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Stability of Two-Dimensional Collision-Free Plasmas

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Abstract

The stability of plasma equilibria is analysed in the framework of two-dimensional Vlasov theory, where the perturbations are constant along the direction of the equilibrium current. The method makes use of several general properties of the equations, thus covering a rather large class of different configurations. Several necessary and sufficient stability criteria and an expression for the growth rates are derived. In particular, the results are applicable to equilibria containing neutral points. A number of earlier results (e.g. the tearing mode) are recovered by appropriate specialisation.

1. Introduction

Considerable work has been devoted to Vlasov stability of plasma equilibria which vary in one direction only. (Furth 1962, Pfirsch 1962, Schindler 1966, Laval et al. 1966, Schindler and Soop 1968, Freidberg and Morse 1969). Of particular interest are situations where the magnetic field magnitude vanishes at some plane (neutral sheet). In that case there exist instabilities which are not contained in ordinary (infinite conductivity) magnetohydrodynamics, the most prominent example being the tearing mode (Furth 1962).

Neutral sheets are belived to be present in a number of important astrophysical structures, such as the tail of the geomagnetosphere (Ness 1965), the interplanetary medium (Wilcox and Ness 1965), solar flares (e.g. Piddington 1969) and flare stars (Gershberg and Pikelner 1972). There is speculation that neutral sheets play a significant role also in the interstellar medium (Piddington 1969) as well as in active galactic nuclei and supernovae envelopes (Syrovatskii 1966). Similar processes can occur in a number of laboratory experiments like theta pinches with reverse magnetic field, belt pinches or tokamaks.

In view of the wide-spread occurrence of neutral sheets it seems of interest to consider a more general class of equilibria, allowing for spatial variation in one more direction. A characteristic feature of such two-dimensional equilibria is that neutral sheets are replaced by regions of weak magnetic field which may contain neutral lines.

Biskamp and Schindler (1971), in anticipating some of our present results, have shown that this leads to qualitatively new features.

Because of the mathematical complexity of the Vlasov equations it seems hopeless to aim at solutions in any explicit form. It turns out, however, that the stability analysis does not require such solutions. We approach the problem by discussing general proporties of the mathematical operators involved and demonstrate the existence of a variational principle. From these properties we derive several necessary and sufficient stability criteria. The method can be regarded as being an extension of the previous work by Pfirsch (1962) and by Schindler (1966), although the present approach is more explicit and even allows to estimate growth rates.

2. The equilibrium

Consider the class of two-dimensional collision-free plasma equilibria where the electric current has only an x_3-component in an orthogonal coordinate frame (x_1, x_2, x_3) the metric being defined by

$$ds^2 = h_1^2\, dx_1^2 + h_2^2\, dx_2^2 + h_3^2 dx_3^2 \tag{1}$$

h_1, h_2, h_3, the magnetic vector potential $\underline{A}_o$ and the electrostatic potential ϕ_o do not depend on x_3. These properties hold for instance for plane equilibria with the electric field $\underline{E}_o = -\nabla\phi_o$ and the magnetic field $\underline{B}_o = \nabla \times \underline{A}_o$ lying in the x,y-plane of a cartesian coordinate system (x,y,z) and for cylindrical coordinates (r,z,θ) with $\underline{E}_o$ and $\underline{B}_o$ lying in the r,z -plane.

Since we are dealing with collision-free plasmas we use the Vlasov theory, i.e. Liouville equations for the one-particle distribution functions f together with Maxwell's equations for the electric and magnetic field $\underline{E}$ and $\underline{B}$

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{e}{m} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{B} = \mu_o \Sigma e \int \underline{v}\, f\, d^3v$$

$$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_o} \Sigma e \int f\, d^3v$$

where e and m denote particle charge and mass and Σ sums over the particle species. Since we are interested in low-frequency non-relativistic phenomena we neglect the displacement current. A guiding center approach does not seem adequate because in the presence of neutral points non-gyroscopic particle orbits in regions of small magnetic field are importent.

For reasons of simplicity we confine ourselves to the above-mentioned case of Cartesian coordinates. We admit the possibility of adding a constant magnetic field component in the z-direction, which is consistent with our assumptions.

