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Flow Fields and Temperature Fields in a Wall Stabilized Arc with Transverse Magnetic Field. Solution of the Coupled System of Equations by Means of Green's Functions

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Abstract

The coupled distributions of viscous flow and temperature in a wall stabilized arc with transverse magnetic field are calculated on the basis of the conservation equations for mass density, momentum, and energy. By means of Green's function of the biharmonic and the Laplace operator, the relevant equations are converted to integral equations which can be solved by an alternating iteration procedure. An example involving an argon arc is given and compared with experimental results.

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1. INTRODUCTION

The problem of the interaction of plasmas with magnetically driven viscous flows is a very interesting one, but the theoretical treatment is difficult because of the strong coupling between the relevant equations. Thus, in order to study such phenomena, it is convenient to look for a relatively simple plasma configuration. A model which allows essentially analytical calculations is the wall-stabilized arc, burning in the axial direction in a tube with cylindrical cross section and constant wall temperature. By applying an external transverse magnetic field a mass flow normal to the arc axis is set in motion owing to the Lorentz forces, thus causing enthalpy transport. The maximum of the temperature distribution (which, in the absence of a magnetic field, is of cylindrical symmetry) is thus shifted in the direction of the magnetic forces. This, in turn, modifies the distribution of the magnetic forces via the temperature dependence of the electrical conductivity. Mathematically this means a coupling between the balance equations for momentum and energy. The arc is deflected to a new equilibrium position which is determined by a consistent solution of the conservation equations for mass density, momentum, and energy in the steady state case.

This paper describes a method of solution based on the method of Green's functions for elliptic boundary value problems. The momentum balance can be reduced to an extended form of the biharmonic equation for the stream function. In the case of incompressible flow the use of Green's function for the biharmonic equation allows a straight-forward solution when the temperature field is known, whereas the compressible case leads to an integral equation which can be solved iteratively /1, 2/. For a given flow field on the other hand the temperature field can be calculated from the energy equation by converting it to an integral equation by means of Green's function of the Laplace operator /3/. Thus, both the momentum equation and the energy equation can be solved iteratively and must be iterated alternately in order to get consistent solutions of the stream function and temperature.

2. BASIC EQUATIONS, ASSUMPTIONS, GEOMETRY

The calculations are based on a one-fluid model under steady state conditions $\left(\frac{\partial}{\partial t} = 0\right)$. The relevant equations are the three conservation laws for mass density, momentum, and energy.

2.1. Continuity Equation

In the steady state case, we get

$$\mathrm{div}\,(\rho_m \vec{V}) = 0 \tag{1}$$

where ρ_m is the mass density, and $\vec{V}$ the flow velocity.

2.2. Momentum Equation

With current density $\vec{j}$, magnetic induction $\vec{B}$, pressure p and constant viscosity η_z the momentum balance in the steady state case is

$$\rho_m \vec{V} \cdot \mathrm{grad}\,\vec{V} = \vec{j} \times \vec{B} - \mathrm{grad}\,p + \eta_z \left(\frac{4}{3}\,\mathrm{grad}\,\mathrm{div}\,\vec{V} - \mathrm{curl}\,\mathrm{curl}\,\vec{V}\right). \tag{2}$$

2.3. Energy Equation

In the steady state case we obtain for the energy balance

$$\rho_m \vec{V} \cdot \mathrm{grad}\,h + \mathrm{div}\,\vec{W} = \vec{j} \cdot \vec{E}\,, \tag{3}$$

with static enthalpy h, heat flux density $\vec{W}$ and electrical field strength $\vec{E}$.

2.4. Assumptions

The basic equations (1), (2), (3) are valid or will be used under the following assumptions:

a) steady state conditions, $\partial/\partial t = 0$, as already mentioned,

b) $|\vec{E}| \gg |\vec{V} \times \vec{B}|$, Ohm's law in its most simple form: $\vec{j} = \sigma \vec{E}$,

c) radiation neglected,

d) p = constant in the material functions such as h, σ , ρ_m , etc.,

e) magnetic field of the arc current neglected,

f) no influence of the magnetic field on the material functions,

g) constant viscosity, as already mentioned.

2.5. Geometry

From Eqs.(1), (2), (3) the fields of the flow velocity $\vec{V}$ and temperature T have to be calculated for the circular cross section of the arc tube, which is assumed to be infinitely long. The geometry and the coordinate system are shown in Fig.1. The flow field is two-dimensional:

$$V_z = 0 . \tag{4}$$

There is no change of any quantity in the axial direction (z-axis):

$$\partial/\partial z = 0 . \tag{5}$$

R is the tube radius, the current density is in the (-z)-direction

$$\vec{j} = - \vec{e}_z j ,$$

the external magnetic field is homogeneous and is assumed to be in the (+y)-direction

$$\vec{B} = \vec{e}_y B .$$

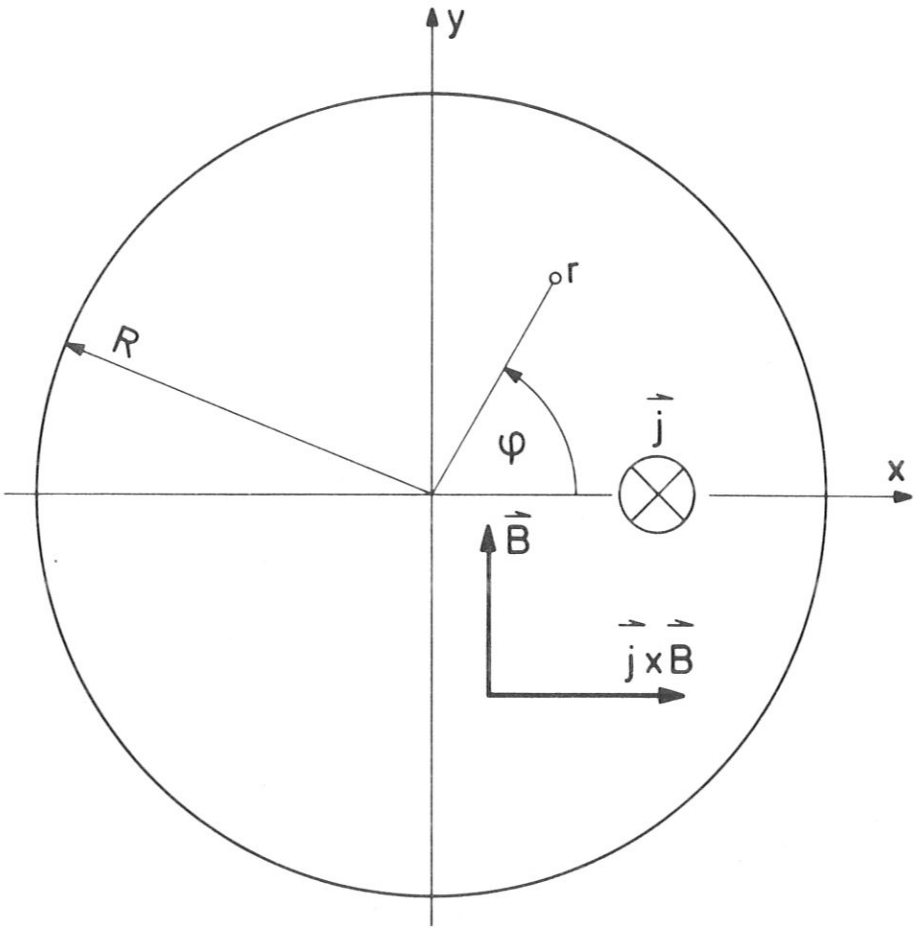

Fig.1 Geometrie and coordinate system

Therefore, the Lorentz forces have the (+x)-direction

$$\vec{j} \times \vec{B} = \vec{e}_x \, j \, B . \tag{6}$$

Further on, we introduce normalized coordinates by

$$\varrho = r/R \quad , \quad \eta = y/R .$$

2.6. Input Quantities

For the numerical computation the following input data must be given:

a) Constant quantities:
   - Tube radius R,
   - magnetic induction B,
   - viscosity η_z,
   - wall temperature T_w,
   - the value $S(O)$ of the heat flux potential S in the tube axis $r = o$ (see Section 3.3.1.),
   - normalization values ϱ_{m_0} for mass density and σ_0 for electrical conductivity,
   - furthermore, numerical data such as error limits, relaxation factors etc.;

b) Tables of the temperature dependent material functions:
   - Electrical conductivity $\sigma(T)$,
   - thermal conductivity $\kappa(T)$,
   - specific heat for constant pressure $c_p(T)$.

From these tables the following functions, the significance of which will be explained later on, have to be calculated:

- Heat flux potential $S(T)$,
- normalized heat flux potential $s(T) = S(T)/S(O)$,
- enthalpy $h(T)$,
- normalized electrical conductivity $w(T) = \sigma(T)/\sigma_0$,
- the function $t = \varrho_{m_0}/\varrho_m(T)$ (see Section 3.1.).

3. METHOD OF SOLUTION

3.1. Continuity Equation and Stream Function

The dimensionless variable t is introduced by setting

$$\varrho_m / \varrho_{m_0} = 1/t \, , \tag{7}$$

where ϱ_{m_0} is a normalization density (see Section 2.6.). The continuity equation (1) thus becomes

$$div(\tfrac{1}{t} \vec{V}) = 0 \, . \tag{8}$$

Eq. (8) allows the introduction of a vector potential $\vec{A}$, which, according to Eqs. (4) and (5), can be assumed to be in the (constant) axial direction:

$$\tfrac{1}{t} \vec{V} = curl \, \vec{A} \, , \quad \text{with} \quad \vec{A} = \vec{e}_z \, \Psi \, . \tag{9}$$

Therefore, we get for the flow velocity

$$\vec{V} = t \cdot curl \, (\vec{e}_z \, \Psi) = -\vec{e}_z \times t \, grad \, \Psi \, , \tag{9a}$$

or, written in cylindrical coordinates, corresponding to the coordinate system in the tube cross section:

$$\begin{aligned} \vec{V}_r &= t \cdot \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \varphi} = \frac{1}{R} \cdot t \cdot \frac{1}{\varrho} \cdot \frac{\partial \Psi}{\partial \varphi} \, , \quad and \\ \vec{V}_\varphi &= - t \frac{\partial \Psi}{\partial r} = - \frac{1}{R} \cdot t \cdot \frac{\partial \Psi}{\partial \varrho} \, . \end{aligned} \tag{9b}$$

Thus, we are able to express the two components of the flow velocity in terms of the function $\Psi(\varrho, \varphi)$, which is the well-known stream function for the divergency-free flow field $\frac{1}{t} \vec{V} = \frac{\varrho_m}{\varrho_{m_0}} \vec{V}$. The streamlines are represented by the curves Ψ = const in the $\varrho - \varphi$ - plane. Therefore, in the following, our aim will be to calculate the field $\Psi(\varrho, \varphi)$, together with the corresponding temperature field T (ϱ, φ).

