligned}
u(x,\varphi) &= \sum_{m=0}^{\infty} A_m\, u_m(x)\, x^{-\frac{m}{2}} \cos\varphi\,, \\
v(x,\varphi) &= \sum_{m}^{\infty} B_m\, v_m(x)\, x^{-\frac{m}{2}} \cos\varphi
\end{aligned}
$$

is used in Appendix A to solve eqs. (10) and then the general solutions of u and v regular for all x and with the asymptotic behaviour $x^{\frac{1}{2}}\cos\varphi$ are determined. Satisfying the same conditions, one gets the solution for $n_e^{(1)}$, $n_i^{(1)}$ with (A 8) and (9)

$$
(12)\qquad
\begin{aligned}
n_e^{(1)} &= n_e^{(0)}\left\{A_0 u_0(x) + A_1 u_1(x)\, x^{-\frac{1}{2}}\cos\varphi - B_1 \gamma^2 f(1-f)\, v_1(x)\, x^{-\frac{1}{2}}\cos\varphi\right\}, \\
n_i^{(1)} &= n_i^{(0)}\left\{A_0 u_0(x) + A_1 u_1(x)\, x^{-\frac{1}{2}}\cos\varphi + B_1 (\gamma^2 f-1)\, v_1(x)\, x^{-\frac{1}{2}}\cos\varphi\right\},
\end{aligned}
$$

with

$$
(13)\qquad
\begin{aligned}
u_m(x) &= 1 - \frac{2}{m+1}\,\frac{1}{x+1}\,, \qquad m = 0, 1 \\
v_1(x) &= e^{\pi\{2\lambda - 1/4\}^{1/2}} \left[\sum_{n=0}^{\infty} \frac{\prod_{m=0}^{n}(m^2-m+2\lambda)}{(n+1)!\, n!}\,(1+x)^{-n}\left\{\ln(1+x) + \sum_{\ell=n}^{\infty}\left(\frac{1+2\ell}{\ell^2+\ell+2\lambda} - \frac{1}{\ell+1} - \frac{1}{\ell+2}\right)\right\} - (x+1)\right].
\end{aligned}
$$

The constant B_1 is determined by comparing equations (12) and (8), which gives the two relations

$$
(14)\qquad
\begin{aligned}
B_1 &= -\frac{2k_i}{e E_{max}}\, e^{-\pi\{2\lambda - 1/4\}^{1/2}}\, \frac{1}{(1-f)(\gamma^2 f - 1)}\,, \\
k_i &= -f k_e\,.
\end{aligned}
$$

The first of equations (14) connects B_1 to the external force k_i on the ions, while the second again gives relation (3). E_{max} is the maximum field strength of the unaccelerated electron beam. To determine the constants A_0 and A_1 in (12), it can be seen that the term with A_0 leads only to a axisymmetric deformation of the density profile and is not related to the acceleration. Omitting this term and using (14), one obtains the following expressions for the electron and ion densities:

$$
\begin{aligned}
n_e(x,\varphi) &= n_e^{(0)}\left\{1 + A_1 u_1(x)\, x^{-1/2}\cos\varphi + \frac{2k_i}{eE_{max}}\,\frac{\gamma^2 f}{\gamma^2 f-1}\, e^{-\pi\{2\lambda-1/4\}^{1/2}}\, x^{-1/2} v_1(x)\cos\varphi\right\}, \\
n_i(x,\varphi) &= n_i^{(0)}\left\{1 + A_1 u_1(x)\, x^{-1/2}\cos\varphi - \frac{2k_i}{eE_{max}}\,\frac{1}{1-f}\, e^{-\pi\{2\lambda-1/4\}^{1/2}}\, x^{-1/2} v_1(x)\cos\varphi\right\},
\end{aligned}
\tag{15}
$$

where as before $x := \frac{r^2 A^2}{a^2}$.

