Stability of exact toroidal MHD equilibrium with non-circular cross section

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IPP 1/136 Nov. 1973

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Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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(in English)

November 1973

Abstract

For a toroidal plasma with vertically elongated cross section, an exact solution of the MHD equilibrium equation is constructed which satisfies the Mercier criterion on the magnetic axis. In the particular example treated in this note, the ellipticity parameter is $E = 0.15$ which corresponds to a ratio of the half axis $\bar{b}$ to a_p of the elliptical plasma cross section of 1.2. This equilibrium is Mercier-stable against localized modes for $(Rj/2B)^2_{axis} < 1.25$, $(\iota/2\pi)_{axis} < 1.23$.

Recently, a class of exact axisymmetric solutions of the ideal MHD equilibrium equation has been found by the authors. If only the simplest type of solution is used for describing a toroidal plasma with vertically elongated cross section, e.g. a belt pinch, it is found that this equilibrium is MHD unstable according to Mercier's criterion. However, using an appropriate superposition of two exact solutions, a weak triangular deformation of the plasma cross section is achieved in such a way that Mercier's criterion for stability can be satisfied.

In recent experiments on toroidal plasma confinement, the aspect ratio (= ratio of major to minor radius of the torus) is relatively small, so that the usual expansions of equilibrium solutions in powers of the inverse aspect ratio are not applicable. A class of exact solutions has recently been found by the authors [1, 2]. A very simple particular solution has a bell-shaped current distribution and seems useful for the description of equilibria of the tokamak or pinch type. In the present note we investigate the stability of this particular solution against localized modes by applying Mercier's criterion [3] in the neighborhood of the magnetic axis. We shall show that this solution is unstable for a plasma with vertically elongated cross section, as for instance the belt pinch [4]. We then show that it is possible to construct a stable solution by an appropriate superposition of two exact solutions.

Let us first introduce the general solution of the equilibrium equation. Using cylindrical coordinates, we write the axisymmetric magnetic field in the form

$$\vec{B} = \frac{R_o}{R} I_A \vec{e}_\varphi + \frac{R_o}{R} \vec{e}_\varphi \times \nabla F \tag{1}$$

where R_o measures the large radius of the toroidal plasma. I_A and the pressure p are functions of F only. We choose these functions in the form

$$p = p_1 + \frac{P}{2R_o^2} F^2 , \; I_A^2 = B_o^2 + \frac{M}{R_o^2} F^2 . \tag{2}$$

The equilibrium equation, $\vec{j} \times \vec{B} = \nabla p$, then takes the following well-known form:

$$\frac{\partial^2 F}{\partial R^2} - \frac{1}{R} \frac{\partial F}{\partial R} + \frac{\partial^2 F}{\partial z^2} = - \left(\frac{R^2}{R_o^2} \frac{P}{R_o^2} + \frac{M}{R_o^2} \right) F . \tag{3}$$

We now introduce the new variable

$$\rho = \frac{1}{2} \sqrt{P} \frac{R^2}{R_o^2} . \tag{4}$$

Looking for solutions in the form $F = H(\rho)\cos(kz/R_o)$ we find that $H(\rho)$ satisfies the differential equation of the Coulomb wave functions [5]. Thus, the general solution of the equilibrium equation has the form

$$F(R,z) = \alpha[F_o(\eta,\rho) + \gamma G_o(\eta,\rho)] \cos (kz/R_o) \tag{5}$$

where F_o and G_o are the regular and irregular Coulomb wave functions of order zero, with

$$\eta = \frac{k^2 - M}{4\sqrt{P}} . \tag{6}$$

We now prescribe the following boundary conditions on $F(R,z)$: $F = 0$ for $R = 0$ and for $z = \pm b$. From the first condition we get $\gamma = 0$, the second condition determines the possible values of k:

$$k = (2n + 1) \frac{\pi}{2} \frac{R_o}{b} = k_n, \quad (n = 0,1,2...). \tag{7}$$

It is now possible to construct various types of equilibrium configurations by superposition of solutions with different values of n. For our purpose it is sufficient to consider

$$F(R,z) = \sum_{n=0,1} \alpha_n F_o (\eta_n,\rho) \cos (k_n z/R_o) \tag{8}$$

where η_n is given by eq. (6) for $k = k_n$.

We take $\alpha_1/\alpha_o > 0$ in order to have a magnetic axis (and not a hyperbolic point) in the plane z=0.

