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INJECTION OF RELATIVISTIC ELECTRON BEAMS WITH
ARBITRARY RADIAL PROFILES INTO WARM COLLISIONAL PLASMAS

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ABSTRACT

The results of previous authors for the injection of relativistic electron beams into unmagnetized collisional plasmas are generalized by allowing for arbitrary radial beam profiles and finite electron temperature. For $\nu t' \ll 1$ (ν = collision frequency of plasma electrons, t' = time after passing of the beam front) the magnetic field B depends strongly on the beam profile, smooth or sharp in units of the plasma skin depth c/ω_p . For $\nu t' \approx \nu t_D = (R\omega_p/c)^2 \gg 1$ (R = beam radius) the return current has diffused out of the beam and magnetic shielding is lost. For beam currents well above the Alfvén current the beam pinches before reaching t_D . For beams with currents from 1 to 100 times the Alfvén current and with $n'/n_e = 10^{-1}$ or 10^{-3} (n' = beam density, n_e = plasma density) critical times for which the beam is seriously affected or destroyed by pinching are calculated numerically. A parabolic and a step function profile are used for comparison and the critical times are found to be only weakly dependent on the profile. The return current heating time, which determines the maximum free path of the beam, is also calculated. Finite plasma temperature does not affect the magnetic field in this linearized theory. Its validity is estimated by evaluating critical times for the return current as well and comparing them with the critical beam times.

1. Introduction

Upon injection of an electron beam into a plasma a reverse current is induced in the plasma, which strongly reduces the beam's azimuthal magnetic field. Beam currents well above the Alfvén current may therefore be transported in plasmas without difficulty. HAMMER and ROSTOKER (1970) have shown that without dissipative effects the screening factor for the magnetic field is $s = R\omega_p/c \gg 1$ provided the plasma density is high compared with the beam density (R = beam radius, ω_p = plasma frequency of the background plasma). A step function profile was assumed by these authors for the radial beam current distribution. More recently it was shown by ROSINSKII, RUKHADZE, RUKHLIN and EPEL'BAUM (1972) and in detail by KÜPPERS, SALAT and WIMMEL (1973) that for more realistic, smooth beam profiles (gradient length \gg skin depth c/ω_p) screening is still considerably stronger, with s of order $(R\omega_p/c)^2$. At later times, with collisional effects, the return current diffuses out of the beam and the magnetic field increases. This process was investigated by RUKHADZE and RUKHLIN (1971) and, with inclusion of an external magnetic field by LEE and SUDAN (1971), LEE (1971) ROSINSKII and RUKHLIN (1973).

These investigations suffer from two restrictions. First, only a step-like radial profile of the beam was considered. This restricts one to a rather narrow class of situations. POUKEY and TOEPFER (1972) considered a parabolic profile, but the numerical solutions were obtained only for small beam currents and particular experimental conditions. Second, the investigations mentioned assume that the beams are rigid and parallel and that they disturb the plasma only slightly.

These two assumptions are satisfied initially owing to current screening; but both may break down for overcritical beams after, possibly short, times as soon as the magnetic field reaches certain critical values. Obviously, a nonlinear theory is needed in order to solve this problem exactly. Although this is not attempted here, we do get critical times for beam propagation and for propagation of the return current.

This paper is arranged as follows. In Section 2, with the assumption $n' \ll n_e$ (n' = beam density, n_e = plasma electron density) the equation of motion of the plasma is linearized. Although heating of the plasma is excluded as a nonlinear effects we allow for a finite plasma temperature. We consider a highly relativistic rigid beam of arbitrary radial profile, with a simple injection phase and a subsequent steady phase.

In Section 3 the azimuthal magnetic field is discussed.

A scaling law for the magnetic field which holds in the dissipation - controlled phase is derived. Numerical plots are presented which compare the magnetic field for two representative profiles, a step function profile and a parabolic profile as time evolves. These two profiles are also used in Section 4 for a set of plots in which six critical times are calculated and discussed for each profile. Each critical time corresponds either to violation of a criterion for regular beam propagation, e.g. the Alfvén criterion (ALFVÉN, 1939), or concerns the perturbation of the return current, or is generally of interest, such as the diffusion time of the return current. The electric field is discussed in Section 5 and final conclusions are presented in Section 6.

2. Induced charges, currents and fields

We consider a current $\underline{j}'(\underline{r}, t)$ of highly relativistic electrons injected into a fully ionized unbounded plasma with infinitely heavy ions. No external electric or magnetic fields are present. The induced charges, currents \underline{j}_e and fields are connected through Maxwell's equations

$$\begin{aligned} \text{curl } \underline{B} &= \frac{1}{c} \frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c} (\underline{j}_e + \underline{j}') \\ \text{curl } \underline{E} &= -\frac{1}{c} \frac{\partial \underline{B}}{\partial t} \end{aligned} \quad (1)$$

$$\text{div } \underline{E} = 4\pi\rho; \quad \text{div } \underline{B} = 0$$

together with the equation of motion for the plasma electrons

$$n_e m_e \frac{d\underline{v}_e}{dt} = q_e n_e (\underline{E} + \frac{1}{c} [\underline{v}_e \times \underline{B}]) - \nabla (n_e T_e) - \nu n_e m_e \underline{v}_e \quad (2)$$

the equation of continuity

$$\frac{\partial n_e}{\partial t} + \text{div} (n_e \underline{v}_e) = 0 \quad (3)$$

and the definitions

$$\underline{j}_e = q_e n_e \underline{v}_e; \quad \rho = q_e n_e + q_i n_i + q_e n' \quad (4)$$

where $q_e = -|e|$; n_e , v_e , T_e are plasma electron density, velocity and temperature, assumed to be non-relativistic, $n_i = \text{const}$ is the ion density, and ν is an effective collision frequency.

On the assumption that the beam is a small disturbance to the plasma the equations (1) - (4) may be linearized. If we go over to Fourier transformed quantities

$$A(\underline{r}, t) = \frac{1}{(2\pi)^4} \int d^3k d\omega e^{-i(\underline{k} \cdot \underline{r} - \omega t)} A(\underline{k}, \omega), \quad (5)$$

we have

$$-i [\underline{k} \times \underline{B}^1] = \frac{i\omega}{c} \underline{E}^1 + \frac{4\pi}{c} (\underline{j}_e^1 + \underline{j}'^1)$$

$$[\underline{k} \times \underline{E}^1] = \frac{\omega}{c} \underline{B}^1 \quad (6)$$

$$\underline{k} \cdot \underline{E}^1 = 4\pi i \rho^1; \quad \underline{k} \cdot \underline{B}^1 = 0$$

and

$$(i\omega + \nu^0) n_e^0 \frac{v_e^1}{v_e} = \frac{q_e}{m_e} n_e^0 \underline{E}^1 + i \underline{k} n_e^1 \frac{\alpha - 1}{\alpha} \frac{T_e^0}{m_e} = 0 \quad (7)$$

$$\omega n_e^1 - \underline{k} n_e^0 \frac{v_e^1}{v_e} = 0; \quad \underline{j}_e^1 = q_e n_e^0 \frac{v_e^1}{v_e}$$

where $n_e T_e^\alpha = \text{const}$ has been assumed. Equations (6) and (7) can easily be solved with the result

$$\underline{B}(\underline{k}, \omega) = -4\pi c \frac{i\omega + \nu}{(k^2 c^2 - \omega^2)(\omega - i\nu) + \omega \omega_p^2} [\underline{k} \times \underline{j}'(\underline{k}, \omega)] \quad (8)$$

$$[\underline{k} \times \underline{E}(\underline{k}, \omega)] = -4\pi \frac{\omega(i\omega + \nu)}{(k^2 c^2 - \omega^2)(\omega - i\nu) + \omega \omega_p^2} [\underline{k} \times \underline{j}] \quad (9)$$

$$\underline{k} \cdot \underline{E}(\underline{k}, \omega) = 4\pi i \rho(\underline{k}, \omega) = \frac{4\pi i}{\omega} \frac{\omega(\omega - i\nu) - k^2 v_{th}^2}{\omega(\omega - i\nu) - k^2 v_{th}^2 - \omega_p^2} \underline{k} \cdot \underline{j}' \quad (10)$$

$$[\underline{k} \times \underline{j}_e(\underline{k}, \omega)] = \frac{q_e^2 n_e^0}{m_e} \frac{1}{i\omega + \nu} [\underline{k} \times \underline{E}] \quad (11)$$

$$\underline{k} \cdot \underline{j}_e(\underline{k}, \omega) = \frac{q_e^2 n_e^0}{m_e} \frac{\omega}{(i\omega + \nu)\omega - ik^2 v_{th}^2} \underline{k} \cdot \underline{E} \quad (12)$$

where the index 1 has been dropped, $\nu \equiv \nu^0$, and

$$v_{th} = \left(\frac{\alpha - 1}{\alpha} \frac{T_e^0}{m_e} \right)^{1/2} \ll c \quad (13)$$

essentially is the thermal velocity of the plasma electrons.