The crucial point is now to find an upper limit of the force k_1 acting on the ions without destroying the beam and determine how this limit will depend on γ and f. The solutions of the linearized equations do not give a limitation in a direct way. Nevertheless an attempt will be made to find an approximation to the limit by the following procedure: As eqs. (15) show, the electron and ion densities will be negative for large x and $\cos\varphi < 0$ and $\cos\varphi > 0$, resp., even for an arbitrarily small k_1. The contributions of the negative densities are limited by the following conditions on the maximum permissible value of k_1, which is assumed positive

$$
\begin{aligned}
\int_0^\infty n_e(x,\varphi=0)\,dx &= \eta \int_0^\infty n_e^{(0)}(x)\,dx, \\
\int_0^\infty n_i(x,\varphi=\pi)\,dx &= \eta \int_0^\infty n_i^{(0)}(x)\,dx .
\end{aligned}
\tag{16}
$$

There η, $0<\eta<1$ is a parameter, which will be determined as follows. Putting (15) in (16) one can express A_1 and k_i/eE_{max} as functions of η. It holds for k_i/eE_{max} that

$$
k_i = eE_{max}(1-\eta)\, e^{\pi\{2\lambda-1/4\}^{1/2}}\, J^{-1}\, \frac{1}{\frac{1}{1-f}+\frac{\gamma^2 f}{\gamma^2 f-1}}, \tag{17}
$$

where $\lambda = \frac{(\gamma^2-1)f}{(\gamma^2 f-1)(1-f)}$ and is the integral

(18) $$J = -\int_0^\infty \frac{x^{-1/2}\,\vartheta_1(x)}{(1+x)^2}\,dx = \frac{\pi^3}{\lambda}\,\frac{2}{1+e^{-2\pi(2\lambda-1/4)^{1/2}}}\left[\frac{1}{\Gamma(1/4)^2}\prod_{n=0}^{\infty}\frac{2(n+2n^2+\lambda)^2}{(1/2+2n)^2}\right]^2$$

In fig.1 the force k_i/eE_{max} is plotted versus f for different values of γ. The curves have their maxima at f = $1/\gamma$ and the maxima increase monotonically to a finite limit as $\gamma \to \infty$. To fix η, we use the result of Barkhudaryan et al.. They calculated the maximum force on the ions for the special case $T_e = T_i$, which corresponds to f = $1/\gamma$, and obtained the result $k_i/eE_{max} = 0.4$. Using this result to fit the curves for $\gamma \to \infty$ one gets for the value $\eta = 0.62$.

As the plot shows, the maximum acceleration only weakly depends on f and γ over a wide range of f for not too small γ. This means that, for equilibria without external focusing, acceleration near f = 1 and f = $1/\gamma^2$ is not possible. Near these Budker limits one gets the expressions for k_i/eE_{max} using the asymptotic expansion of J for large λ : $J = \frac{\pi}{2}\,e^{\pi(2\lambda-1/4)^{1/2}}(2\lambda)^{-1/2}$

(19) $$\frac{k_i}{eE_{max}} = (1-\eta)\frac{\sqrt{8}}{\pi}\times \begin{cases}(1-f)^{1/2} & \text{for } f\approx 1\\ (\gamma^2 f-1)^{1/2} & \text{for } f = 1/\gamma^2\end{cases}$$

In fig.2 the density distributions of ions and electrons are plotted versus $\sqrt{x}$, ($\varphi = 0, \pi$) for the case of maximum k_i/eE_{max} and different values of f and γ. The contributions of negative densities are small, but are nevertheless present. The Bennett beam density goes to zero for $x \to \infty$. It seems to be sure, that the difficulty with the negative density arises from the high-energy tail of the Maxwellian velocity distribution in the Bennett beam. It is plausible, that in the case of a beam with finite transverse dimension and bounded transverse energy of the particles the difficulty would not have arisen. Assuming that the results do not depend very strongly on the particle distribution, the calculation given here should be a fairly good approximation.