3.2. Momentum Equation

3.2.1. Vorticity

In order to establish a differential equation for the stream function, we introduce the vorticity

$$\vec{\omega} = \operatorname{curl} \vec{V} \tag{1o}$$

which, according to Eqs. (4) and (5), is in the axial direction:

$$\vec{\omega} = \vec{e}_z \, \omega . \tag{1oa}$$

The gradient terms are now eliminated by taking the curl of the momentum equation (2) and, finally, the curl of the inertia forces is neglected. So we get from Eq. (2):

$$\operatorname{curl}(\vec{j} \times \vec{B}) = \eta_z \operatorname{curl} \operatorname{curl} \vec{\omega}$$

and, with div $\vec{\omega}$ = o and Eqs. (6) and (1oa):

$$\Delta \omega = \frac{1}{\eta_z} B \frac{\partial j}{\partial y} . \tag{11}$$

Eq. (11) is a Poisson equation for the vorticity field $\omega(\varrho, \varphi)$, which, however, cannot be treated immediately because there is no information on the boundary condition $\omega(\varrho = 1; \varphi)$. Therefore, we have to express ω in terms of the stream function ψ.

3.2.2. Relation between Vorticity and Stream Function

Eqs. (9) and (1o) are combined:

$$\vec{\omega} = \operatorname{curl}(t \operatorname{curl} \vec{A}) = t \operatorname{curl} \operatorname{curl} \vec{A} - (\operatorname{curl} \vec{A}) \times (\operatorname{grad} t)$$

$$= t (\operatorname{grad} \operatorname{div} \vec{A} - \nabla^2 \vec{A}) - (\operatorname{curl} \vec{A}) \times \operatorname{grad} t .$$

Furthermore, using Eqs. (5) and (9), we have

$$\operatorname{div} \vec{A} = 0 \quad \text{and} \quad \nabla^2 \vec{A} = \vec{e}_z \, \Delta \psi .$$

Thus, we get

$$\vec{\omega} = \vec{e}_z \omega = -\vec{e}_z t \Delta \psi + (\vec{e}_z \times \operatorname{grad} \psi) \times \operatorname{grad} t =$$

$$= -\vec{e}_z t \Delta \psi - \vec{e}_z (\operatorname{grad} \psi \cdot \operatorname{grad} t) + \operatorname{grad} \psi (\vec{e}_z \cdot \operatorname{grad} t).$$

The right-hand term $\vec{e}_z \cdot \operatorname{grad} t$ is zero since grad t is a vector in the plane of the tube cross section, i.e. perpendicular to the axial unit vector $\vec{e}_z$. Therefore, we obtain:

$$\omega = -t \Delta \psi - \operatorname{grad} \psi \cdot \operatorname{grad} t , \quad \text{or}$$
$$\omega = -\operatorname{div}(t \operatorname{grad} \psi) . \qquad (12)$$

3.2.3. Differential Equation for the Stream Function

Inserting Eq. (12) into (11) we finally arrive at an extended biharmonic differential equation for the stream function ψ :

$$\Delta \operatorname{div}(t \operatorname{grad} \psi) = -\frac{1}{\eta_z} B \frac{\partial j}{\partial y} . \qquad (13)$$

The flow velocity has to be zero at the tube wall:

$$\vec{V}(r=R) \equiv 0 .$$

For this reason (see Eq. (9b)) and because of symmetry, the stream function ψ has to satisfy the following boundary conditions:

$$\psi(r=R, \varphi) = 0 \; ; \quad \left(\frac{\partial \psi}{\partial r}\right)_{r=R,\varphi} = 0 . \qquad (13a,b)$$

Let us now assume that the temperature field $T(r,\varphi)$ is known from the energy equation (or from an experiment). Then, via the temperature dependence of the electrical conductivity, the distribution of the current density $j(r,\varphi)$ is also given, i.e. we know the right-hand side of Eq.(13). In the following we shall solve the elliptic boundary value problem (13, 13a,b) by means of Green's function of the biharmonic equation. First we shall look for a solution $\psi_1(\varrho,\varphi)$ for constant mass density and in the next step we shall treat the general case of variable mass density. The method is described in the next two sections.

3.2.4. Solution for Incompressible Flow

In this section we derive a solution $\psi_1(\varrho,\varphi)$ for incompressible flow, i.e. $\varrho_m \equiv \varrho_{m_0}$ ($t \equiv 1$). Eq.(13) then becomes the pure biharmonic equation, which can be written in terms of the normalized co-ordinates $\varrho = r/R$ and $\eta = y/R$:

$$\Delta \Delta_{(\varrho)} \psi_1 = - R^3 \frac{1}{\eta_z} B \frac{\partial j}{\partial \eta} . \tag{14}$$

The boundary conditions are the same as in Eqs.(13a,b), i.e.

$$\psi_1(\varrho=1,\varphi)=0 \; ; \quad \left(\frac{\partial \psi_1}{\partial \varrho}\right)(\varrho=1,\varphi) = 0 . \tag{14a,b}$$

We thus have to solve the first boundary value problem of the biharmonic equation for the interior of the unit circle. This can be done by Green's method: Green's function of the biharmonic equation for the interior of the unit circle is given by the following expression /1, 2, 4/:

$$G_\psi(\varrho,\varphi;\varrho',\varphi') = (1-\varrho^2)(1-\varrho'^2) - a \ln(a/b) , \tag{15}$$

where

$$a = \varrho^2 + \varrho'^2 - 2\varrho\varrho' \cos(\varphi'-\varphi) , \tag{15a}$$

$$b = 1 + (\varrho\varrho')^2 - 2\varrho\varrho' \cos(\varphi'-\varphi) , \tag{15b}$$

with the field point coordinates $\rho = \frac{r}{R}$, φ, and the source point coordinates $\rho' = \frac{r'}{R}$, φ'.

The solution of the system (14, 14a,b) is then given by a parameter integral /1, 2/:

$$\Psi_1(\rho,\varphi) = -\frac{R^3 B}{16\pi \eta_z} \int_{\varphi'=0}^{2\pi} \int_{\rho'=0}^{1} G_\psi(\rho,\varphi;\rho',\varphi') \frac{\partial j}{\partial \eta'}(\rho',\varphi')\rho' d\rho' d\varphi'. \tag{16}$$

By integrating by parts Eq. (16) can be transformed to an expression which is more suitable for numerical computation:

$$\Psi_1(\rho,\varphi) = +\frac{R^3 B}{16\pi \eta_z} \int_0^{2\pi} \int_0^1 j(\rho',\varphi') \frac{\partial G_\psi}{\partial \eta'}(\rho,\varphi;\rho',\varphi')\rho' d\rho' d\varphi', \tag{16a}$$

where we used the property $G_\psi(\rho,\varphi;1,\varphi') = 0$ of Green's function at the boundary.

In the following, we introduce some abbreviations and definitions:

a) Normalized electrical conductivity:

$$w = \frac{\sigma}{\sigma_0} = \frac{j}{j_0} \quad , \text{ where } \sigma_0 \text{ is a} \tag{17}$$

normalization conductivity (see Section 2.6.a). We thus get for the arc current:

$$J = \int_0^{2\pi} \int_0^R j(r',\varphi') r' dr' d\varphi' = R^2 \cdot j_0 \cdot (INT) \quad , \text{ where} \tag{18}$$

$$(INT) = \int_0^{2\pi} \int_0^1 w(\rho',\varphi')\rho' d\rho' d\varphi' \quad . \tag{18a}$$

b) Normalization value for the stream function ψ:

$$\psi_0 = \frac{R\,J\,B}{8\pi\,\eta_a} \quad . \tag{19}$$

c) Kernel function /2/:

$$\begin{aligned} g^*(\varrho,\varphi;\varrho',\varphi') &= \frac{1}{2}\varrho'\frac{\partial G_\psi}{\partial \eta'}(\varrho,\varphi;\varrho',\varphi') \\ &= g_1(\varrho,\varphi;\varrho',\varphi')\sin\varphi - g_2(\varrho,\varphi;\varrho',\varphi')\sin\varphi' \, , \end{aligned} \tag{2o}$$

with

$$g_1 = \varrho\varrho'\left[\frac{a}{b} - 1 - \ln\frac{a}{b}\right] \qquad \text{and} \tag{2oa}$$

$$g_2 = \varrho'^2\left[\varrho^2\left(\frac{a}{b}-1\right) - \ln\frac{a}{b}\right] . \tag{2ob}$$

(For a, b, see Eqs. (15a,b)).

Inserting Eqs.(17), (18), (19), (2o) into (16a) finally yields the following expression for the stream function ψ_1 in the incompressible case:

$$\frac{\psi_1(\varrho,\varphi)}{\psi_0} = \frac{\int_0^{2\pi}\int_0^1 w\,g^*\,d\varrho'\,d\varphi'}{\int_0^{2\pi}\int_0^1 w\,\varrho'\,d\varrho'\,d\varphi'} = \frac{\phi_1}{(INT)} \quad , \tag{21}$$

where

$$\phi_1 = \int_0^{2\pi}\int_0^1 w\,g^*\,d\varrho'\,d\varphi' \quad . \tag{21a}$$

Eq.(2o) shows that, in the case of constant mass density $\varrho_m = \varrho_{m_0}$, we are now able to calculate the flow field in terms of the stream function $\psi_1(\varrho,\varphi)$ if the function $w(\varrho',\varphi')$ is known from the temperature field.