For arbitrary radial and axial beam profiles $\underline{j}'(\underline{r}, t)$ the net currents, charges and fields may be found by Fourier inversion of equations (8) - (12). Equations (8)(9) and (10) show that as long as the linearization holds the magnetic field and the nonpotential part of the electric field are not affected by finite plasma electron temperature, while the space charges are.

Considerable simplification occurs for highly relativistic beams for which the velocity of propagation u , assumed to be in the z -direction,

may be put equal to c , the velocity of light, and if the beam quantities depend on time and the axial coordinate only in the combination $t - z/c$. In this case

$$\underline{j}'(\underline{k}, \omega) = 2\pi\delta(k_z - \frac{\omega}{c}) \underline{j}'(\underline{k}_\perp, \omega) \quad (14)$$

where $\underline{j}'(\underline{k}_\perp, \omega)$ is the two-dimensional Fourier transform with respect to the perpendicular coordinates x, y . The results of RUKHADZE and RUKHLIN(1971) show that the assumption $u=c$ is not essential for the validity of the results as long as $u \approx c$. We introduce cylindrical coordinates:

$$\underline{r} = (r \cos \phi, r \sin \phi, z); \quad \underline{k} = (k_\perp \cos \theta, k_\perp \sin \theta, k_z) \quad (15)$$

and consider axially symmetric beams only.

Then \underline{B} is in the ϕ -direction, while \underline{E} has components E_r and E_z .

In the following, for simplicity we confine ourselves to the study of the magnetic field $B \equiv B_\phi$ and the total space charge ρ in order to investigate current screening and charge screening. From equations (8) - (15) we get

$$B(\underline{k}, \omega) = 4\pi ic \ j'_z(\underline{k}, \omega) k_\perp \cos(\theta - \phi) \frac{\omega - i\nu}{(\omega - i\nu)k_\perp^2 c^2 + \omega\omega_p^2} \quad (16)$$

$$\rho(\underline{k}, \omega) = \frac{1}{c} \ j'_z(\underline{k}, \omega) \frac{\omega(\omega - i\nu) - k_\perp^2 v_{th}^2}{\omega(\omega - i\nu) - k_\perp^2 v_{th}^2 - \omega_p^2} \quad (17)$$

We choose a simple ansatz for the axial profile with an injection phase of time τ and a subsequent stationary phase:

$$j'_z(\underline{r}, t) = \begin{cases} j'_z(r) (1 - e^{-t'/\tau}); & t' > 0 \\ 0 & ; t' < 0 \end{cases}; \quad t' = t - z/c \quad (18)$$

Effects due to the rear end of the beam are not considered. Only a small overall damping $e^{-\epsilon t'}$, $\epsilon \rightarrow 0$, has to be added in order to make the Fourier transform converge. In Fourier space one gets

$$j'_z(\underline{k}, \omega) = \frac{2\pi}{i} \delta(k_z - \frac{\omega}{c}) j'_z(k_{\perp}) \left[\frac{1}{\omega} - \frac{1}{\omega - i/\tau} \right] \quad (19)$$

Through the δ -function B and ρ also depend on t and z in the combination $t' = t - z/c$ only. From equations (16), (17) and (19) one obtains

$$B(\underline{k}_{\perp}, t') = 4\pi i c j'_z(k_{\perp}) k_{\perp} \cos(\theta - \phi) \frac{1}{2\pi i} \int d\omega e^{i\omega t'} \cdot \frac{\omega - i\nu}{(\omega - i\nu) k_{\perp}^2 c^2 + \omega \omega_p^2} \left[\frac{1}{\omega} - \frac{1}{\omega - i/\tau} \right] \quad (20)$$

$$\rho(\underline{k}_{\perp}, t') = \frac{1}{c} j'_z(k_{\perp}) \frac{1}{2\pi i} \int d\omega e^{i\omega t'} \cdot \frac{\omega(\omega - i\nu) - k_{\perp}^2 v_{th}^2}{\omega(\omega - i\nu) - k_{\perp}^2 v_{th}^2 - \omega_p^2} \left[\frac{1}{\omega} - \frac{1}{\omega - i/\tau} \right] \quad (21)$$

The integrals may be done with the help of the residues at $\omega = 0; i/\tau; i\sqrt{k_1^2/(k_1^2+a^2)}$; $i\omega/2 \pm f$ and the result for $t' > 0$ is

$$B(\underline{k}_1, t') = \frac{4\pi i}{c} k_1 \cos(\theta-\phi) j'_z(k_1) \cdot \left\{ \frac{1}{k_1^2} - \frac{1-v\tau}{k_1^2(1-v\tau) + a^2} e^{-t'/\tau} - \frac{a^2}{k_1^2} \frac{1}{k_1^2(1-v\tau)+a^2} e^{-\frac{k_1^2 v t'}{k_1^2+a^2}} \right\} \quad (22)$$

$$\rho(\underline{k}_1, t') = \frac{1}{c} j'_z(k_1) \left\{ \frac{k_1^2 v_{th}^2}{k_1^2 v_{th}^2 + \omega_p^2} - \frac{1-v\tau+k_1^2 v_{th}^2 \tau^2}{1-v\tau+(k_1^2 v_{th}^2 + \omega_p^2) \tau^2} e^{-t'/\tau} \right. \\ \left. + \frac{\tau}{f} \frac{\omega_p^2}{k_1^2 v_{th}^2 + \omega_p^2} \frac{1}{1-v\tau+(k_1^2 v_{th}^2 + \omega_p^2) \tau^2} e^{-v t'/2} \cdot S(t') \right\} \quad (23)$$

with

$$S(t') = \left(f^2 - \frac{v^2}{4} + \frac{v}{2\tau} \right) \sin(f t') - \left(v - \frac{1}{\tau} \right) f \cos(f t') \quad (24)$$

and

$$a = \omega_p/c; \quad f^2 = k_1^2 v_{th}^2 + \omega_p^2 - v^2/4 \quad (25)$$

3. The magnetic field

Without collisions, $\nu = 0$, we obtain

$$B(\underline{k}_\perp, t'; \nu = 0) = \frac{4\pi i}{c} k_\perp \cos(\theta - \phi) j'_z(k_\perp) \frac{1}{k_\perp^2 + a^2} (1 - e^{-t'/\tau}) \quad (26).$$

Because of axial symmetry we find

$$\begin{aligned} B(r, t'; \nu=0) &= \frac{2}{c} \int_0^\infty dk_\perp \frac{k_\perp^2}{k_\perp^2 + a^2} J_1(k_\perp r) j'_z(k_\perp) \cdot (1 - e^{-t'/\tau}) \\ &= B(r) \cdot (1 - e^{-t'/\tau}) \end{aligned} \quad (27)$$