Acknowledgements:

The author wishes to acknowledge many fruitful discussions with Prof. A. Schlüter.

Appendix A

Inserting the ansatz (11) into the differential equations (10), one obtains the following equations for the functions $u_m(x)$ and $v_m(x)$:

$$\text{(A1)}\qquad \begin{aligned} &\left\{ x\frac{d^2}{dx^2} + (1-m)\frac{d}{dx} + \frac{2}{(x+1)^2} \right\} u_m(x) = 0, \\ &\left\{ x\frac{d^2}{dx^2} + (1-m)\frac{d}{dx} - \frac{2\lambda}{(x+1)^2} \right\} v_m(x) = 0. \end{aligned}$$

Both equations have two independent solutions $u_m^{(1)}(x)$, $u_m^{(2)}(x)$ and $v_m^{(1)}(x)$, $v_m^{(2)}(x)$ for m = 0,1,... [KAMKE 1967 , MAGNUS, OBERHETTINGER 1966] , which are

$$\text{(A2)}\qquad \begin{aligned} &u_m^{(1)}(x) = 1 - \frac{2}{m+1}\,\frac{1}{(x+1)}, \qquad m = 0,1,2,\ldots \\ &u_0^{(2)}(x) = \frac{x-1}{x+1}\ln x - \frac{4}{x+1}, \\ &u_1^{(2)}(x) = \frac{x}{x+1}\ln x + \frac{x-1}{2}, \\ &u_m^{(2)}(x) = \sum_{n=0}^{m-2} \frac{(2)_n\,(2-m)_n}{n!\,(n+3)!}\,(x+1)^{n+2}, \qquad m = 2,3,\ldots \end{aligned}$$

There $(a)_n$ is defined as $(a)_n = a\cdot(a-1)\cdots(a-n+1)$, $(a)_0 = 1$ and

$$\text{(A3)}\qquad \begin{aligned} &v_m^{(1)}(x) = \mathrm{Re}\left\{ \frac{\Gamma(\alpha)\,\Gamma(\alpha+m)}{\Gamma(2\alpha)\,\Gamma(m)}(x+1)^{\alpha}\,{}_2F_1(\alpha,\alpha-m,2\alpha;x+1) \right\}, \\ &v_m^{(2)}(x) = \mathrm{Im}\left\{ \frac{\Gamma(\alpha)\,\Gamma(\alpha+m)}{\Gamma(2\alpha)\,\Gamma(m)}(x+1)^{\alpha}\,{}_2F_1(\alpha,\alpha-m,2\alpha;x+1) \right\}, \\ &\text{with} \qquad \alpha = \tfrac{1}{2}\left[1 + i\{8\lambda-1\}^{1/2}\right]. \end{aligned}$$

$\Gamma(\alpha)$ is the gamma function and ${}_2F_1$ the hypergeometric function, which can be represented by the following infinite series. It holds that for $|x| < 1$

$$\text{(A4)}\qquad \begin{aligned} {}_2F_1(\alpha,\alpha-m,2\alpha;x+1) = {} & \frac{\Gamma(2\alpha)\,\Gamma(m)}{\Gamma(\alpha)\,\Gamma(\alpha+m)} \sum_{n=0}^{m-1} \frac{(\alpha)_n\,(\alpha-m)_n}{(1-m)_n\,n!}(-x)^n - \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\,\Gamma(\alpha-m)}\,x^m \times \\ & \times \sum_{n=0}^{\infty} \frac{(\alpha+m)_n\,(\alpha)_n}{n!\,(n+m)!} \left\{ \ln x + \psi(\alpha+m+n) + \psi(\alpha+n) - \psi(n+1) - \psi(n+m+1) + i\pi \right\} (-x)^n \end{aligned}$$