3.2.5. Flow with Variable Mass Density

If the temperature dependence of the mass density $\varrho_m = \varrho_m(T)$ is taken into account, we have to start with the extended biharmonic equation (13), where t is now a function of the coordinates ϱ, φ :

$$t(\varrho,\varphi) = \frac{\varrho_{m_0}}{\varrho_m[T(\varrho,\varphi)]} \quad .$$

This more complicated case can, however, also be treated by means of the biharmonic Green's function, Eq. (15). It can be shown /1/, that it is possible to convert Eq. (13) to an integral equation for the stream function Ψ :

$$\Psi(\varrho,\varphi) = \frac{1}{t(\varrho,\varphi)} \cdot \Psi_1(\varrho,\varphi) - \\ - \frac{1}{t(\varrho,\varphi)} \cdot \frac{1}{16\pi} \int_0^{2\pi} \int_0^1 \Psi(\varrho',\varphi') \{ \text{grad}\, t(\varrho',\varphi') \cdot \text{grad}\, \Delta G_{\Psi}(\varrho,\varphi;\varrho',\varphi') \} \varrho' d\varrho' d\varphi' , \tag{22}$$

where $\Psi_1(\varrho,\varphi)$ is the solution for constant mass density, given by Eq. (21). The partial derivatives in Eq. (22) have to be taken with respect to the normalized source point coordinates $\varrho' = \frac{r'}{R}$.

It may be pointed out that it has been possible to "roll" the derivatives from the non-analytically given parts in the integrands to the analytically known Green's function by employing integration by parts and the properties of Green's function. This method has consistently been used in formulating both Eq. (16a) and (22) in order to avoid numerical differentiation.

Eqs. (15), (15a,b) permit analytic calculation of the Kernels in Eq. (22). Furthermore, we wish to write the integral equation (22) in terms of the normalized stream function ϕ, as was also done in Eq. (21). For this purpose we again use the quantities Ψ_0 and (INT) from Eqs. (19) and (18a).

In this way, Eq. (22) takes the form

$$\phi_{i+1}(\rho,\varphi)=\frac{1}{t(\rho,\varphi)}\left\{\phi_1-\frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \phi_i(\rho',\varphi')\Lambda(\rho,\varphi;\rho',\varphi')\rho'd\rho'd\varphi'\right\}, \tag{23}$$

with the kernel function

$$\Lambda(\rho,\varphi;\rho',\varphi')=M(\rho,\varphi;\rho',\varphi')\frac{\partial t(\rho',\varphi')}{\partial\rho'}+N(\rho,\varphi;\rho',\varphi')\frac{1}{\rho'}\frac{\partial t(\rho',\varphi')}{\partial\varphi'}, \tag{23a}$$

where

$$M=\frac{a'}{a}+\frac{1}{b}\left\{2c^{*\prime}-b'\left(1+2\frac{c^*}{b}\right)\right\}, \tag{23b}$$

and

$$N=2\rho\sin\vartheta\left\{\frac{1}{a}-\frac{1}{b}\left(\rho^2+2\frac{c^*}{b}\right)\right\}, \tag{23c}$$

where $\vartheta=\varphi'-\varphi$,

$$\left.\begin{array}{ll} a=\rho^2+\rho'^2-2\rho\rho'\cos\vartheta, & a'=\dfrac{\partial a}{\partial\rho'}=2(\rho'-\rho\cos\vartheta), \\ b=1+(\rho\rho')^2-2\rho\rho'\cos\vartheta, & b'=\dfrac{\partial b}{\partial\rho'}=2(\rho^2\rho'-\rho\cos\vartheta), \\ c^*=(1-\rho^2)(1-\rho\rho'\cos\vartheta), & c^{*\prime}=\dfrac{\partial c^*}{\partial\rho'}=-(1-\rho^2)\rho\cos\vartheta. \end{array}\right\} \tag{23d}$$

ϕ_1 and (INT) are taken from Eqs.(21) and (18a), and $\emptyset_i$ is defined in the same way as in Eq.(21), i.e.

$$\frac{\psi_i}{\psi_0}=\frac{\phi_i}{(INT)}\quad . \tag{23e}$$

(The subscript "i" marks the step number in the iteration procedure; see Section 4.).

It is convenient to introduce the abbreviation

$$I_I(\rho,\varphi) = \int_0^{2\pi}\int_0^1 \phi_i \Lambda \rho' d\rho' d\varphi' \qquad (24)$$

for the integral in Eq.(23). Thus, the integral equation can finally be written in the form

$$\phi_{i+1} = \frac{1}{t}\left\{\phi_1 - \frac{1}{4\pi} I_I\right\} . \qquad (25)$$

We are now able to calculate the flow field from a given temperature field even in the case of temperature-dependent mass density. For this purpose Eq.(25) will be solved by means of an iteration procedure (see Section 4.).

The relevant integrals in the momentum equation are

a) (INT), see Eq.(18a),
b) ϕ_1, see Eq.(21a) with Eqs.(2o),(2oa,b),
c) I_I, see Eq.(24) with Eqs.(23a-d).

3.3. Energy Equation

3.3.1. Conversion to an Integral Equation

Instead of the temperature T we introduce the heat flux potential

$$S(T) = \int_{T_A}^{T} K(T)\, dT \quad , \qquad (26)$$

where $K(T)$ is the thermal conductivity and T_A is the wall temperature, so that $S(r=R) = o$.

Furthermore, we use Fourier's law for the heat flux density

$$\vec{W} = -K(T)\,\mathrm{grad}\,T = -\mathrm{grad}\,S \tag{27}$$

and Ohm's law

$$\vec{j} = \sigma \vec{E}\,. \tag{28}$$

Thus, with Eqs. (26), (27), (28) we can write the energy equation (3) as a partial differential equation for the heat flux potential S :

$$\Delta S + \sigma(S) E^2 = \rho_m \vec{V} \cdot \mathrm{grad}\, h(S) \tag{29}$$

(For the material functions σ (S) and h(S), see Section 2.6b.).

Eq. (29) has to be solved with the boundary conditions $S(r = R, \varphi) = o$. This can formally be done by employing Green's method again. We see from Eq. (29) that we now need Green's function of the Laplace operator for the interior of the unit circle, which is given by the following expression /3/:

$$G_s(\rho, \varphi; \rho', \varphi') = \ln \frac{a}{b}\,,$$

where $a(\rho, \varphi; \rho', \varphi')$ and $b(\rho, \varphi; \rho', \varphi')$ are given by Eqs. (15a,b).

Thus, we get formally from Eq. (29)

$$S(\rho, \varphi) = -\frac{R^2 E^2}{4\pi} \int_0^{2\pi}\int_0^1 \sigma[S(\rho', \varphi')] \cdot G_s(\rho, \varphi; \rho', \varphi')\, \rho' d\rho' d\varphi' + K(\rho, \varphi)\,. \tag{3o}$$

$K(\rho, \varphi)$ is the convection term which is represented by

$$K(\rho, \varphi) = \frac{R}{4\pi} \int_0^{2\pi}\int_0^1 \rho_m G_s \vec{V} \cdot \mathrm{grad}\, h\, \rho' d\rho' d\varphi'\,, \tag{31}$$

where the gradient of h(S) is taken with respect to the normalized coordinates ϱ'. Eq.(31) can be modified by

a) replacing $\vec{V}$ by the stream function ψ according to Eqs.(7) and (9a),

b) "rolling" the gradient from h to q_s (see Section 3.2.5.):

$$K(\varrho,\varphi)=\frac{R}{4\pi}\,\varrho_{m_0}\int_0^{2\pi}\!\!\int_0^{1} q_s\left(\frac{1}{t}\,\vec{V}\cdot \mathrm{grad}_{(\varrho')}h\right)db' \;,$$

where $\quad db'=\varrho'\,d\varrho'\,d\varphi' \;,\quad$ or

$$K(\varrho,\varphi)=-\frac{\varrho_{m_0}}{4\pi}\int_0^{2\pi}\!\!\int_0^{1} q_s\,(\vec{e}_z\times \mathrm{grad}_{(\varrho')}\psi)\cdot \mathrm{grad}_{(\varrho')}h\;db' =$$

$$=-\frac{\varrho_{m_0}}{4\pi}\int_0^{2\pi}\!\!\int_0^{1} q_s\,\vec{e}_z\cdot(\mathrm{grad}\,\psi\times \mathrm{grad}\,h)\,db' =$$

$$=+\frac{\varrho_{m_0}}{4\pi}\int_0^{2\pi}\!\!\int_0^{1} q_s\,\vec{e}_z\cdot \mathrm{curl}\,(h\,\mathrm{grad}\,\psi)\,db' =$$

$$\left.\begin{aligned}&=\cancel{\vec{e}_z\cdot\frac{\varrho_{m_0}}{4\pi}\int_0^{2\pi}\!\!\int_0^{1} \mathrm{curl}\,(q_s h\,\mathrm{grad}\,\psi)\,db'} \;+\\ &\quad+\vec{e}_z\cdot\frac{\varrho_{m_0}}{4\pi}\int_0^{2\pi}\!\!\int_0^{1} h\,(\mathrm{grad}\,\psi\times \mathrm{grad}\,q_s)\,db'\end{aligned}\right\} \qquad (32)$$

The first term in Eq.(32) can be converted to a contour integral by means of Stokes' law and vanishes since $q_s\,h\,\mathrm{grad}\,\psi$ is zero at the boundary $\varrho' = 1$. We thus obtain

$$K(\varrho,\varphi)=-\frac{\varrho_{m_0}}{4\pi}\int_0^{2\pi}\!\!\int_0^{1} h(S)\left[(\vec{e}_z\times \mathrm{grad}\,q_s)\cdot \mathrm{grad}\,\psi\right]\varrho'\,d\varrho'\,d\varphi' .$$

The factor $\vec{e}_z$ x grad q_s can be determined analytically and we get the convection term in the following form:

$$K(\varrho,\varphi) = -\frac{\varrho_{m_0}}{4\pi}\int_0^{2\pi}\int_0^1 h(S)\left[P\frac{\partial\Psi}{\partial\varrho'} + Q\frac{\partial\Psi}{\partial\varphi'}\right]d\varrho' d\varphi' , \qquad (33)$$

where

$$P(\varrho,\varphi;\varrho',\varphi') = -\frac{\partial G_S}{\partial\varphi'} = -2\varrho\varrho'\sin\vartheta\left(\frac{1}{a}-\frac{1}{b}\right) , \qquad (33a)$$

$$Q(\varrho,\varphi;\varrho',\varphi') = +\frac{\partial G_S}{\partial\varrho'} = \frac{2}{a}(\varrho'-\varrho\cos\vartheta) - \frac{2}{b}(\varrho^2\varrho'-\varrho\cos\vartheta) , \qquad (33b)$$

with $\vartheta = \varphi' - \varphi$ and a,b according to Eqs. (15a,b).