For a step-like profile of the beam,

$$j'_z(r) = \begin{cases} j'_1; & r < R \\ 0; & r > R \end{cases}; \quad j'_z(k_\perp) = j'_1 \frac{2\pi R}{k_\perp} J_1(k_\perp R) \quad (28)$$

one easily recovers the result (HAMMER and ROSTOKER, 1970)

$$B(r, \nu=0) = \frac{4\pi j'_1}{\omega_p} X_1 \begin{cases} K_1(X_1) I_1(X); & X \leq X_1 \\ I_1(X_1) K_1(X); & X \geq X_1 \end{cases} \quad (29)$$

where $X = r\omega_p/c$, $X_1 = R\omega_p/c$. For a parabolic profile,

$$j'_z(r) = \begin{cases} j'_1 \cdot (1 - r^2/R^2); & r < R \\ 0 & ; r > R \end{cases}; \quad j'_z(k_\perp) = j'_1 \frac{4\pi}{k_\perp^2} J_2(k_\perp R), \quad (30)$$

the Fourier integral of equation (27) gives (GRADSHTEYN and RYZHIK, 1965)

$$B(\mathbf{r}; \nu=0) = \frac{8\pi j_1'}{\omega_p} \frac{1}{X_1} \left\{ \begin{array}{ll} X/X_1 - X_1 K_2(X_1) I_1(X); & X \leq X_1 \\ X_1 I_2(X_1) K_1(X); & X \geq X_1 \end{array} \right\} \quad (31)$$

which agrees with KÜPPERS, SALAT and WIMMEL (1973).

The fields (29 and (31) are listed for later reference.

For weak collisions, $\nu t' \ll 1$, the effect of collisions is small (see Appendix A). Hence, in the following we only consider times t' such that $\nu t' \gg 1$. Also, we separately consider the build-up phase of the beam, $t' \ll \tau$, and stationary phase $t' \gg \tau$.

In the build-up phase, $t' \ll \tau$, it holds that $\nu t' \gg 1$, which may be used to simplify B, equation (22):

$$B(\underline{k}_1, t') = \frac{4\pi i}{c} k_1 \cos(\theta - \phi) j_z'(k_1) \frac{1}{k_1^2} \cdot \left[1 - \frac{1}{y - t'/\tau} (y e^{-t'/\tau} - \frac{t'}{\tau} e^{-y}) \right]$$

with $y \equiv k_1^2 \nu t' / a^2$. For $t'/\tau \ll 1$ equation (32) approximately reduces to

$$B(\underline{k}_1, t') = \frac{4\pi i}{c} \cos(\theta - \phi) j_z'(k_1) \frac{1}{k_1} \left[1 - \frac{1}{y} (1 - e^{-y}) \right] \cdot \frac{t'}{\tau} \quad (33)$$

In the stationary beam phase, $t' \gg \tau$, the second term of equation (22) may be dropped. (The singularity that occurs at $k_1^2(1 - \nu\tau) + a^2 = 0$ for $\nu\tau > 1$ disappears when the third term is taken into account). The third term may be simplified by noting that only such k_1^2 contribute effectively for which $k_1^2 \lesssim a^2/(\nu t' - 1) \ll a^2/(\nu\tau - 1)$. As a result

$$B(\underline{k}_1, t') = \frac{4\pi i}{c} \cos(\theta - \phi) j_z'(k_1) \frac{1}{k_1} (1 - e^{-y}) \quad (34)$$

The interpretation of the equation (33) and (34) is obvious:

The first term is the magnetic field that would be produced by the beam j_z' alone. This follows from equation (22) for example, with the plasma frequency $\omega_p = ac$ put equal to zero. Consequently, the second term of equations (33) and (34) is the contribution B^P of the plasma return current to the magnetic field. In the steady beam phase this contribution is particularly simple. The exponential $\exp(-k_1^2 \nu t'/a^2)$ shows that B^P is determined by the diffusion equation

$$\nabla^2 B^P = \frac{a^2}{\nu} \frac{\partial B^P}{\partial t} \quad (35)$$

Hence for times $t' \gg t_D$ (LOVELACE and SUDAN, 1971), where

$$\nu t_D = \left(\frac{R\omega_p}{c} \right)^2, \quad (36)$$

and R is the radius of the beam, the return current inside the beam has diffused away and screening inside the beam is lost.

Comparison of equations (33) and (34) shows that in the build-up phase the process of field diffusion is slightly different. The function $f(y) \equiv y^{-1} [1 - \exp(-y)]$ decays more slowly for large $y = k_1^2 \nu t'/a^2$ than does $\exp(-y)$. An e -fold decrease of $f(y)$ from $f(y=0)=1$ occurs only at $y \approx 2.5$ instead of $y = 1$. Hence the

diffusion of the return current is slowed down by a factor of approximately 2.5 during the build-up phase. Since apart from this factor there is no great difference to the stationary beam phase, we will not consider the short initial phase any more. In the steady beam phase we get from equation (34)

$$B(r, t') = \frac{2}{c} \int_0^{\infty} dk_{\perp} J_1(k_{\perp} r) j'_z(k_{\perp}) (1 - e^{-k_{\perp}^2 / \rho^2}) \quad (37)$$

where

$$\rho^2 = \frac{a^2}{vt'} = \frac{\omega^2 p}{c^2 vt'} \quad (38)$$

By Fourier inversion of $j'_z(k_{\perp})$ one obtains

$$B(r, t') = \frac{4\pi}{c} \left[\frac{1}{r} \int_0^r dr' r' j'_z(r') - \int_0^{\infty} dr' r' j'_z(r') \cdot \rho \int_0^{\infty} dk J_0(kr' \rho) J_1(kr \rho) e^{-k^2} \right] \quad (39)$$

Another useful representation is obtained by Fourier inversion of $\exp(-k^2)$ and subsequent integration over k (GRADSHTEYN and RYZHIK, 1965):

$$B(r, t') = \frac{4\pi}{c} \frac{1}{r} \left[\int_0^r dr' r' j'_z(r') e^{-(r-r')^2 \rho^2 / 4} - \frac{\rho^2}{2\pi} \int_0^{\infty} dr' r' j'_z(r') \int_{|r-r'|}^{r+r'} dr'' r'' e^{-r''^2 \rho^2 / 4} \arccos \frac{r'^2 + r''^2 - r^2}{2r'r''} \right] \quad (40)$$

Equation (40) shows that the normalized B, apart from the beam profile, depends only on the two variables r/R and q ,

$$q^2 = \frac{R^2 \rho^2}{4} = \frac{R^2 \omega^2}{4c^2 \nu t'} \quad (41)$$

From equation (40) it also follows that asymptotically for large radii the magnetic field goes to zero as

$$B(r, t') \cong 0 \left[e^{-(r-R)^2 \rho^2 / 4} \right] \quad (42)$$

where factors $(r/R)^\alpha$, $\alpha = \text{const}$, have been neglected. The fact that the magnetic field decays more strongly than r^{-1} is important. It proves that the total net current remains zero in spite of dissipative effects acting on the reverse current. This is possible because additional return currents are induced outside the beam. Since the radial e-folding length of B, equation (42), is $\rho^{-1} = \sqrt{\nu t'} \quad c/\omega_p$ it follows that the average radius of the return current grows proportionally to $(\nu t')^{1/2}$.

Obviously, if the plasma is confined in a cavity of radius $r_c \geq R$ the return current distribution will be modified by boundary effects if $\rho^{-1}(t') > r_c$, and the compensation will be lost. This effect has been investigated by, for example, POUKEY and TOEPFER (1972).

Inside the beam, at times $t' \gg t_D$ when we expect the return current to have decayed to a small fraction we find from $r\rho < R\rho \ll 1$ and the small argument expansion of the Bessel function in equation (39)

$$B(r, t') \cong \frac{2}{cr} \left[I'(r) - I'(R) r^2 \rho^2 / 4 \right] \quad (43)$$

where $I'(r)$ is the beam current at radius r ,

$$I'(r) = 2\pi \int_0^r dr' r' j_z'(r'). \quad (44)$$

This shows that the residual return current is only a fraction of order $R^2 \rho^2 \sim (vt')^{-1}$ of the beam current.