and for $|x| > 1$

$$
\begin{aligned}
{}_2F_1(\alpha, \alpha-m, 2\alpha; x+1) &= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha+m)} e^{-i\pi\alpha} \Bigg[(-1)^m (1+x)^{-\alpha+m} \sum_{n}^{m-1} \frac{\Gamma(m-n)}{n!} (\alpha-m)_n (\alpha+m-n)_n (1+x)^{-n} \\
&+ (1+x)^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha-m)_{n+m} (1-m-\alpha)_{n+m}}{n!\,(n+m)!} (1+x)^{-n} \Big\{ \ln(x+1) - \psi(\alpha+n) - \psi(\alpha-n) + \psi(1+n) + \psi(1+n+m) + i\pi \Big\} \Bigg],
\end{aligned}
\tag{A4}
$$

where for m = o the sum is defined as $\sum_{n=0}^{m-1} = 0$. The function $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function.

The particle densities $n_e^{(1)}/n_e^{(0)}$ and $n_i^{(1)}/n_i^{(0)}$ have to be regular for all x and should have the asymptotic behaviour $x^{1/2}\cos\varphi$. Because of (9) the $u(x,\varphi)$ and $v(x,\varphi)$ have to satisfy the same conditions. From (11) and (A2) it follows that the only terms contributing to $u(x,\varphi)$ are $u_0^{(1)}(x)$ and $u_1^{(1)}(x)$.

Because of the behaviour of $v_m^{(1)}(x)$ near x = o

$$
\begin{aligned}
v_0^{(1)}(x) &\underset{x\to 0}{=} -\ln x \\
v_m^{(1)}(x) &\underset{x\to 0}{=} 1 , \quad m = 1, 2, \ldots
\end{aligned}
\tag{A5}
$$

contribution of $v_m^{(1)}(x)$ would lead to a singularity of $v(x,\varphi)$ at the origin x = o. Furthermore, from the asymptotic expansion of $v_m^{(2)}(x)$

$$
\begin{aligned}
v_0^{(2)}(x) &\underset{x\to\infty}{=} e^{\pi\{2\lambda - 1/4\}^{1/2}} \\
v_m^{(2)}(x) &\underset{x\to\infty}{=} e^{\pi\{2\lambda - 1/4\}^{1/2}} \left\{ \frac{(m^2-m+2\lambda)}{m!\,(m-1)!} \ln(1+x) + (-1)^m (1+x)^m \right\}, \quad m = 1, 2, \ldots
\end{aligned}
\tag{A6}
$$

it can be seen that only the term $v_1^{(2)}(x)$ has the desired asymptotic behaviour. At x = o this function $v_1^{(2)}(x)$ is proportional to x:

(A7) $$v_1^{(2)}(x) = -\left\{\pi + \pi \, tgh\, \pi\{2\lambda - 1/4\}^{1/2} + \frac{2(2\lambda - 1/4)^{1/2}}{2\lambda}\right\} x$$

so that, in fact, $v_1^{(2)}(x)$ is the only regular contribution to $v(x,\varphi)$.

Thus, the general solutions for $u(x,\varphi)$ and $v(x,\varphi)$ which satisfy all required conditions are

(A8) $$u(x,\varphi) = A_0 u_0^{(1)}(x) + A_1 x^{-1/2} u_1(x) \cos\varphi$$
$$v(x,\varphi) = B_1 v_1^{(2)}(x)\, x^{-1/2} \cos\varphi$$