3.3.2. Elimination of the Electric Field Strength

It is well known from arc physics that Eq. (3o) represents an eigenvalue problem since the electric field strength E is determined by geometry, material functions and arc current. Therefore, E has to be eliminated and it is convenient to introduce the value S(O) of the heat flux potential in the tube axis $\varrho' = 0$.

From Eqs. (3o) and (33) we get

$$S(0) = -\frac{R^2}{4\pi}E^2\int_0^{2\pi}\int_0^1 \sigma(S)\,G_S(0,\varphi;\varrho',\varphi')\,\varrho' d\varrho' d\varphi' -$$
$$-\frac{\varrho_{m_0}}{4\pi}\int_0^{2\pi}\int_0^1 h(S)\left[P(0,\varphi;\varrho',\varphi')\frac{\partial\Psi}{\partial\varrho'} + Q(0,\varphi;\varrho',\varphi')\frac{\partial\Psi}{\partial\varphi'}\right]d\varrho' d\varphi' ,$$

where

$$G_S(0,\varphi;\varrho',\varphi') = \ln\varrho'^2 = 2\ln\varrho' ,$$
$$P(0,\varphi;\varrho',\varphi') = 0 ,$$
$$Q(0,\varphi;\varrho',\varphi') = \frac{2}{\varrho'} .$$

We thus obtain

$$S(0) = -\frac{R^2}{4\pi} E^2 \int_0^{2\pi}\int_0^1 \sigma(S)\, \rho' \ln \rho'^2 \, d\rho' d\varphi' -$$

$$- \frac{\rho m_0}{4\pi} \int_0^{2\pi}\int_0^1 h(S) \frac{2}{\rho'} \frac{\partial \Psi}{\partial \varphi'} \, d\rho' d\varphi'$$

or

$$E^2 = \frac{-S(0) - \frac{\rho m_0}{4\pi} \int_0^{2\pi}\int_0^1 h(S) \frac{2}{\rho'} \frac{\partial \Psi}{\partial \varphi'} \, d\rho' d\varphi'}{\frac{R^2}{4\pi} \int_0^{2\pi}\int_0^1 \sigma(S)\, \rho' \ln \rho'^2 \, d\rho' d\varphi'} . \tag{34}$$

Finally, in order to obtain an integral equation suitable for numerical computation (inhomogeneous Fredholm type of the second kind) G_S is replaced by a new Kernel function:

$$K_S(\rho,\varphi;\rho',\varphi') = \rho' G_S(\rho,\varphi;\rho',\varphi') - \rho' \ln \rho'^2 =$$

$$= \rho' \ln \frac{1 + \left(\frac{\rho}{\rho'}\right)^2 - 2\left(\frac{\rho}{\rho'}\right) \cos \vartheta}{1 + (\rho\rho')^2 - 2(\rho\rho') \cos \vartheta} . \tag{35}$$

By combining Eqs. (3o), (33), (34), and (35) we arrive at the nonlinear integral equation

$$S(\rho,\varphi) = \left[S(0) + \frac{\rho m_0}{4\pi} \int_0^{2\pi}\int_0^1 h(S) \frac{2}{\rho'} \frac{\partial \Psi}{\partial \varphi'} \, d\rho' d\varphi' \right] \cdot \left[1 + \frac{\int_0^{2\pi}\int_0^1 \sigma(S) K_S \, d\rho' d\varphi'}{2\int_0^{2\pi}\int_0^1 \sigma(S) \rho' \ln \rho' \, d\rho' d\varphi'} \right]$$

$$- \frac{\rho m_0}{4\pi} \int_0^{2\pi}\int_0^1 h(S) \left[P \frac{\partial \Psi}{\partial \rho'} + Q \frac{\partial \Psi}{\partial \varphi'} \right] d\rho' d\varphi' . \tag{36}$$

Again, the Kernels are known analytically. Thus, Eq. (36) permits the computation of the S-distribution (i.e. of the temperature field) if the mass flow field (i.e. the stream function ψ) is known from the corresponding integral equation for the momentum balance, Eq.(22), (23) or (25).

3.3.3. Normalization

In order to obtain a dimensionless form of Eq.(36) (like Eq.(25)), we define the following functions:

$$\mathcal{S}(\varrho,\varphi) = \frac{S(\varrho,\varphi)}{S(0)} \tag{37}$$

(normalized heat flux potential), and

$$\Gamma(s) = \frac{h(S)}{h_0} \tag{38}$$

(normalized enthalpy), with the normalization enthalpy

$$h_0 = \frac{4\pi S(0)\cdot(INT)}{\varrho_{m_0}\,\Psi_0} \tag{38a}$$

(INT) and Ψ_0 are taken from Eqs.(18a) and (19).

Furthermore, we again use the relation $\psi = \Psi_0 \frac{\phi}{(INT)}$ and the normalized electrical conductivity $w = \frac{\sigma}{\sigma_0}$ (see Eqs.(23e) and (17)).

Using these abbreviations, we obtain from Eq.(36):

$$\mathcal{S}(\varrho,\varphi) = \left[1 + \frac{I_1^E(\varrho,\varphi)}{2\,I_E^2}\right]\left[1 + I_3^E\right] - I_4^E(\varrho,\varphi) - I_5^E(\varrho,\varphi) \quad . \tag{39}$$

This is the dimensionless form of the integral equation for the energy balance. The relevant five integrals are then given by

$$I_1^E(\varrho,\varphi) = \int_0^{2\pi}\int_0^1 w(S)\,K_S(\varrho,\varphi;\varrho',\varphi')\,d\varrho'\,d\varphi' \quad , \tag{39a}$$

$$I_2^E = \int_0^{2\pi}\int_0^1 w(S)\,\rho' \ln\rho' \, d\rho' \, d\varphi' \; ; \tag{39b}$$

$$I_3^E = \int_0^{2\pi}\int_0^1 \Gamma \frac{\partial\phi}{\partial\varphi'} \cdot \frac{2}{\rho'} \, d\rho' \, d\varphi' - I_5^E(\rho = 0) \; ; \tag{39c}$$

$$I_4^E(\rho,\varphi) = \int_0^{2\pi}\int_0^1 \Gamma \frac{\partial\phi}{\partial\rho'} \, P(\rho,\varphi;\rho',\varphi') \, d\rho' \, d\varphi' \; ; \tag{39d}$$

$$I_5^E(\rho,\varphi) = \int_0^{2\pi}\int_0^1 \Gamma \frac{\partial\phi}{\partial\varphi'} \, Q(\rho,\varphi;\rho',\varphi') \, d\rho' \, d\varphi' \quad . \tag{39e}$$

The relation between the electric field strength E and the value of S(O) in the tube axis, Eq.(34), then takes the form:

$$E^2 = - \frac{S(0)\left[1 + I_3^E\right]}{\frac{R^2}{4\pi}\sigma_0 \, 2 I_2^E} \quad . \tag{4o}$$

In order to calculate arc characteristics, we also need the ex pression for the arc current, which, according to Eq.(18), is

$$J = R^2 \sigma_0 E \cdot (INT) \tag{4oa}$$

where E has to be taken from Eq.(4o).

4. COMPUTATION

4.1. General Remarks

In order to find consistent solutions for the heat flux potential $S(\varrho,\varphi)$ and the corresponding stream function $\psi(\varrho,\varphi)$, the two governing integral equations (25) and (39) must be solved iteratively and alternately iterated. Eq.(23) can be solved for a given distribution of $S(\varrho',\varphi')$ and hence for given distributions of $\partial t/\partial\varrho'$ and $\partial t/\partial\varphi'$. The solution $\emptyset_1(\varrho,\varphi)$ for incompressible flow has to be calculated from Eq.(21a) and it is convenient to use it as a first approximation for $\emptyset$ on the right-hand side of Eq.(23). In the following, each approximation $\emptyset_i$ inserted in the right-hand side of Eq.(23) produces a better approximation $\emptyset_{i+1}$. This procedure converges rapidly within 5 to 1o iterations /2/. If the distribution of $\emptyset$, (i.e. ψ), is known, a corresponding iteration process is used to solve Eq.(39) for s, (i.e. S). The convergence is the same as with Eq.(23). The whole scheme, with typically two iterations of each of the Eqs.(25) and (39), must be repeated about ten times in order to converge within a maximum error of about $1o^{-4}$. The convergence can be accelerated by successive overrelaxation with relaxation factors of about 1.5.