We calculated numerically the magnetic field distribution from equation (40) as a function of r/R and the dimensionless time $q^{-2} = 4c^2 vt' / (R\omega_p)^2$ both for a step-like radial profile of the beam current, equation (28), and a parabolic profile, equation (30). Figure 1 shows the early phases of diffusion, $t' \ll t_D$, together with the analytic curves for $vt'=0$ taken from equations (29) and (31), respectively, with the arbitrary choice $R\omega_p/c = 25$, corresponding to $q = 12,5$ for $vt'=1$. The differences in the amplitude and radial distribution of B between both profiles are very pronounced. (Normalization is with respect to equal currents $I'=I'(R)$ in both profiles.) At later times, see Figure 2, the dependence on the profile is much weaker. At the diffusion time t_D , $q = 0.5$, the return current inside the beam has almost vanished so that the magnetic field approximately agrees with the relevant beam vacuum fields. Outside the beam however, the return current is strong, and B decreases much faster than r^{-1} .

4. Beam criteria and return current criteria

We made use of the magnetic field as given by equation (40) for the numerical evaluation of several criteria relevant to the propagation of overcritical electron beams in plasmas. Again, both a step function profile of the beam current and a parabolic profile were considered. The beam currents I' were taken to range from one up to one hundred times the Alfvén current $I_A = m_e c^3 \gamma' / e$. Under these conditions the magnetic field initially screened by the plasma reverse current, grows by diffusion of the return current until at some critical time $t' = t - z/c$ it is strong enough to modify appreciably the propagation of the beam itself or the propagation of the return current. Six characteristic times which were discussed in detail in KÜPPERS, SALAT and WIMMEL, 1973a were considered:

- a) The "Alfvén time" t_A . At $t' = t_A$ the gyroradius R'_e of the beam electrons in the azimuthal magnetic field has become equal to the radial (half-) width δr of the magnetic field region, and parallel motion of the beam is perturbed.
- b) The "pinch time" t_p , which is defined by the virial condition (KÜPPERS, SALAT and WIMMEL, 1973a)

$$I^2 = 2I_A I', \quad (45)$$

where $I = I(R)$ is the net current at $r = R$.

For $t > t_p$ the magnetic pressure is larger than the maximum possible transverse beam pressure $p'_{\max} = n' m_e c^2 \gamma'$. The criteria a) and b) are

equivalent for beams with smooth radial profiles in vacuum, but not for beams in plasmas. $t' > t_p$ leads to catastrophic pinching, while for $t_p > t' > t_A$ propagation is still possible, in principle, if the beam manages to change to a new self-consistent structure.

- c) The "diffusion time" t_D , equation (36). For $t' > t_D$ the magnetic field in the beam region is almost constant (up to 20 %, see Figure 2).
- d) The "heating time" t_H (LOVELACE and SUDAN, 1971). While all other criteria give critical times $t' = t - z/c$, t_H actually corresponds to a critical length $L_H = ct_H$. L_H is the length after which the beam has lost all its kinetic energy by dissipation of the return current.
- e) The "gyro time" t_G , the duration of the first gyroperiod of the plasma electrons. It may be used as a lower limit to the time when the nonlinear magnetic force on the return current becomes important.
- f) The "Lee time" t_L , defined by $\Omega_e = \omega_p$. LEE and SUDAN (1971) have shown that an external magnetic field \underline{B}_0 does not interfere with the return current propagation provided $\Omega_{e0} < \omega_p$ because plasma electrons set up electric fields which compensate the $[\underline{v}_e \times \underline{B}_0]/c$ term in the equation of motion. When finite ion mass is included (LEE, 1971) however, ion drift in a plane geometry leads to an additional rapid deterioration of the return current. The relevance of these results to propagation of a cylindrical beam in the self-consistent inhomogeneous field B is unknown. We have included t_L , which may be useful as an upper limit to the time at which magnetic field effects on the return current flow become crucial.

The criteria a)-f) were evaluated with I'/I_A and $n'(r=0)/n_e$ as free parameters (see Figure 3). Two values of n'/n_e were considered, $n'/n_e = 10^{-3}$ (see Figures 3a and 3b, and $n'/n_e = 10^{-1}$ (see Figures 3c and 3d).

The procedure to determine the critical times was as follows. From the definition of $q^2 = (R \omega_p / 2c)^2 / (\nu t')$ it holds that

$$\nu t' = \frac{b\gamma'}{q^2} \cdot \frac{I'}{I_A} \cdot \frac{n_e}{n'(r=0)} \quad (46)$$

where $b = 2$ for the parabolic and $b = 1$ for the step function profile. All criteria except criterion e) may be expressed in terms of q directly or in terms of the normalized magnetic field, which again is a function of q (and the radius) only, and the parameters I'/I_A and $n'(0)/n_e$. Then, the q_{critical} were determined numerically and inserted in equation (46). For the gyroperiod t_G the parameter ω_p/ν has to be fixed independently, and $\omega_p/\nu = 100$ was chosen somewhat arbitrarily (see GUILLORY and BENFORD, 1972). It is easy to see that while t_G alone increases νt_G decreases with increasing ω_p/ν . The characteristic parameter $R\omega_p/c$ is determined by

$$\frac{R\omega_p}{c} = \left(b\gamma' \frac{I'}{I_A} \frac{n_e}{n'(0)} \right)^{1/2}. \quad (47)$$

From the graphs (Figure 3) we get the following results:

The Alfvén time t_A is well below the diffusion time t_D as soon as the beam current is appreciably larger than the Alfvén current. From this one might expect large differences between the parabola and the step function profile. This, however, does not occur. The two t_A never disagree by more than a factor of two. The reason for this is the following. The step function profile indeed needs less time to reach a certain magnetic field level; additional time is used, however, to quench the gyroradius to the small value δr of the B region, typical of this profile.

For very large beam currents one would not take the violation of the Alfvén criterion for the step function profile too seriously since it occurs in a very thin sheath only. Figures 3b) and 3d), however, show that in this case the pinch time t_P is not too wide of the Alfvén time, any way. The pinch times for the parabolic profile (Figures 3a) and 3c)) are consistently somewhat higher than for the step-like profile. The ratio of the two pinch times, which grows with increasing current, never exceeds a factor of three, however. For small beam currents the pinch time curves run into the diffusion time curves, which indicates that catastrophic pinching no longer occurs. Of course, the criteria that have been used are based on simplifications so that factors of order one should be allowed for all critical times.

For the step function profile the magnetic field maximum and the half-width δr grow in time in a particularly simple manner (roughly in proportion to q^{-1}) for times far from the diffusion time. This may be used to derive

simple analytic expressions for the Alfvén time and the pinch time; see Appendix B.

The heating time for the parabolic profiles lies between the Alfvén time and the pinch time, while for step-like profiles the Alfvén and the pinch time are both larger than the heating time.

The smallest of all times plotted turns out to be the gyroperiod t_G , with νt_G of order one. For later times this indeed implies the existence of a radial electric field to compensate for the nonlinear magnetic forces on the plasma current. Figures 3a and 3b show that the compensation could work for low-current beams or for sufficiently small values of n'/n_e where the Lee time t_L is not reached before the magnetic diffusion time. For $n'/n = 10^{-1}$, however, t_L is of the order of the pinch time t_P , so that the return current is influenced by the magnetic field certainly not later than the beam pinches.

Instead of the absolute value of the critical times t_c we plotted the products νt_c , which do not depend on the collision frequency ν any more (except for the gyroperiod). The effective collision frequency will be an anomalous one if the return current velocity exceeds a critical value; see, for example, GUILLORY and BENFORD (1972) and GUILLORY (1972).

5. The electric field

Electron beams injected into fully ionized plasma are initially charge neutralized very effectively since owing to the induction of the reverse current almost as many plasma electrons are thrown out of the common volume of the beam and plasma as are replaced by beam electrons. Section 3 shows that the global balance between inflow and outflow is also maintained when collisions modify the return current distribution. Since dissipative effects act only on currents, but not on a static distribution of charges, the role of collisions should be different for current neutralization and charge neutralization, respectively. This is borne out in equation (23) for the charge density ρ : Plasma oscillations which are induced in the beam front are damped by collisions for $v t' \gg 1$, the radial profile of ρ is modified somewhat (second term) by collisions, but there is no diffusion-like behaviour as in the corresponding equation (22) for B.