We now assume that there is a magnetic axis at $R = R_a$, $z = 0$. If we denote the partial derivatives of F with respect to R and z by the corresponding indices, we may write the equation of the magnetic surfaces in the neighborhood of the magnetic axis in the following form:

$$(F_{RR})_a \; (R-R_a)^2 + (F_{zz})_a \; z^2 + (F_{Rzz})_a \; (R-R_a) \; z^2 + (F_{RRR})_a \; (R-R_a)^3 = \text{const.} \tag{9}$$

(the index a indicates that the derivatives are taken on the magnetic axis).

For equilibria characterized by functions p(F) and $I_A(F)$ of the form (2), Mercier's stability criterion applied to the neighborhood of a magnetic axis is

$$\left[\frac{Rj}{2B}\right]_a^2 < \frac{1}{(1+E)\;(1+E/2)} \left\{1 + \frac{3}{4}\,E\left[1-R_a\left(\frac{\frac{1}{3}\,F_{zz}F_{RRR}-F_{RR}F_{Rzz}}{F_{zz}\;\;F_{RR}}\right)_a\right] - \frac{P}{(R_a/R_o)P+(R_o/R_a)M}\,f\,(E)\right\} \tag{10}$$

where

$$E = \left[\frac{F_{RR} - F_{zz}}{F_{RR} + F_{zz}}\right]_a \tag{11}$$

and

$$f(E) = \frac{4E^2}{2+E} \cdot \frac{1}{1+\sqrt{1-E^2} - E^2/(2+E)} \; . \tag{12}$$

The expression on the left-hand side of the criterion (10) is related to the rotational transform on the magnetic axis by

$$\left(\frac{\iota}{2\pi}\right)_a = \sqrt{1-E^2}\left(\frac{Rj}{2B}\right)_a . \qquad (13)$$

The last term in the curly brackets of (10) is the ballooning term which is always destabilizing and increases in magnitude with increasing pressure. The other terms in the curly bracket correspond to the mean magnetic well.

a) We first consider the case $\alpha_1 = 0$. This corresponds to the case studied by Herrnegger [1]. We then have

$$(F_{Rzz})_a = 0, \quad (F_R)_a \quad (F_{zz})_a = 0$$

since $(F_R)_a = 0$. For the other partial derivatives we find

$$R_o^2 (F_{RR})_a = - P \left(\frac{R_a}{R_o}\right)^2 \left(1 - \frac{2\eta_o}{\rho_a}\right) F_o (\eta_o, \rho_a)$$

$$R_o^2 (F_{zz})_a = - k_o^2 F_o(\eta_o, \rho_a)$$

$$R_o^3 (F_{RRR})_a = - P \frac{R_a}{R_o} \left(3 - \frac{2\eta_o}{\rho_a}\right) F_o (\eta_o, \rho_a).$$

Here $\rho_a = (1/2) \sqrt{P} \; (R_a^2/R_o^2)$.

The stability criterion then becomes

$$\left(\frac{Rj}{2B}\right)_a^2 < \frac{1}{(1+E)\;(1+E/2)} \left\{1 - \frac{3}{2} E \frac{\rho_a}{\rho_a - 2\eta_o} - \frac{P}{\frac{R_a}{R_o} P + \frac{R_o}{R_a} M} f(E)\right\}. \qquad (14)$$

The expression $\rho_a/(\rho_a-2\eta_o)$ in the second term of the right-hand side of (14) is of the order of the aspect ratio, as we can see in the following way: The differential equation for the Coulomb wave function [5] shows that for $\rho = 2\eta_o$, the second derivative of $F_o(\eta_o, \rho)$ vanishes. Hence, for this value of ρ $|dF_o/d\rho|$ is a maximum and therefore $\rho = 2\eta_o$ corresponds approximately to the boundary of the plasma. Denoting by R_b the radial coordinate of the inner plasma boundary and by a_p the typical small plasma radius we may write

$$\frac{\rho_a}{\rho_a - 2\eta_o} \approx \frac{R_a^{\ 2}}{R_a^{\ 2} - R_b^{\ 2}} \approx \frac{R_a}{2a_p} . \tag{15}$$

If the ballooning term in (14) is negligible (low pressure) then stability is achieved if

$$1 - \frac{3}{2} E \frac{\rho_a}{\rho_a - 2\eta_o} > 0. \tag{16}$$

This is clearly satisfied for $E < 0$ (horizontally elongated plasma cross section). But according to (15), the stability condition (16) is not satisfied for $E > a_p/R_a$ (vertically elongated plasma cross section).