A problem always occurring when Green's functions are used for treating differential equations by transforming them to integral equations is caused by the field point singularities, e.g. in our problem the functions M and N in Eq.(23) with (23a) and K_s, P, Q in Eq.(39) with (39a-e). Therefore, in order to avoid difficulties through numerical integration over these singularities, the parts of the integrands which are not given analytically [i.e. $w(\varrho',\varphi')$, $(\phi\frac{\partial t}{\partial\varrho'})(\varrho',\varphi')$, $(\phi\frac{\partial t}{\partial\varphi'})(\varrho',\varphi')$, $(r\frac{\partial\phi}{\partial\varrho'})(\varrho',\varphi')$, $(r\frac{\partial\phi}{\partial\varphi'})(\varrho',\varphi')$] are expanded in Fourier series and the coefficients thereof are expanded in polynomial series. These expansions have to be carried out numerically after each step in the iteration procedure. Thus, all integrals in Eqs.(21a), (25), (39), (4o), and (4oa) can be calculated analytically and written in terms of double sums. In this way, no numerical integration procedure is necessary. This method of integration shall be described in the following sections.

4.2. Fourier Expansions

We use the following Fourier expansions, whereby we have to ensure that the functions being expanded must be regular and differentiable in the origin:

a) $$W(\varrho',\varphi') = \sum_{n=0}^{\infty} c_{Wn}(\varrho') \cos n\varphi' \; ; \tag{41a}$$

b) $$\varrho' \mu \frac{\partial \phi}{\partial \varrho'}(\varrho',\varphi') = \sum_{n=1}^{\infty} c_{E\varrho n}(\varrho') \sin n\varphi' \; ; \tag{41b}$$

c) $$\mu \frac{\partial \phi}{\partial \varphi'}(\varrho',\varphi') = \sum_{n=0}^{\infty} c_{E\varphi n}(\varrho') \cos n\varphi' \; ; \tag{41c}$$

d) $$\varrho' \phi \frac{\partial t}{\partial \varrho'}(\varrho',\varphi') = \sum_{n=1}^{\infty} c_{I\varrho n}(\varrho') \sin n\varphi' \; ; \tag{41d}$$

e) $$\phi \frac{\partial t}{\partial \varphi'}(\varrho',\varphi') = \sum_{n=0}^{\infty} c_{I\varphi n}(\varrho') \cos n\varphi' \; ; \tag{41e}$$

As stated above, the Fourier expansions are carried out numerically. Because the functions to be expanded are only known in the grid points of our coordinate system, we have to replace the integrals which determine the Fourier coefficients by finite sums. To achieve this aim, it is not necessary to approximate the original integrals by a numerical integration furmula of any kind because the sine and cosine functions are orthogonal not only with respect to integration but also to summation /5, 6/.

4.3. Polynomial Expansions of the Fourier Coefficients

As indicated in Eqs.(41a-e), the Fourier coefficients are functions of the radial coordinate ϱ'. In principle, these functions of ϱ' can be represented by polynomials, the number of terms in these being equal to the number of radial grid points.

Unfortunately, the matrix of the system of linear equations which determines the polynomial coefficients is very ill-conditioned. With the number of radial grid points N_S necessary in our calculations $(N_S > 10)$ it is not possible in practice to invert the coefficient matrix. Therefore, we have to use a number of terms in the polynomial expansion which is less than the number of grid points. In order to keep the error below a certain limit in a controllable way, we use an expansion in terms of orthogonal polynomials. Because we are dealing with a finite sum, we again choose orthogonal functions which are orthogonal with respect to finite summation. A suitable choice in our case is the so-called Forsythe polynominals /7, 8/. By rearranging these expansions with respect to the powers of ϱ' we arrive at the following sums representing our Fourier coefficients:

$$C_{Wn}(\varrho') = \sum_{i=n}^{m} a_{ni}\, \varrho'^{(i)} \quad , \tag{42a}$$

$$C_{E\varrho n}(\varrho') = \sum_{i=n}^{m} B_{ni}\, \varrho'^{(i)} \quad , \tag{42b}$$

$$C_{E\varphi n}(\varrho') = \sum_{i=n}^{m} C_{ni}\, \varrho'^{(i)} \quad , \tag{42c}$$

$$C_{I\varrho n}(\varrho') = \sum_{i=n}^{m} D_{ni}\, \varrho'^{(i)} \quad , \tag{42d}$$

$$C_{I\varphi n}(\varrho') = \sum_{i=n}^{m} E_{ni}\, \varrho'^{(i)} \quad . \tag{42e}$$

The lower index of summation i = n is always equal to the order of the Fourier coefficient being considered. The upper index m is equal to n plus the maximum order of the Forsythe polynomials chosen.

4.4. Integration in the Azimuthal Direction

By inserting Eqs.(41a-e) into the relevant integrals (18a), (21a), (24) of the momentum equation and into the integrals (39a-c) of the energy equation and integrating with respect to the azimuthal coordinate φ', we arrive at the expressions to be listed in the following sections.

4.4.1. Integrals of the Momentum Equation

a) $$(INT) = \int_0^{2\pi}\int_0^1 w(\rho',\varphi')\rho' d\rho' d\varphi' = 2\pi \int_{\rho'=0}^{1} c_{w_0}(\rho')\rho' d\rho' , \tag{43}$$

b) $$\begin{aligned}\phi_1(\rho,\varphi) &= \int_0^{2\pi}\int_0^1 w(\rho',\varphi')\, g^*(\rho,\varphi;\rho',\varphi')\, d\rho' d\varphi' = \\ &= \phi_{10} + \phi_{11} + \sum_{n=2}^{\infty} \phi_{1n} ,\end{aligned} \tag{44}$$

with
$$\begin{aligned}\phi_{10} = &-2\pi \sin\varphi \int_{\rho'=0}^{1} c_{w_0}(\rho')(\rho\rho')(1-\rho^2)(1-\rho'^2)\, d\rho' - \\ &-2\pi \sin\varphi \int_{\rho'=0}^{\rho} c_{w_0}(\rho')\left[\rho\rho' \ln\rho^2 - \rho^3\left(\rho - \frac{1}{\rho}\right)\right] d\rho' - \\ &-2\pi \sin\varphi \int_{\rho'=\rho}^{1} c_{w_0}(\rho')\left[\rho\rho' \ln\rho'^2 - \rho\rho'^2\left(\rho' - \frac{1}{\rho'}\right)\right] d\rho' ;\end{aligned} \tag{44a}$$

$$\phi_{11} = -2\pi \sin\varphi \cos\varphi \int_{\rho'=0}^{1} c_{w_1}(\rho')(\rho\rho')^2(1-\rho^2)(1-\rho'^2)\, d\rho' -$$

$$-2\pi \sin\varphi \cos\varphi \int_{\rho'=0}^{\rho} c_{w1}(\rho')\left[\rho\rho'^2\left(\rho-\frac{1}{\rho}\right)-\frac{1}{2}\rho'^4\left(\rho^2-\frac{1}{\rho^2}\right)\right]d\rho' -$$

$$-2\pi \sin\varphi \cos\varphi \int_{\rho'=\rho}^{1} c_{w1}(\rho')\left[\rho^2\rho'\left(\rho'-\frac{1}{\rho'}\right)-\frac{1}{2}(\rho\rho')^2\left(\rho'^2-\frac{1}{\rho'^2}\right)\right]d\rho' , \quad (44b)$$

and

$$\Phi_{1n} = \pi \sin(n+1)\varphi\, F_n^+(\rho) + \pi \sin(n-1)\varphi\, F_n^-(\rho) , \quad (44c)$$

$(n>1)$

with

$$F_n^+(\rho) = -\int_{\rho'=0}^{1} c_{wn}(\rho')(1-\rho^2)(1-\rho'^2)(\rho\rho')^{n+1}\, d\rho' -$$

$$-\frac{1}{n}\int_{\rho'=0}^{\rho} c_{wn}(\rho')\rho\rho'^{(n+1)}(\rho^n-\rho^{-n})\, d\rho' +$$

$$+\frac{1}{n+1}\int_{\rho'=0}^{\rho} c_{wn}(\rho')\cdot\rho'^2\rho'^{(n+1)}\left(\rho^{(n+1)}-\rho^{-(n+1)}\right) d\rho' -$$

$$-\frac{1}{n}\int_{\rho'=\rho}^{1} c_{wn}(\rho')\rho^{(n+1)}\rho'\left(\rho'^n-\rho'^{(-n)}\right) d\rho' +$$

$$+\frac{1}{n+1}\int_{\rho'=\rho}^{1} c_{wn}(\rho')\rho'^2\rho^{(n+1)}\left(\rho'^{(n+1)}-\rho'^{-(n+1)}\right) d\rho' ,$$

$$F_n^-(\varrho) = \frac{1}{n}\int_{\varrho'=0}^{\varrho} c_{wn}(\varrho')\varrho\varrho'^{(n+1)}(\varrho^n - \varrho^{-n})\, d\varrho' -$$

$$- \frac{1}{n-1}\int_{\varrho'=0}^{\varrho} c_{wn}(\varrho')\varrho'^2\varrho'^{(n-1)}(\varrho^{(n-1)} - \varrho^{-(n-1)})\, d\varrho' +$$

$$+ \frac{1}{n}\int_{\varrho'=\varrho}^{1} c_{wn}(\varrho')\varrho'\varrho^{(n+1)}(\varrho'^n - \varrho'^{(-n)})\, d\varrho' -$$

$$- \frac{1}{n-1}\int_{\varrho'=\varrho}^{1} c_{wn}(\varrho')\varrho'^2\varrho^{(n-1)}(\varrho'^{(n-1)} - \varrho'^{-(n-1)})\, d\varrho',$$

finally

c) $$I_I(\varrho,\varphi) = \int_0^{2\pi}\int_0^1 \phi_i(\varrho',\varphi')\Lambda(\varrho,\varphi;\varrho',\varphi')\varrho'\, d\varrho'\, d\varphi' =$$

$$= \int_0^{2\pi}\int_0^1 (\varrho'\phi\frac{\partial t}{\partial\varrho'})(\varrho',\varphi')\cdot M(\varrho,\varphi;\varrho',\varphi')\, d\varrho'\, d\varphi' +$$