If we neglect plasma oscillations whose dominant modes $k_{\perp} \cong \omega_p/v_{th}$ are of order $(\omega_p \tau)^{-1} \ll 1$ compared with the non-oscillating part, the charge density is given by

$$\rho(\underline{k}_{\perp}, t') = \frac{1}{c} j'_z(\underline{k}_{\perp}) \left[\frac{k_{\perp}^2 v_{th}^2}{k_{\perp}^2 v_{th}^2 + \omega_p^2} - \frac{1 - \nu \tau + k_{\perp}^2 v_{th}^2 \tau^2}{1 - \nu \tau + (k_{\perp}^2 v_{th}^2 + \omega_p^2) \tau^2} e^{-t'/\tau} \right] \quad (48)$$

For a cold plasma, $v_{th}=0$, it follows that ρ is of the form $\rho(\underline{k}_{\perp}, t') = f(t') \cdot j'_z(\underline{k}_{\perp})$ so that the normalized radial profiles of ρ and the beam current are the same and the charge density completely disappears in the steady beam phase $t' \gg \tau$. In the build-up phase the charge density is non-zero owing to the finite inertia of the plasma electrons, which causes a phase shift between inflow and outflow of electrons.

For a warm plasma space charges exist even in the steady state because electric fields may be balanced by a pressure gradient in the plasma.

For $t' \gg \tau$ one gets

$$\rho(r) = \frac{1}{2\pi c} \int_0^{\infty} dk k J_0(kr) j_z'(k) \frac{k^2}{k^2 + b^2} \quad (49)$$

where $b^{-1} = v_{th}/\omega_p$ is the plasma electron Debye length.

From $\text{curl } \underline{B} = 4\pi \underline{j}/c$ and equation (27) for the magnetic field in the collisionless case it follows that

$$\rho(r) = \frac{1}{c} j_z(r; v=0, a \rightarrow b) \quad (50)$$

where the symbol $j_z(r; v=0, a \rightarrow b)$ stands for the current density in the collisionless case with the skin depth a^{-1} replaced by the Debye length b^{-1} . From the symmetries involved and Poisson's equation it follows for the radial electric field E_r

$$E_r(r) = B(r; v = 0, a \rightarrow b). \quad (51)$$

With previous results for B , equation (29) and (31), this means that the electric field is smaller than the initial ($vt'=0$) magnetic field by a factor of order v_{th}/c or $(v_{th}/c)^2$ for the step-function beam profile or a smooth profile, respectively. On the other hand, it has been shown in the previous section that additional radial electric

field have to be set up in the plasma if the return current is to overcome the $[\underline{v}_e \times \underline{B}]$ force. This electric field is obviously of order $(v_{ez}/c)B \cong (n'/n_e)B$ and, depending on the value of $n'/n_e \ll 1$, may well be larger than that due to finite plasma temperature.

6. Summary and conclusions

The evolution of the magnetic field and the space charge of a relativistic electron beam injected into a warm plasma with collisions was investigated for arbitrary radial beam profiles. Initially, there is a large difference in the magnetic field for smooth and sharp beam profiles, compared at equal beam currents. Finite plasma temperature does not affect the magnetic field, but affects the electric field, in this linearized injection theory. The electric screening is more effective than the magnetic screening and persists in time. Collisions have almost no influence upon charge screening.

For overcritical beams, $I' > I_A$ (I_A = Alfvén current), the magnetic field, by diffusion of the return current, becomes strong enough at later times to interfere with a regular propagation of the beam and to influence the return current. The pertinent critical times (KÜPPERS, SALAT and WIMMEL, 1973a) were calculated numerically for a set of beam and plasma parameters. It was found that differences in the beam profile exert only a limited influence on the values of the critical times. The critical times for the beam, i.e. the Alfvén time and the pinch time, are in general well below the magnetic diffusion time; for step-like profiles of the beam they do not differ much from each other. The heating time for parabolic beam profiles lies between the (smaller) Alfvén time and the (larger) pinch time, while for step-like profiles it lies below both times.

The influence of the magnetic field upon the return current was estimated by calculating the gyroperiod of the plasma electrons and the LEE time. It turns out that at the critical beam times an additional radial electric field is needed in order to compensate for the nonlinear magnetic forces on the return current. Such a compensation could happen for sufficiently small values of n'/n_e or I'/I_A such that the LEE time is not reached before the magnetic diffusion time. For $n'/n_e = 10^{-1}$, however, the LEE time is of the order of the pinch time of the beam, and the results of the linear injection theory will be insufficient. Actually, according to LEE(1971) the return current due to ion drift could be quenched by the magnetic field even at times earlier than the LEE time. However, it is not clear to what extent LEE's results can be applied to the present case. A definite answer to these questions can only be obtained from a nonlinear theory of beam injection.

Appendix A: $v t' \ll 1$

Consider equation (22) for the magnetic field. In the build-up stage of the beam, $t' \ll \tau$, the exponentials in both the second and the third term may be expanded. If we keep the first two terms of each expansion, this leads after several cancellations to

$$B(\underline{k}_1, t') = \frac{4\pi i}{c} k_1 \cos(\theta - \phi) j'_z(k_1) \frac{1}{k_1^2 + a^2} \cdot \frac{t'}{\tau} \quad (A1)$$

which is identical to the collisionless case, equation (26), with $t' \ll \tau$.

In the steady phase, $t' \gg \tau$, the second term of equation (22) may be neglected, while the third term may still be expanded, so that

$$B(\underline{k}_1, t') = \frac{4\pi i}{c} k_1 \cos(\theta - \phi) j'_z(k_1) \left\{ \frac{1}{k_1^2 + \tilde{a}^2} + \left(\frac{1}{k_1^2 + a^2} - \frac{1}{k_1^2 + \tilde{a}^2} \right) \cdot \frac{t'}{\tau} \right\} \quad (\text{A2})$$

where

$$\tilde{a}^2 = \frac{a^2}{1 - v\tau} ; \quad a^2 = \frac{\omega_p^2}{c^2} ; \quad \frac{\tilde{\omega}_p^2}{p} = \frac{\omega_p^2}{1 - v\tau} \quad (\text{A3})$$

From $t' \gg \tau$, $v\tau' \ll 1$ it follows that $v\tau \ll 1$, so that equation (A2) may be expanded in powers of $v\tau$ and by comparison with equation (26) one finds

$$B(r, t') \cong B(r, t'; v=0) - \frac{\omega_p}{2} \cdot \frac{\partial B(r, t', v=0)}{\partial \omega_p} \cdot v\tau' \quad (\text{A4})$$

where $B(r, t', v=0)$ is given by equation (27) and is a smooth function of ω_p .

In conclusion, for times t' small compared with a mean collision time collisional effects are either negligible or small.

Appendix B

From RUKHADZE and RUKHLIN (1971) it follows that for the step function profile the maximum of B is at $r = R$ and for $t' \ll t_D$ is given by

$$B = \frac{2I'}{\sqrt{\pi} cRq} , \quad (\text{B1})$$

with $q = R\omega_p / (c\sqrt{vt^2})$. This agrees with our numerical results, which, moreover, approximately yield for the half-width δr :

$$\delta r \cong \frac{R}{3q} . \quad (B2)$$

Inserting this into the Alfvén condition and equation (45) yields

$$vt_A \cong \frac{3\sqrt{\pi}}{2} \frac{\gamma' n_e}{n'} ; \quad vt_p \cong 2\pi \frac{\gamma' n_e}{n'} . \quad (B3)$$

Figures 3b) and 3d) show that these are indeed good approximations if t_A, t_p are well below t_D .

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Figure captions:

Figure 1: Evolution of the radial magnetic field distribution at early times $t' = t - z/c$. Curves with maximum at $r/R = 1$ correspond to a beam with radial step function profile, other curves to parabolic beam profile. Times are the same for each pair of profiles.