b) We now consider the case $\alpha_1/\alpha_o \neq 0$.
We wish to show that a small deformation of the configuration considered in a) is sufficient to achieve stability for $E > 0$ (vertically elongated plasma cross section). For this purpose we consider a particular case by choosing the following parameters:

$$\eta_o = 4, \quad \frac{R_o}{b} = 3.59, \quad \sqrt{P} = 21.2,$$

$$\frac{\alpha_1}{\alpha_o} = 0.93.$$

We then obtain $\eta_1 = 7$, $M = -307$ and $E = 0.15$,

$$R_a \left(\frac{3F_{RR}F_{Rzz} - F_{zz}F_{RRR}}{3F_{RR}\;F_{zz}} \right)_a = 4.11,$$

$$\frac{P}{P+M} = 3.2, \quad f(E) = 0.021.$$

The criterion now is $\left(\frac{Rj}{2B} \right)_a^2 < 1.25$

or taking the square root and using eq. (13):

$$\left(\frac{\iota}{2\pi} \right)_a < 1.23 \;.$$

In Figure 1 the magnetic surfaces of two configurations with vertically elongated cross sections are shown. Fig. 1(a) corresponds to the case a) discussed above, Fig.2(b) corresponds to the case b). In both cases, the magnetic surfaces are approximately elliptical with a triangular deformation.

In case a), the triangle points are towards the symmetry axis, and this case is MHD unstable, as we have shown above. On the contrary, in case b) the triangle points are away from the symmetry axis, and this case is stable for sufficiently low current density.

In conclusion, we have obtained the following result. For a toroidal plasma with vertically elongated cross section, an exact solution of the MHD equilibrium equation can be constructed in such a way that Mercier's criterion is satisfied. This solution is then appropriate for further studies of the stability of configurations like a belt pinch or a sharply curved tokamak with non-circular cross section. In the particular example treated in this note, the ellipticity parameter is E = 0.15, which corresponds to a ratio of the half axis a_p and $\bar{b}$ of the elliptic cross section of

$\bar{b}/a_p \approx \left[(1+E)/(1-E)\right]^{1/2} = 1.16$, with $b/R_o = 0.28$. It is possible to obtain stable solutions with much larger values of E, for instance of the order of the values used in the belt pinch [4]. In this case the stability limit for $(Rj/2B)^2$ can be shifted to values larger than one, so that more efficient ohmic heating might be expected.

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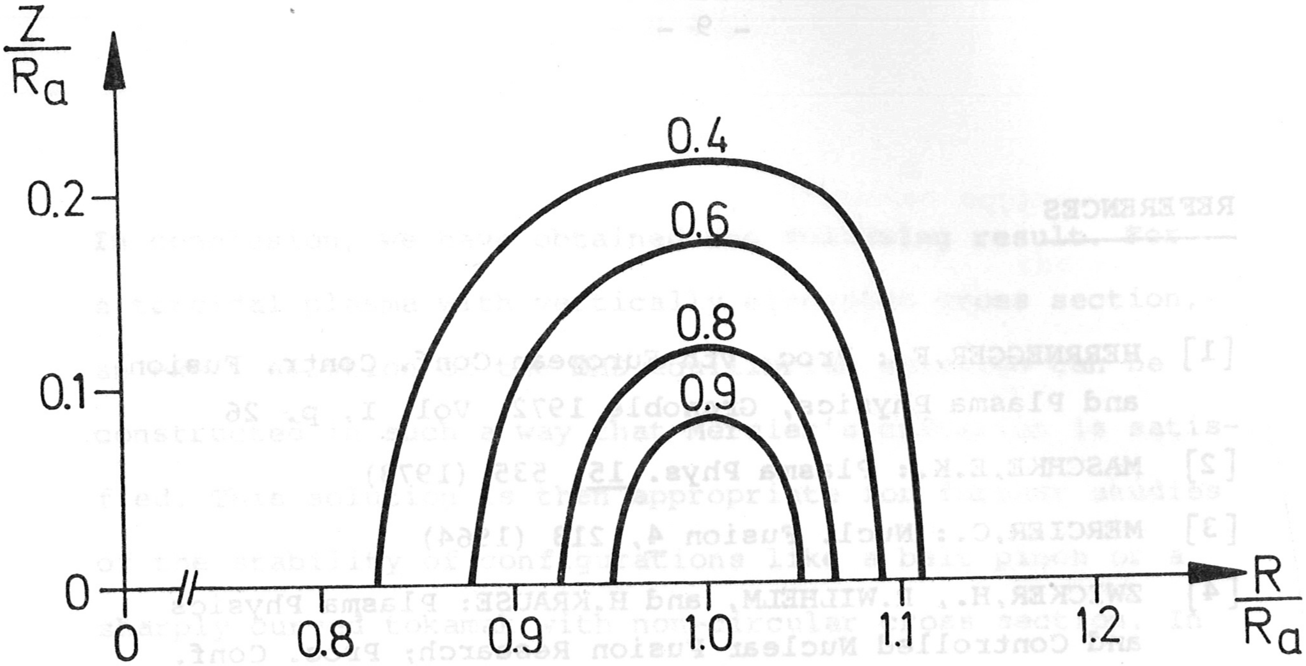