$$+ \int_0^{2\pi}\int_0^1 (\phi\frac{\partial t}{\partial\varphi'})(\varrho',\varphi')\cdot N(\varrho,\varphi;\varrho',\varphi')\, d\varrho'\, d\varphi' =$$

$$= \sum_{n=1}^{\infty} \sin n\varphi\, F_n^I(\varrho) \quad , \qquad (45)$$

where

$$
\left.
\begin{aligned}
F_n^{I}(\varrho) = {} & 2\pi n \varrho^n (1-\varrho^2) \int_{\varrho'=0}^{1} \left[C_{I\varrho n}(\varrho') + C_{I\varphi n}(\varrho') \right] \varrho'^{(n-1)} \, d\varrho' + \\
& + 2\pi (\varrho^n - \varrho^{-n}) \int_{\varrho'=0}^{\varrho} \left[C_{I\varrho n}(\varrho') + C_{I\varphi n}(\varrho') \right] \varrho'^{(n-1)} \, d\varrho' \\
& + 2\pi \varrho^n \int_{\varrho'=\varrho}^{1} \left[C_{I\varrho n}(\varrho') + C_{I\varphi n}(\varrho') \right] \varrho'^{(n-1)} \, d\varrho' \\
& + 2\pi \varrho^n \int_{\varrho'=\varrho}^{1} \left[C_{I\varrho n}(\varrho') - C_{I\varphi n}(\varrho') \right] \varrho'^{(-n-1)} \, d\varrho' .
\end{aligned}
\right\} \quad (45a)
$$

4.4.2. Integrals of the Energy Equation

a) $$I_1^E(\varrho,\varphi) = \int_0^{2\pi}\int_0^1 w(\varrho',\varphi') \, K_s(\varrho,\varphi;\varrho',\varphi') \, d\varrho' \, d\varphi' =$$

$$
\begin{aligned}
& = 4\pi \int_{\varrho'=0}^{\varrho} C_{w_0}(\varrho') \cdot \varrho' \left[\ln\varrho - \ln\varrho' \right] d\varrho' + \\
& + 2\pi \sum_{n=1}^{\infty} \frac{1}{n} \cos n\varphi \left\{ (\varrho^n - \varrho^{-n}) \int_{\varrho'=0}^{\varrho} C_{wn}(\varrho') \varrho'^{(n+1)} \, d\varrho' + \right. \\
& \left. \qquad + \varrho^n \int_{\varrho'=\varrho}^{1} C_{wn}(\varrho') \left[\varrho'^{(n+1)} - \varrho'^{(-n+1)} \right] d\varrho' \right\} ,
\end{aligned}
\quad (46)
$$

b) $$I_2^E = \int_0^{2\pi}\int_0^1 w(\varrho',\varphi') \, \varrho' \ln\varrho' \, d\varrho' \, d\varphi' = 2\pi \int_{\varrho'=0}^{1} C_{w0}(\varrho') \, \varrho' \ln\varrho' \, d\varrho' , \quad (47)$$

c) $$I_3^E = \int_0^{2\pi}\int_0^1 \Gamma \frac{\partial \phi}{\partial \varphi'} \cdot \frac{2}{\varrho'} \, d\varrho' \, d\varphi' = 4\pi \int_{\varrho'=0}^{1} \frac{1}{\varrho'} C_{E\varphi 0}(\varrho') \, d\varrho' \quad , \quad (48)$$

d) $$I_4^E(\varrho,\varphi)=\int_0^{2\pi}\int_0^1 r\frac{\partial\phi}{\partial\varphi'}(\varrho',\varphi')\cdot P(\varrho,\varphi;\varrho',\varphi')\,d\varrho'd\varphi'=$$

$$=2\pi\sum_{n=1}^{\infty}\cos n\varphi\Big\{(\varrho^n-\varrho^{-n})\int_{\varrho'=0}^{\varrho}C_{E\varrho n}(\varrho')\varrho'^{(n-1)}d\varrho'- \tag{49}$$

$$-\varrho^n\int_{\varrho'=\varrho}^{1}C_{E\varrho n}(\varrho')\left[\varrho'^{(-n-1)}-\varrho'^{(n-1)}\right]d\varrho'\Big\},$$

e) $$I_5^E(\varrho,\varphi)=\int_0^{2\pi}\int_0^1 r\frac{\partial\phi}{\partial\varphi'}(\varrho',\varphi')\,Q(\varrho,\varphi;\varrho',\varphi')\,d\varrho'd\varphi'=$$

$$=2\pi\sum_{0}^{\infty}\cos n\varphi\Big\{(\varrho^n-\varrho^{-n})\int_{\varrho'=0}^{\varrho}C_{E\varphi n}(\varrho')\varrho'^{(n-1)}d\varrho'+$$

$$+\varrho^n\int_{\varrho'=\varrho}^{1}C_{E\varphi n}(\varrho')\left[\varrho'^{(n-1)}+\varrho'^{(-n-1)}\right]d\varrho'\Big\}. \tag{5o}$$

4.5. Integration in the Radial Direction

We replace the Fourier coefficients in the integrals, Eqs.(43 - 5o) by the polynomials Eqs.(42a-d). We are thus able to carry out the integration with respect to the radial coordinate ϱ' analytically. In this way, all integrals can be expressed in terms of double sums.

4.5.1. Integrals of the Momentum Equation

a) $$(INT)=2\pi\sum_{i=0}^{m}\frac{a_{0i}}{i+2}, \tag{51}$$

b) $$\Phi_1 = \Phi_{10} + \Phi_{11} + \sum_{n=2}^{\infty} \Phi_{1n} \;, \tag{52}$$

with

$$\Phi_{10} = -2\pi \sin\varphi \sum_{i=0}^{m} a_{0i} \frac{2\rho}{(i+2)(i+4)} \left[(1-\rho^2) - \frac{2}{i+2}\left(1-\rho^{(i+2)}\right) \right] , \tag{52a}$$

$$\Phi_{11} = -\frac{\pi}{2} \sin 2\varphi \sum_{i=1}^{m} \rho^2 a_{1i} \left[\frac{\rho^{(i+1)}-1}{i+1} + 2\,\frac{2-\rho^2-\rho^{(i+1)}}{i+3} + \frac{\rho^{(i+1)}-3+2\rho^2}{i+5} \right] , \tag{52b}$$

$$\Phi_{1n} = 2\pi \Bigg\{ \sin(n+1)\varphi \sum_{i=n}^{m} a_{ni} \frac{\rho^{(n+1)}}{i+n+2} \left[\frac{\rho^2-1}{i+n+4} + \frac{2\left(1-\rho^{(i-n+2)}\right)}{(i+n+4)(i-n+2)} \right] +$$

$$+ \sin(n-1)\varphi \sum_{i=n}^{m} a_{ni} \frac{1}{i+n+2} \left[\frac{\rho^{(i+3)}-\rho^{(n+1)}}{i-n+2} - \frac{\rho^{(i+3)}-\rho^{(n-1)}}{i-n+4} \right] \Bigg\} , \tag{52c}$$

c) $$I_I = 2\pi \sum_{n=1}^{\infty} \sin n\varphi \Bigg\{ \rho^n \ln\rho \,(E_{nn}-D_{nn}) + n\rho^n (1-\rho^2) \sum_{i=n}^{m} \frac{D_{ni}+E_{ni}}{i+n} +$$

$$+ 2 \sum_{i=n+1}^{m} (\rho^n - \rho^i) \frac{i D_{ni} - n E_{ni}}{i^2 - n^2} \Bigg\} . \tag{53}$$

4.5.2. Integrals of the Energy Equation

a) $$I_1^E(\varrho,\varphi) = 4\pi \sum_{i=0}^{m} a_{0i} \frac{\varrho^{(i+2)}}{(i+2)^2} + \tag{54}$$

$$+ 2\pi \sum_{n=1}^{\infty} \frac{1}{n} \cos n\varphi \sum_{i=n}^{m} a_{ni} \left\{ \frac{\varrho^{(i+2)}}{i+n+2} (\varrho^{2n} - 1) + \varrho^n \left(\frac{1-\varrho^{(i+n+2)}}{i+n+2} - \frac{1-\varrho^{(i-n+2)}}{i-n+2} \right) \right\},$$

b) $$\overline{I}_2^E = -2\pi \sum_{i=0}^{m} a_{0i} \frac{1}{(i+2)^2} , \tag{55}$$

c) $$I_3^E = 4\pi \sum_{i=2}^{m} \frac{C_{0i}}{i} = I_5^E(\varrho = 0), \tag{56}$$

d) $$\overline{I}_4^E(\varrho,\varphi) - 2\pi \sum_{n=1}^{\infty} \cos n\varphi \left\{ B_{nn} \varrho^n \ln \varrho + 2n \sum_{i=n+1}^{m} B_{ni} \frac{\varrho^i - \varrho^n}{i^2 - n^2} \right\}, \tag{57}$$

e) $$I_S^E(\varrho,\varphi) = 4\pi \sum_{i=2}^{m} C_{0i}\frac{1}{i}(1-\varrho^i) - \tag{58}$$

$$-2\pi\sum_{n=1}^{\infty}\cos n\varphi \left\{C_{nn}\varrho^n \ln\varrho + \sum_{i=n+1}^{m} C_{ni}\frac{2i(\varrho^i-\varrho^n)}{i^2-n^2}\right\}.$$

5. RESULTS

In this section, we shall give as an example some results for an argon arc with tube radius $R = 5 \times 10^{-3}$ m and mean value of viscosity $\eta_z = 2 \times 10^{-5}$ Ns/m^2. Fig. 2 shows a computer plot