$B_0 = 4I'/(cR)$, $q = R\omega_p/(2c\sqrt{\nu t'})$. I' = beam current, R = beam radius, ν = collision frequency. For the analytic initial values $R\omega_p/c = 25$ was specified arbitrarily.

Figure 2: The magnetic field at later times. Also shown is the asymptotic state with no return current.

Figure 3: Critical times versus beam current for beams with parabolic radial profile - Figures a), c) - and beams with step function profile - Figures b), d) - for density ratios $n'/n_e = 10^{-3}$ and 10^{-1} . t_D = diffusion time, t_P = pinch time, t_L = Lee time, t_A = Alfvén time, t_H = heating time, t_G = gyro time. For t_L not shown the Lee criterion is not violated.

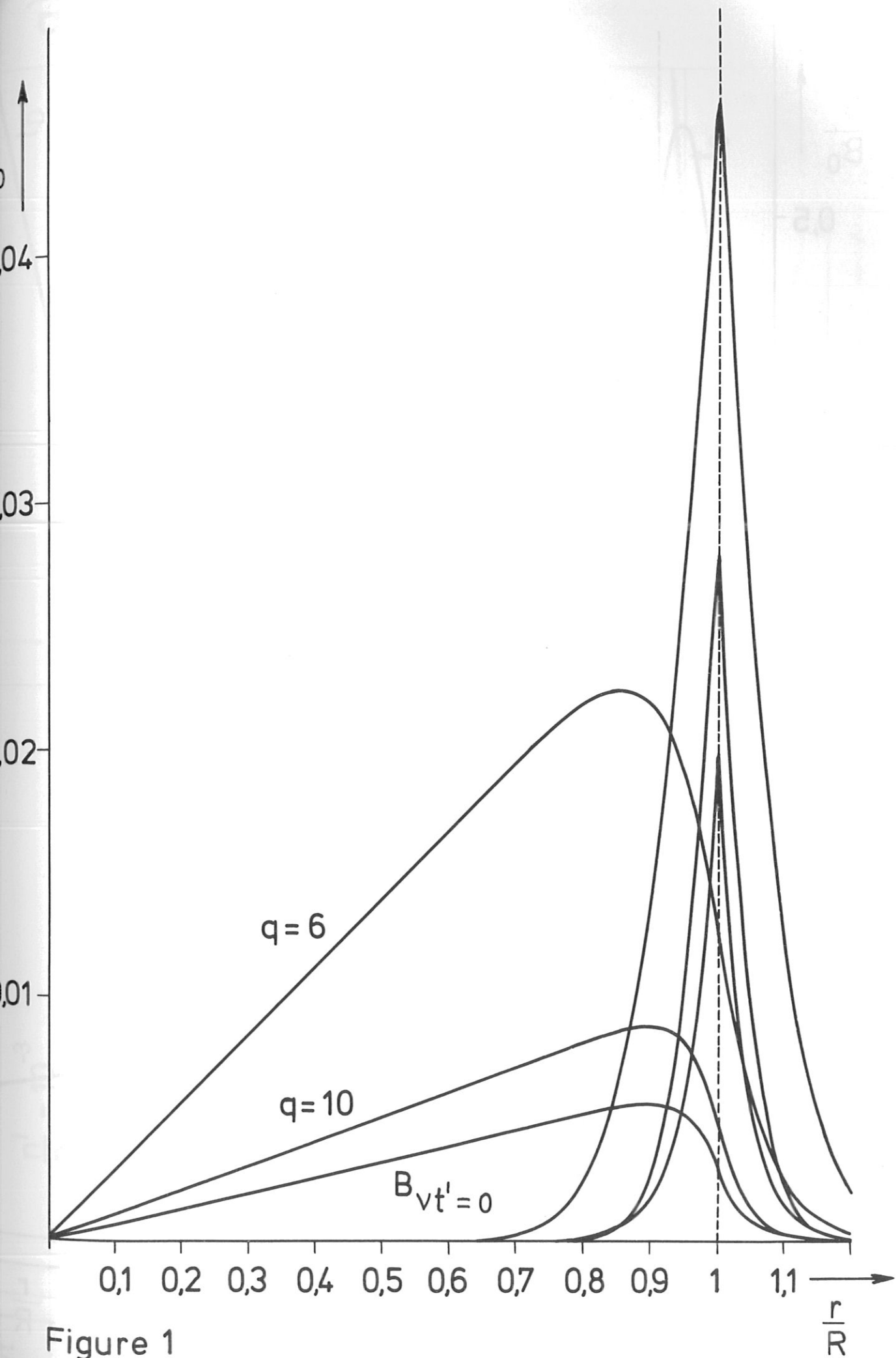


Figure 1

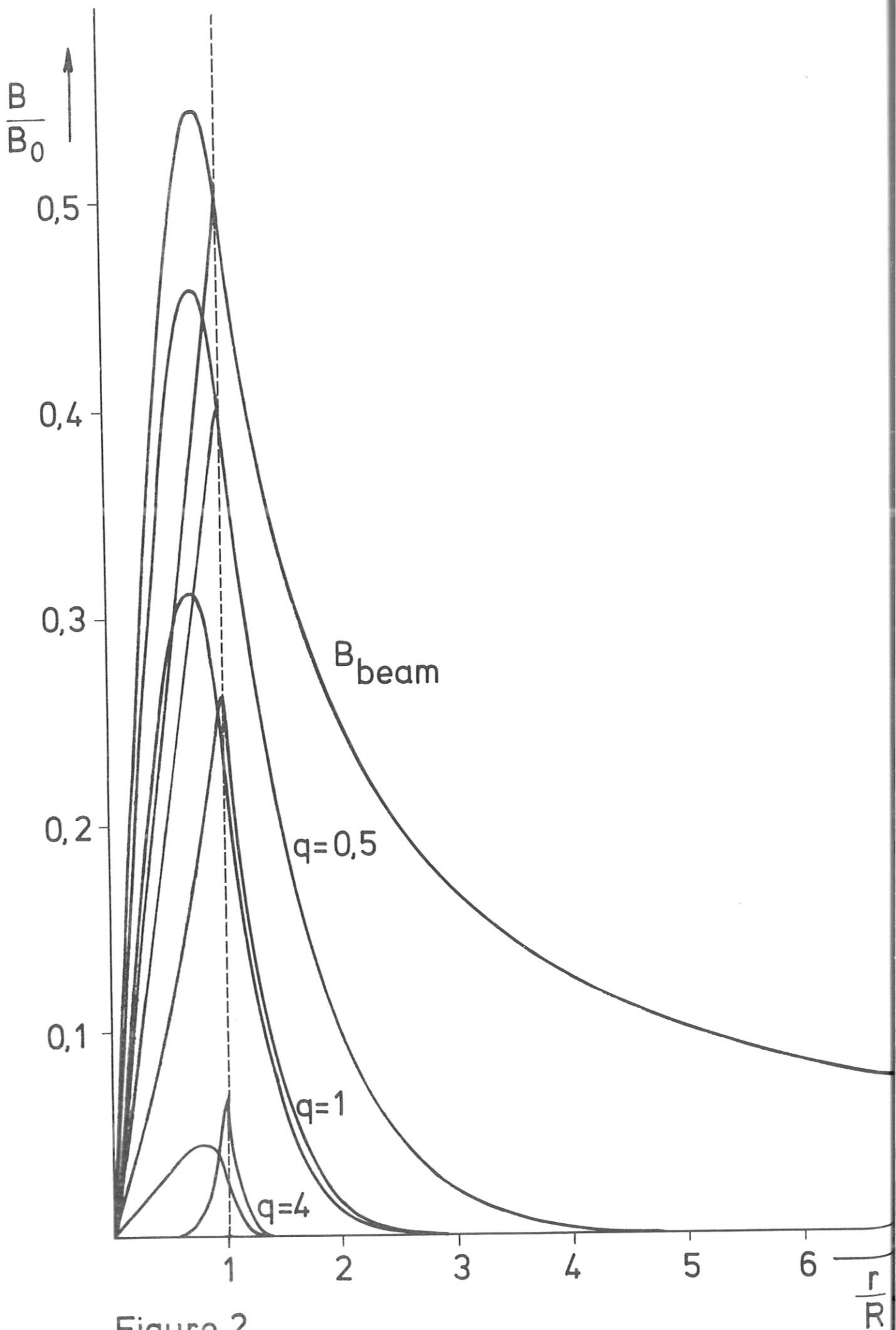
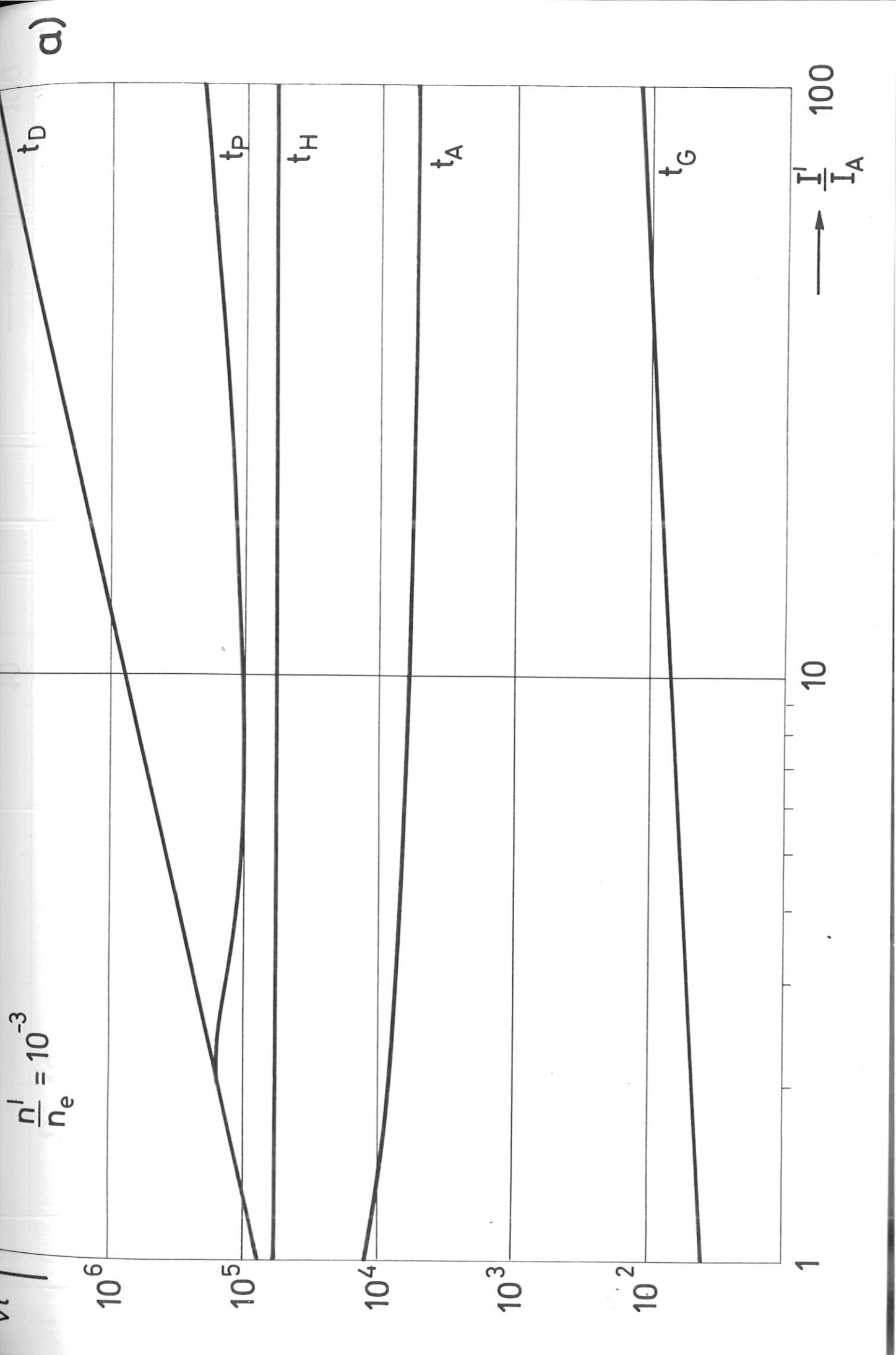
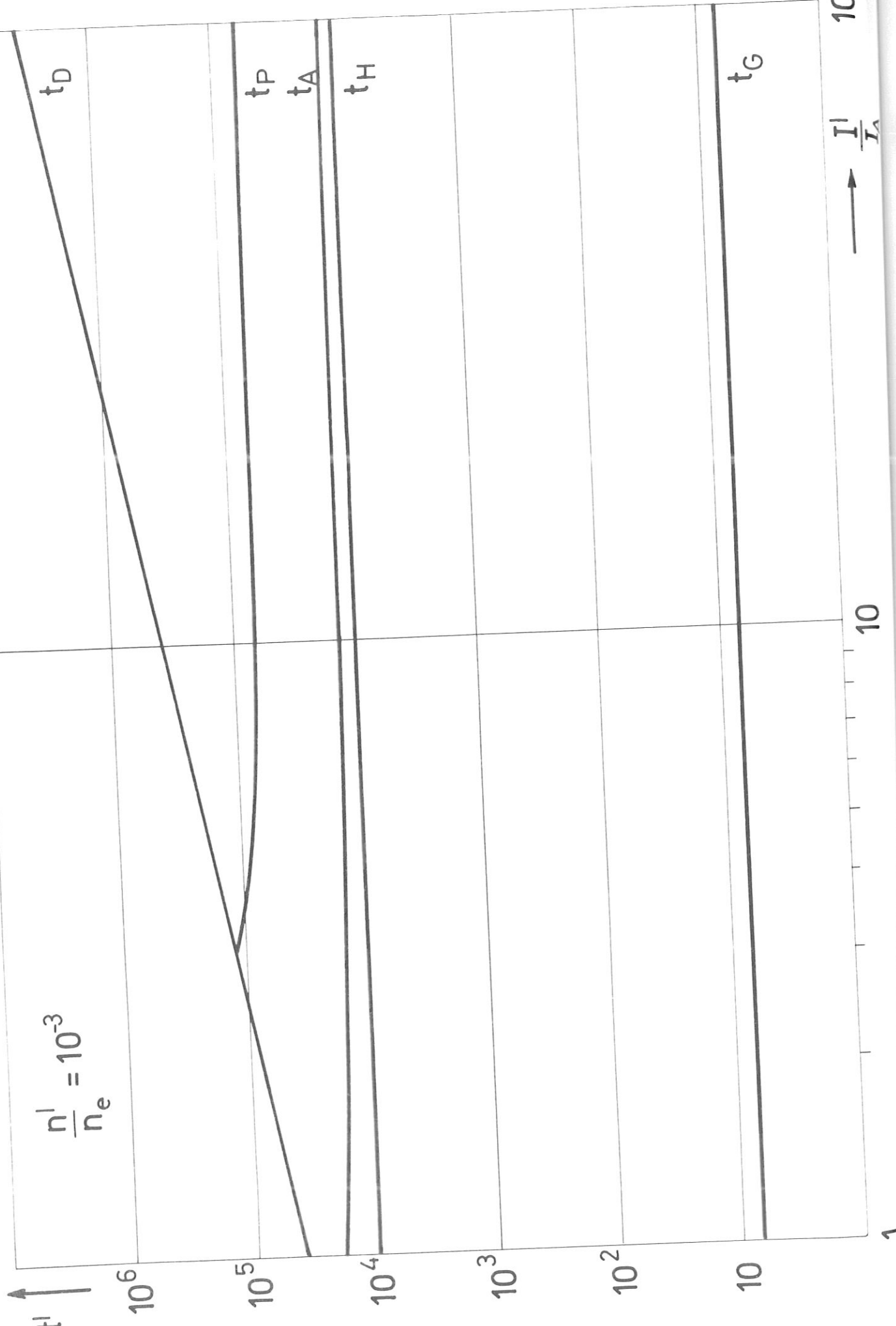
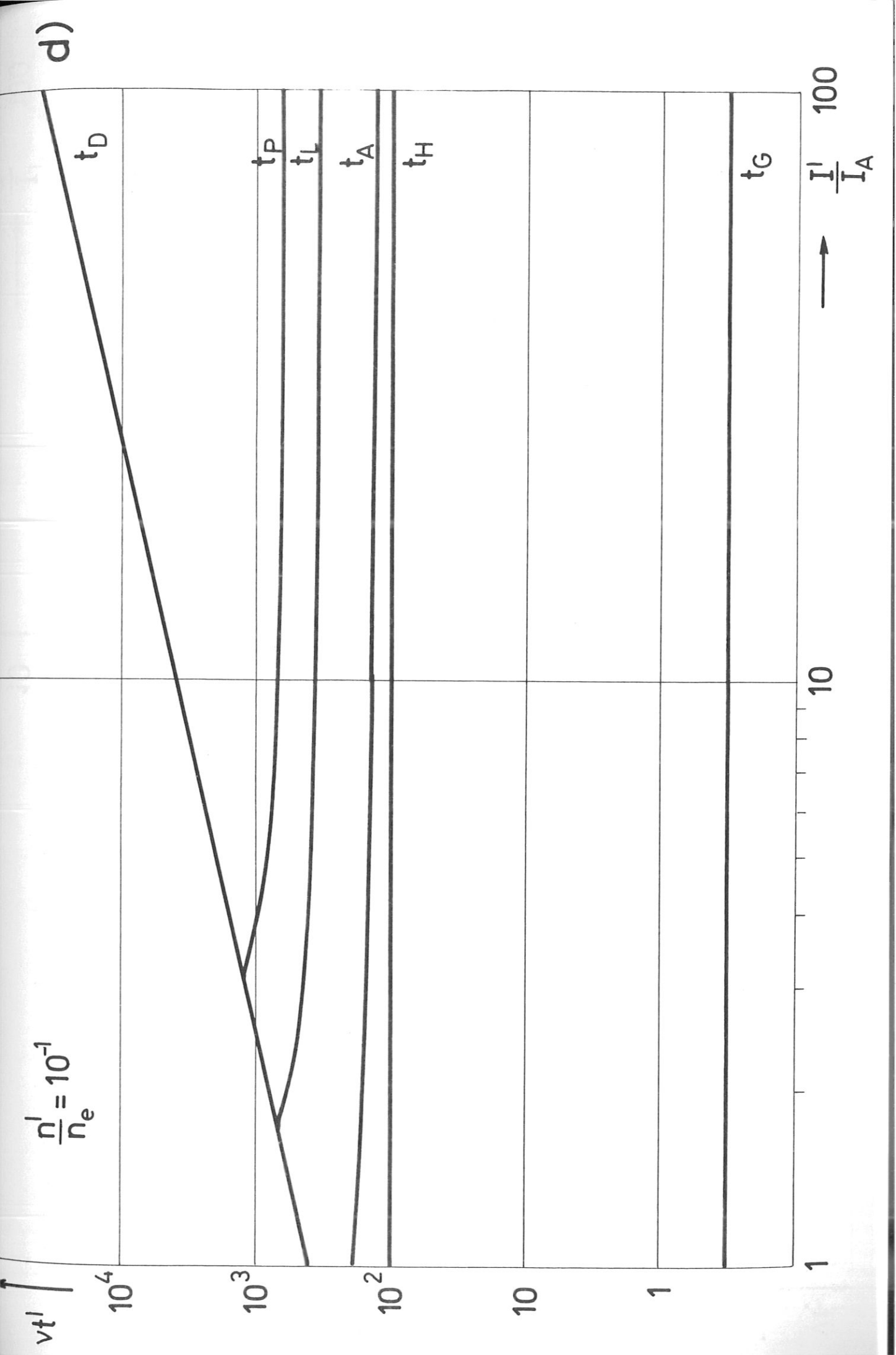