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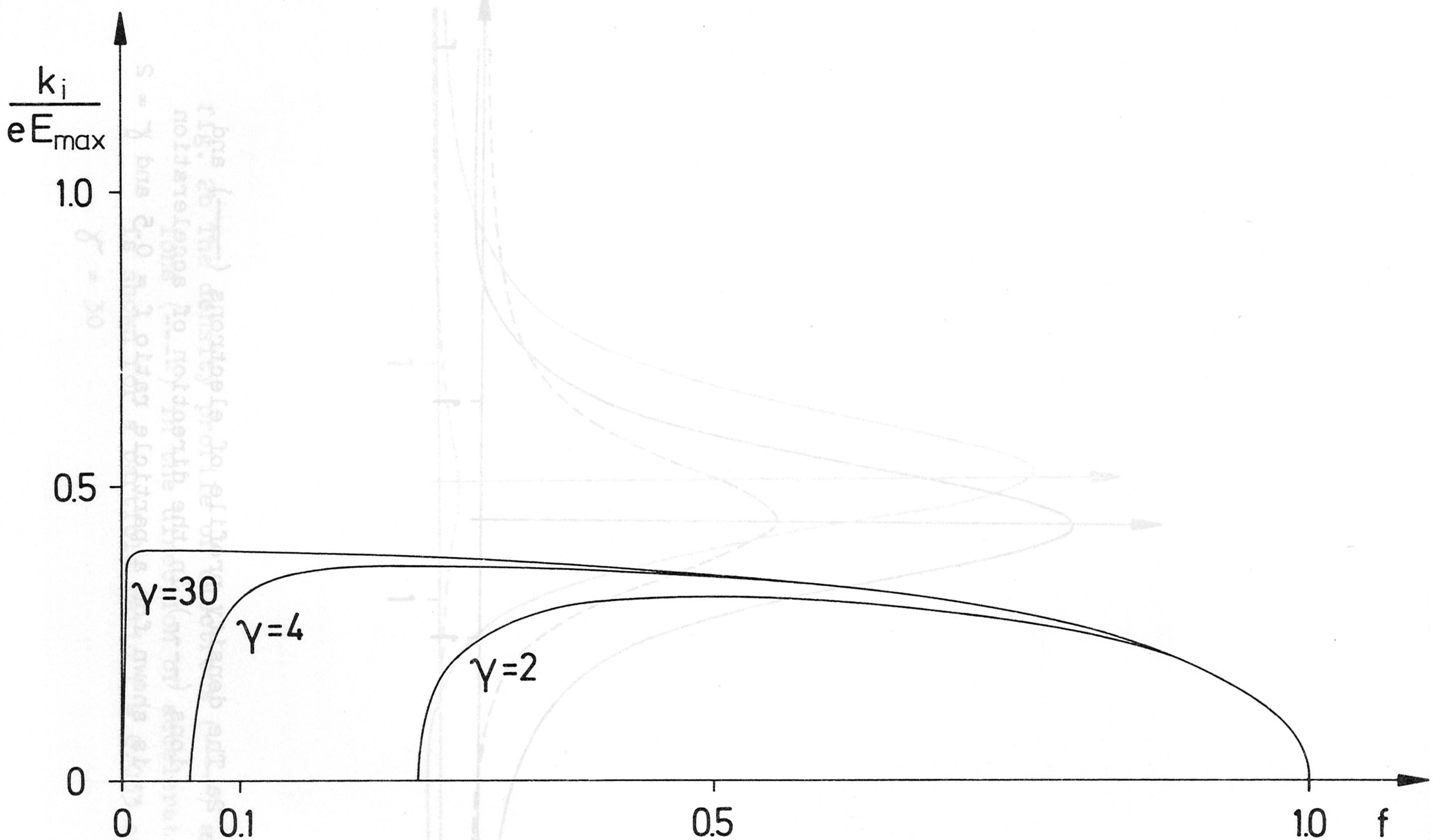

fig. 1 The maximum possible force k_i on the ions compared to the maximum electric field E_{max} produced by the electron is shown as a function of the particle ratio f for different values of γ .