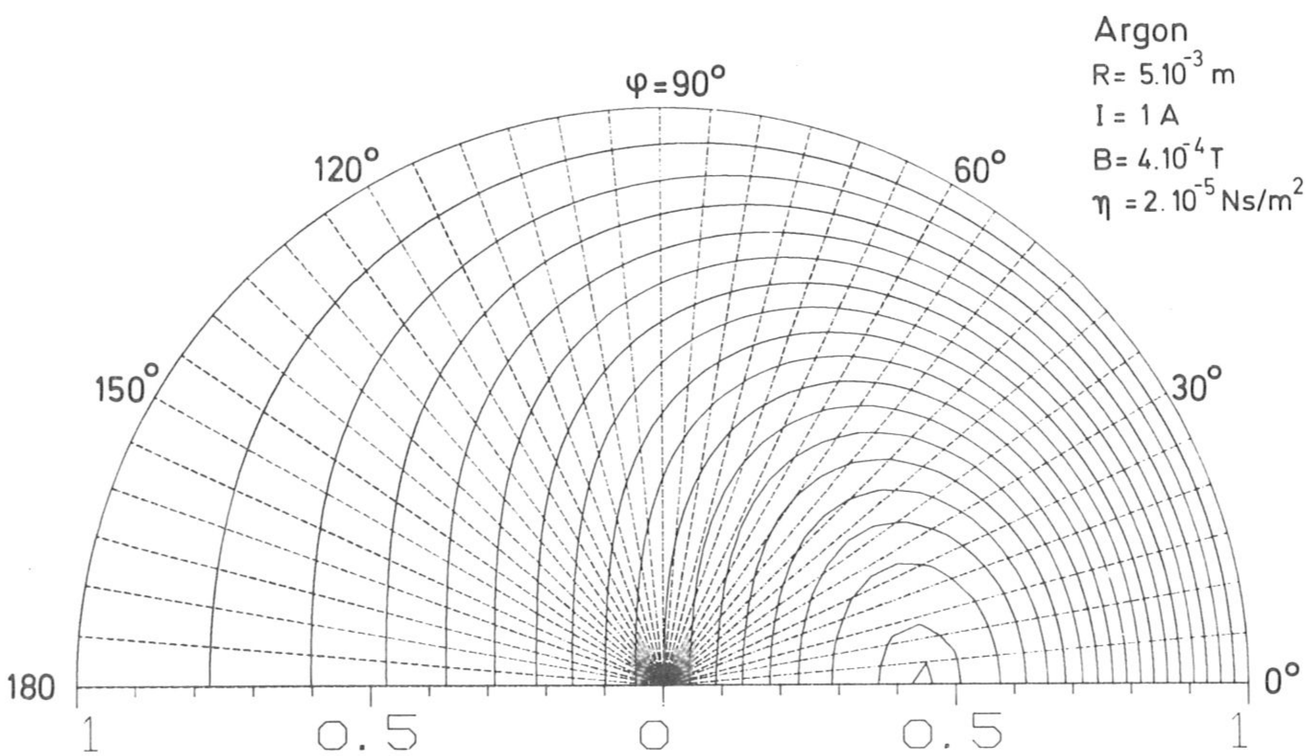

Fig.2 Computer plot of the isotherms S = const in one half of the tube cross section

with isotherms S = const in one half of the tube cross section for an arc current J = 1 A and an external magnetic field of $B = 4 \times 10^{-4}$ tesla. The corresponding flow field is shown in Fig.3. The stream lines ψ = const form a double vortex.

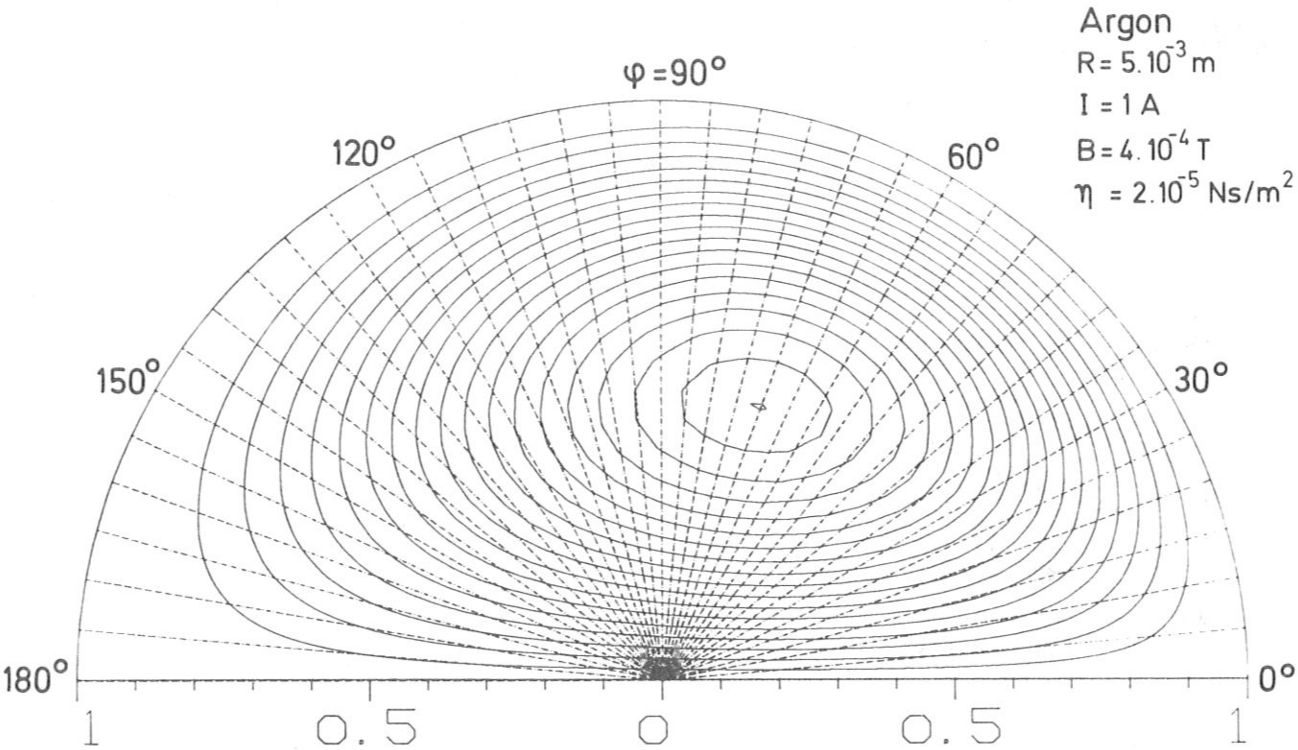

Fig.3 Computer plot of the streamlines ψ = const in one half of the tube cross section

Fig. 4 shows the calculated distribution of the temperature T as a function of radius ϱ along the symmetry axis (x-axis) of the tube cross section. For comparison we have also plotted an experimental temperature distribution measured by Kollmar /9/ by means of a Schlieren method which, however, gives no precise information on the height of the temperature maximum. Fig.5 shows the calculated values of the flow velocity V_x in the symmetry axis (x-axis) and gives a comparison with velocity values measured by Rosenbauer /9, 10/ by means of test particles brought into the flow field of the arc. The observed shift of the experimental velocity distribution towards the theoretical

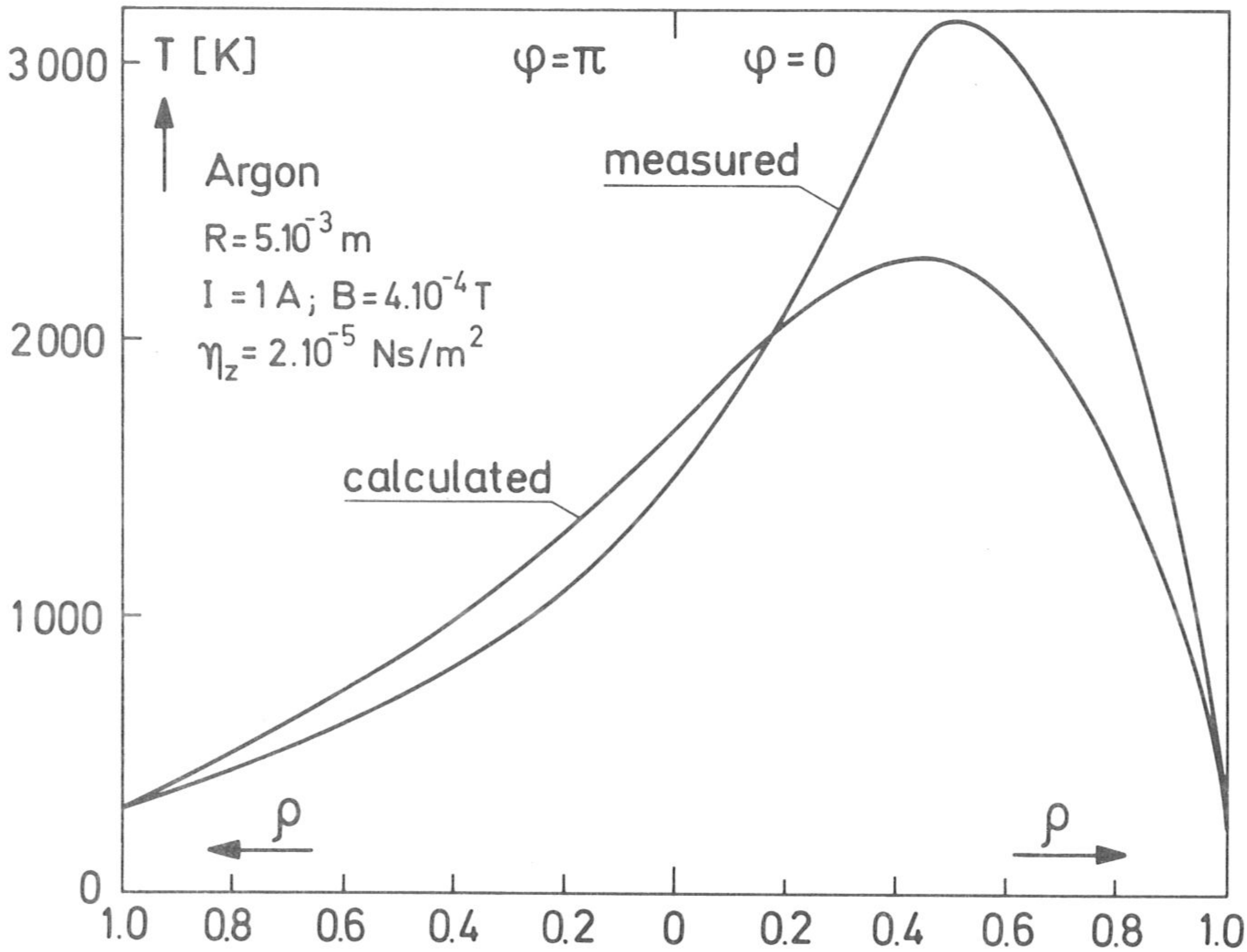

Fig.4 Distribution of the temperature T vs. ϱ along the axis of symmetry (x-axis)