Fig. 1a) Magnetic surfaces of configurations with vertically elongated cross section: $\alpha_1 = 0$, $\eta_o = 4$, $E = 0.51$, $b/R_o = 0.28$, $\sqrt{P} = 20.5$, $F(R_a, z=o) = 1$; unstable for localized modes.

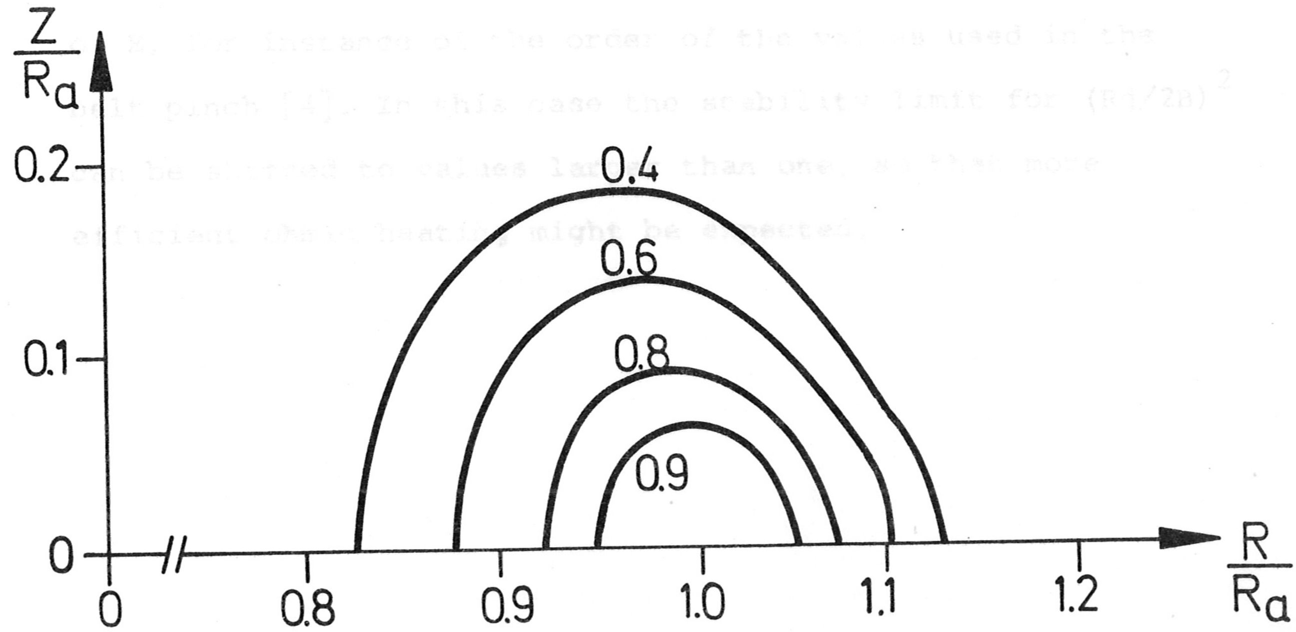

Fig. 1b) Magnetic surfaces of configurations with vertically elongated cross section:
$\alpha_1 = 0.93$, $\eta_o = 4$, $b/R_o = 0.28$, $\sqrt{P} = 21.2$, $E = 0.15$, $F(R_a, z=0) = 1$; stable against localized modes.