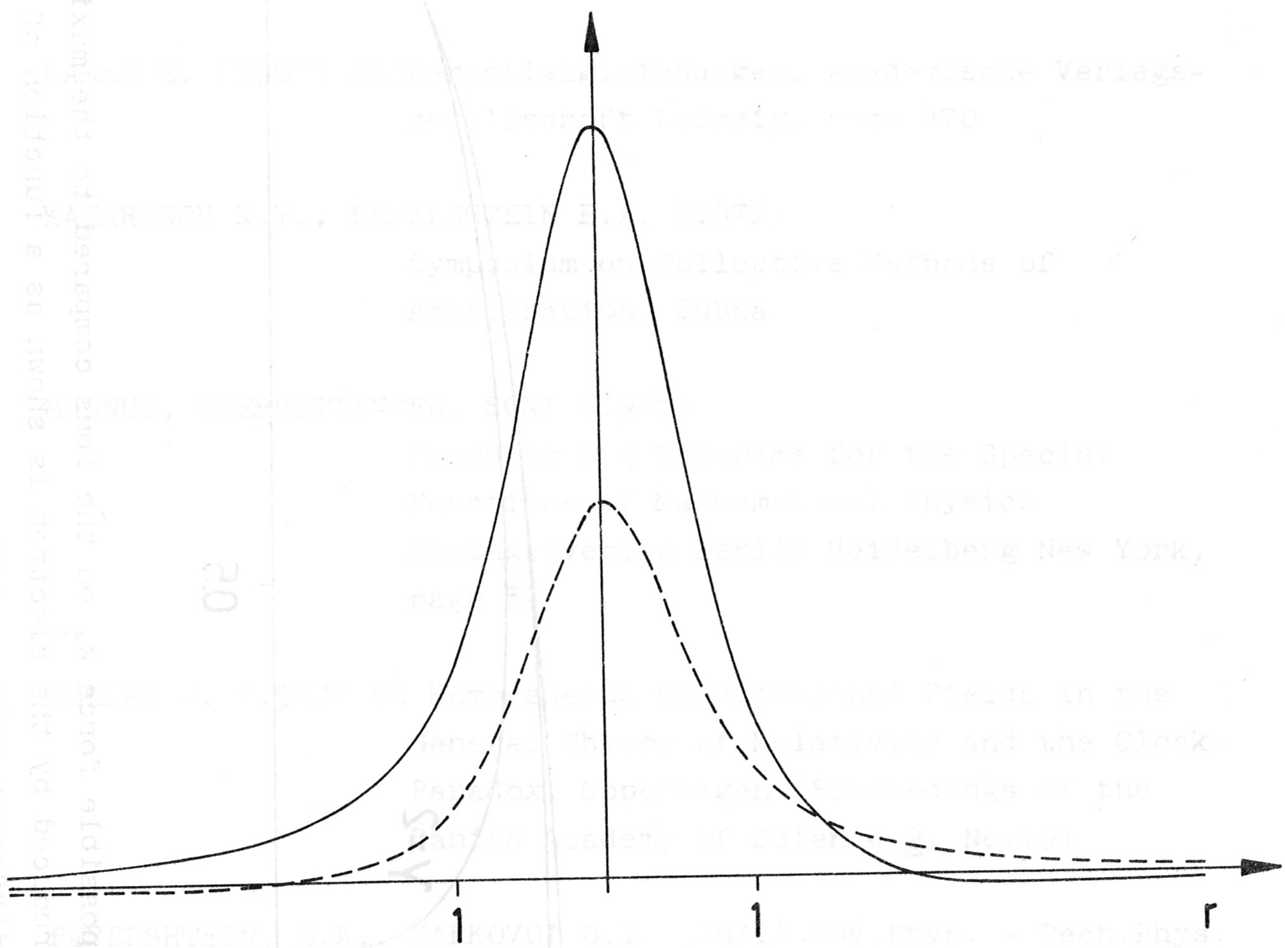

fig. 2a The density profile of electrons (——) and ions (----) in the direction of acceleration is shown for a particle ratio f = 0.5 and γ = 2