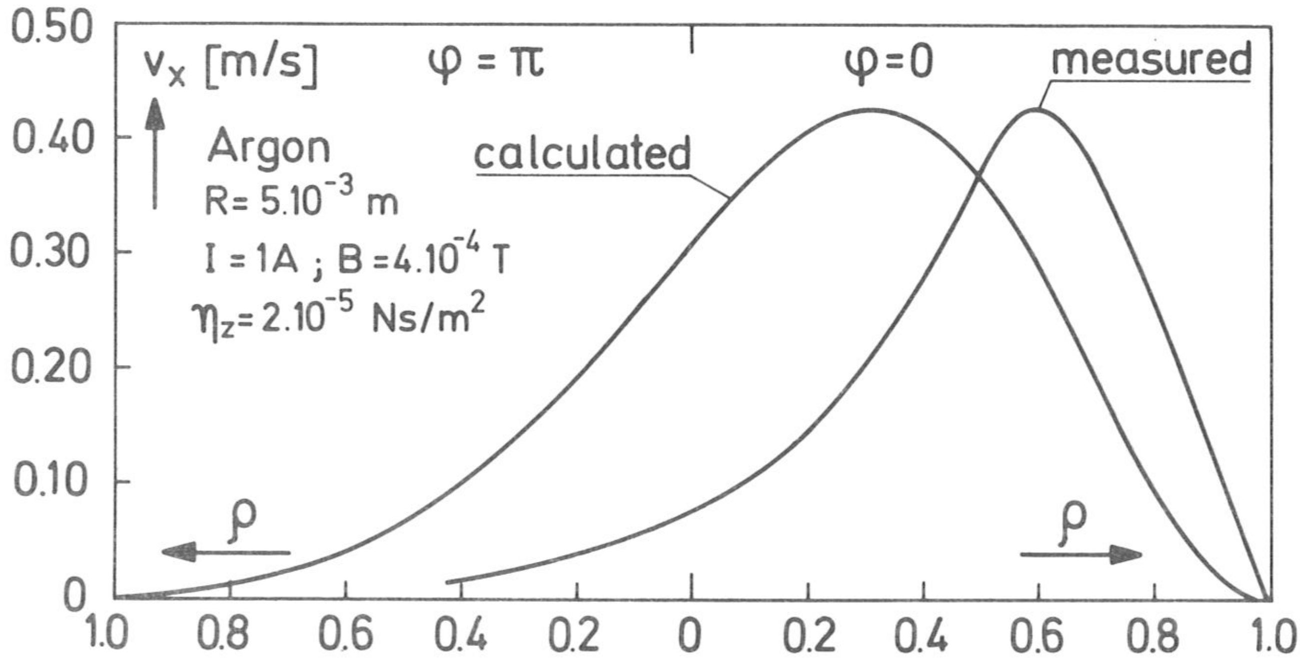

Fig.5 Distribution of the velocity V_x vs. ϱ along the axis of symmetry (x-axis)

curve can be assumed to be due to the inertia forces, the curl of which is as yet neglected in this theoretical treatment. Fig.6 finally shows characteristics of the arc, calculated from Eqs. (4o) and (4oa), i.e. plots of the electric field strength E versus the arc current J with the magnetic induction B as parameter. For a constant arc current, the electric field strength increases with increasing magnetic field. This increase of power input per unit length has to compensate the additional losses due to the viscous mass flow.

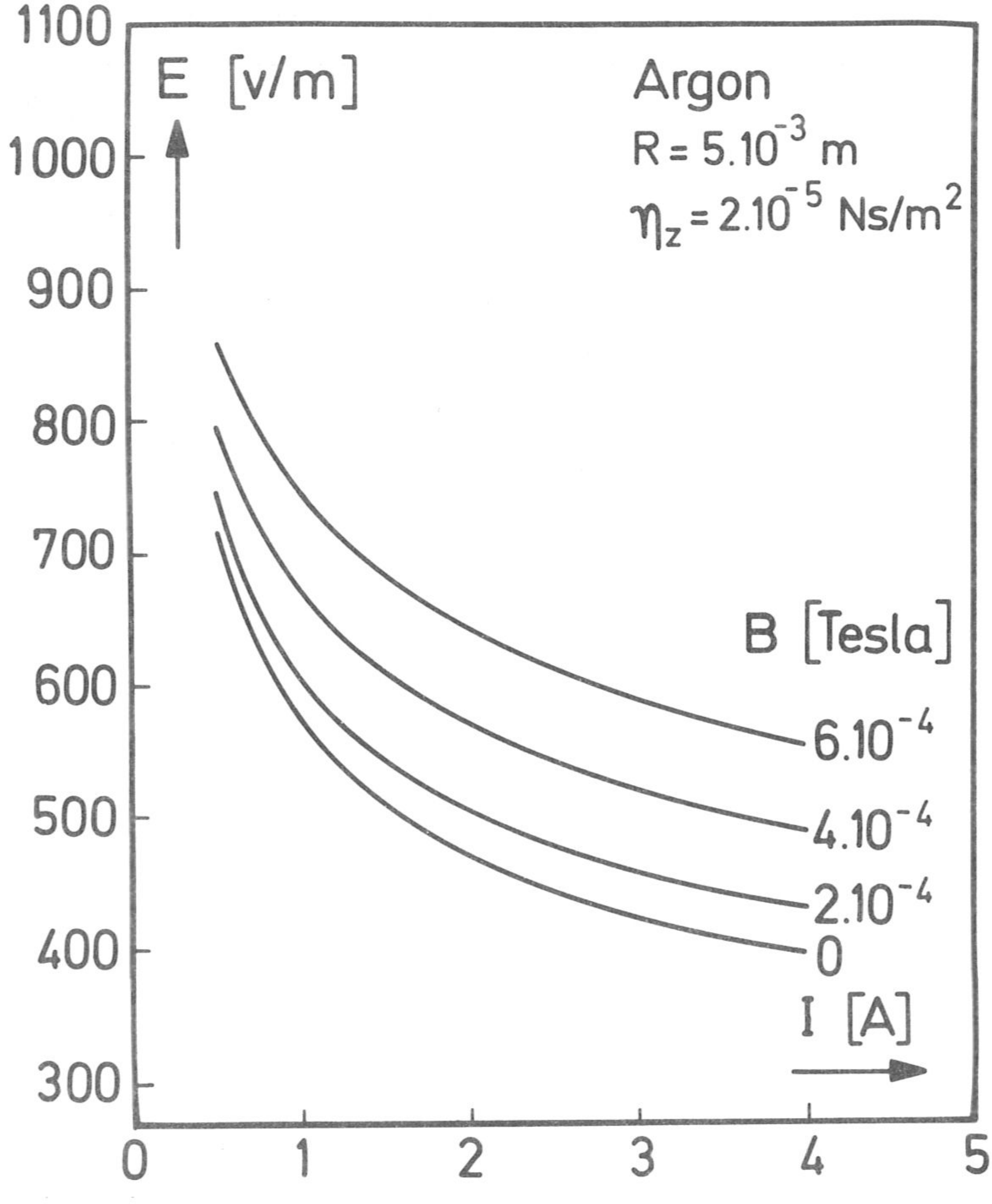

Fig.6 Arc characteristics E(J) for different values of the magnetic induction B

6. CONCLUDING REMARKS

In the case of a wall-stabilized magnetically deflected arc the advantage of Green's function method in treating the relevant equations and thereby deriving flow and temperature fields has been demonstrated. The calculated fields show good overall agreement with experimental measurements except in the region near the arc core, where inertia forces begin to play a dominant role. The influence of these inertia forces can, in principle, also be taken into account by the same method of Green's function. This leads to an additional term in the integral equation of the momentum balance. Such theoretical formulations are being carried out at the present time.

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REFERENCES

1 G. Seeger, Z. Angew. Phys. 25, 23 (1968)

2 G. Seeger, Z. Angew. Phys. 29, 357 (197o)

3 J. Raeder, Z. Naturforsch 23a, 423 (1968)

4 S. Bergman and M. Schiffer, Kernel Functions and Elliptic Differential Equations in Mathematical Physics, Academic Press Inc. Publ., New York, 1953 (p.239)

5 A. Ralston, H. Wilf, "Mathematical Methods for Digital Computers", John Wiley and Sons, New York, 196o, Chapter 24 (p.258)

6 R.W. Hamming, "Numerical Methods for Scientists and Engineers", McGraw Hill Book Company, Inc., New York, San Francisco, Toronto, London, 1962, Chapter 6 (p. 67)

7 G.E. Forsythe, Journal Society Industrial and Applied Mathematics 5, 74 (1957)

8 T. Goemans, "Curve Fitting Techniques", Report Nr THD-KR-2oo, Department of Mechanical Engineering, Delft University of Technology, June 1969

9 W. Kollmar, H. Rosenbauer and G. Seeger in "Proceedings of the 9. Int. Conf. on Phenomena in Ionized Gases", Contributed, Edited by G. Musa et al. (Inst. of Physics, Bukarest, Romania, 1969), p. 318

1o H. Rosenbauer, Z. Physik 245, 295 (1971)