Figure 2

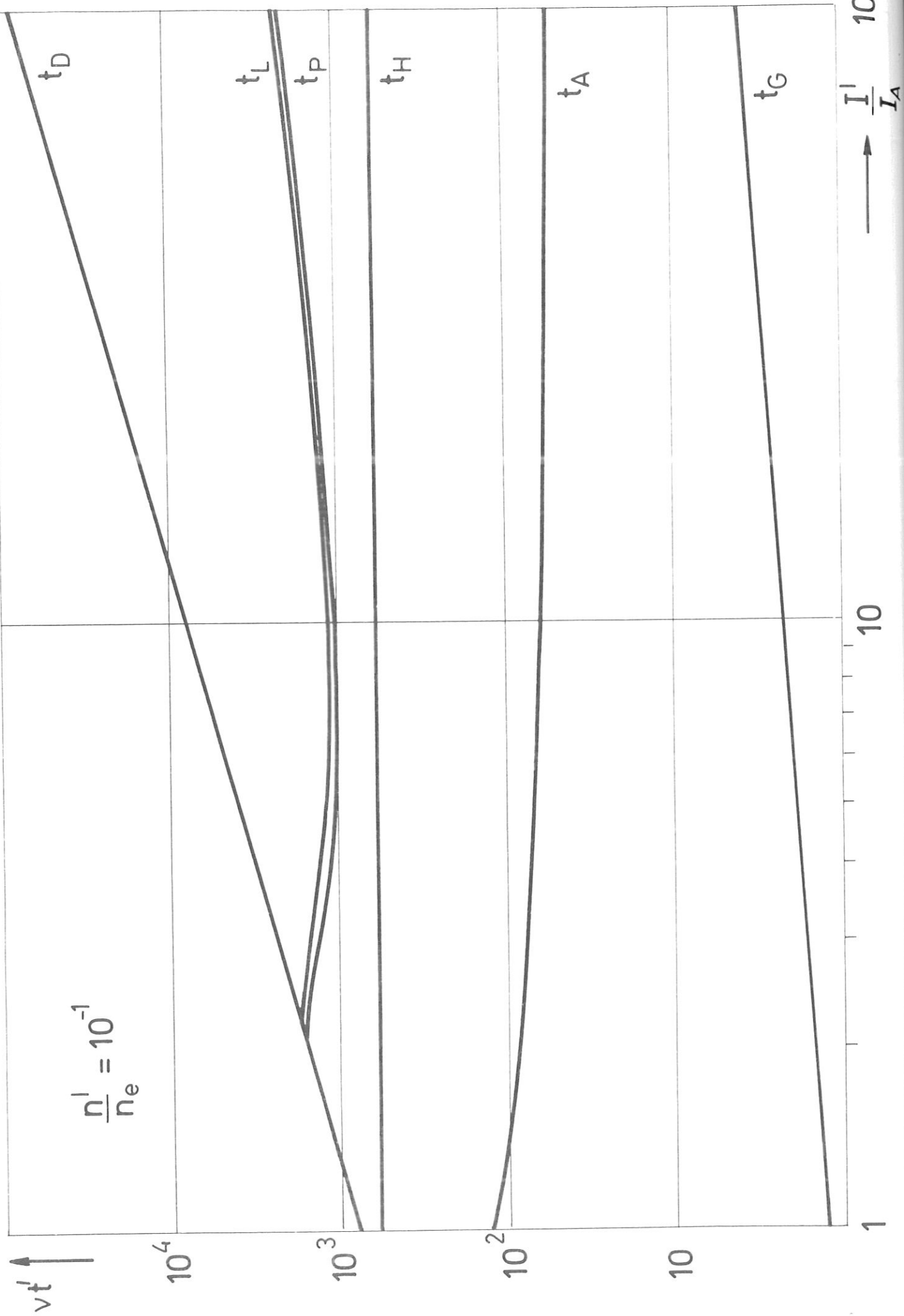


b)





c)



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