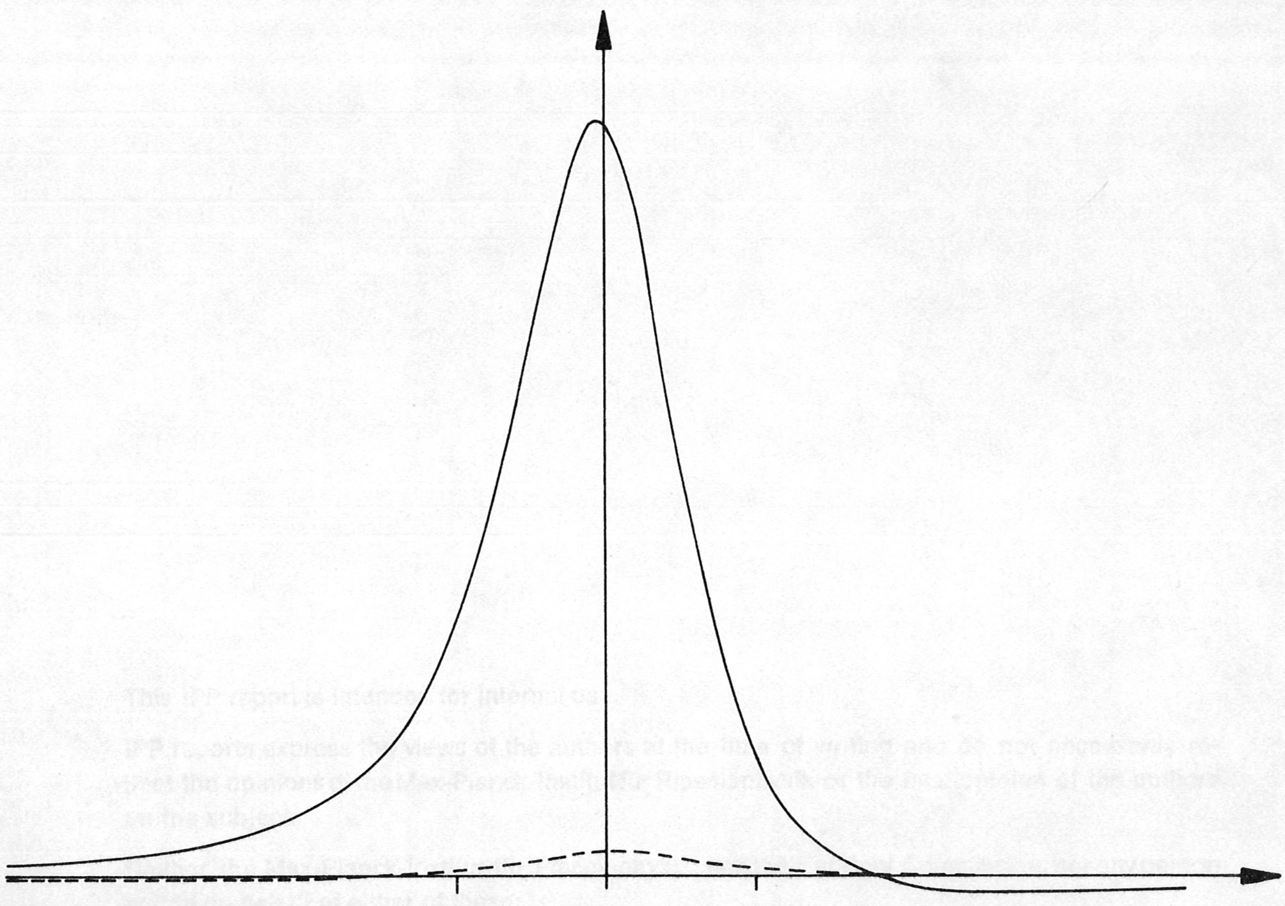

fig. 2b The density profile of electrons (———) and ions (----) in the direction of acceleration is shown for a particle ratio $f = 0.033$ and $\gamma = 30$