Under these circumstances Liouvilles equation is solved by

$$f_o(\underline{x}, \underline{v}) = F_o(H_o, p_z) \tag{2}$$

where H_o is the equilibrium Hamiltonian

$$H_o = \frac{m}{2}(v_x^2+v_y^2+v_z^2) + e\phi_o(x,y) \tag{3}$$

and p_z the generalised momentum in z-direction

$$p_z = mv_z + eA_o(x,y) \tag{4}$$

where A_o is the z-component of $\underline{A}_o$.

It will be convenient to assume that F_o is a monotonically decreasing function of

$$\frac{\partial F_o}{\partial H_o} \equiv F_o' < 0 \tag{5}$$

This choice excludes several microinstabilities which may be discussed separately.

The equilibrium potentials A_o and ϕ_o are found from Maxwell's equations, which are for the special geometry chosen

$$- \nabla^2 \phi_o = \frac{1}{\varepsilon_o} \rho_o (A_o, \phi_o) \tag{6}$$

$$- \nabla^2 A_o = \mu_o j_o (A_o, \phi_o)$$

with

$$\rho_o = \Sigma e \int F_o \, d^3 v$$

$$j_o = \Sigma e \int v_z F_o \, d^3 v \tag{7}$$

In the quasi-neutral approximation we compute $\phi_o (A_o)$ from $\rho_o (A_o, \phi_o) = 0$ and A_o from

$$- \nabla^2 A_o = \mu_o j_o (A_o) \tag{8}$$

In order to make (6) a well-posed problem we impose boundary conditions on the boundary $\dot{D}$ of a domain D in the x,y-plane. A convenient choice is to keep

$$A_o, \phi_o \text{ constant on } \dot{D}. \tag{9}$$

Under such conditions it can be established by standard mathematical techniques (e.g. Courant and Hilbert 1937) that there exists always a solution of (6) provided ρ_o and j_o are bounded functions of A_o and ϕ_o. But even in the case of unbounded functions j_o

there exist solutions. An example is $j_o(A_o) = e^{A_o}$. With this choice equation (8) has the solutions

$$A_o = -2 \ln (1+\frac{1}{8}(x^2+y^2)) \tag{10a}$$

(cylindrical pinch, Bennet 1934)

$$A_o = -2 \ln \cosh \frac{x}{\sqrt{2}} \tag{10b}$$

(sheet pinch, Harris 1962)

$$A_o = -2 \ln \left[(1+\alpha^2)^{1/2} \cosh \frac{x}{\sqrt{2}} + \alpha \cos \frac{y}{\sqrt{2}}\right] \tag{10c}$$

(periodic sheet pinch, Schmid-Burgk, 1965)

Equilibrium solutions relevant for geomagnetospheric structures have been studied by Soop and Schindler (1972).

There is a useful relation between charge density ρ_o and electric current density j_o . From (2) we find

$$\frac{dF_o}{dv_z} = mv_z F_o' + \frac{m}{e} \frac{\partial F_o}{\partial A_o} \tag{11}$$

By integrating over relocity space we find

$$\Sigma \int e^2 v_z F_o' d^3v = - \Sigma \int e \frac{\partial F_o}{\partial A_o} d^3v$$

which is the same as

$$\frac{\partial j_o}{\partial \phi_o} = - \frac{\partial \rho_o}{\partial A_o} \tag{12}$$

This relation says that j_o and ρ_o can be derived from a generating function $W(A_o, \phi_o)$

$$j_o = \frac{\partial W}{\partial A_o} , \quad \rho_o = - \frac{\partial W}{\partial \phi_o} \tag{13}$$

For one-dimensional equilibria these relations were derived by Völk (1961).

The distribution function $F_o(H_o, p_z)$ is not necessarity a single-valued function of its arguments. There may exist non-overlapping regions in the x,y-domain which contain particle orbits with the same values of H_o and p_z. In that case different values of F can be assigned to the different regions. When necessary we will take this possibility into account explicitly by writing the distribution function (2) in the form

$$f_o(\underline{x}, \underline{v}) = F_{oi}(H_o, p_z) \tag{14}$$

where the subscript i counts the non-overlapping orbits with the same value of H_o and p_z . Clearly, i can be considered to be a constant of motion.

We will assume that (14) is the general solution to the two-dimensional equilibrium problem. This means that almost all particles in a domain i will ergodically cover the subdomain traced out by the particle's values of H_o and p_z. For strictly one-dimensional equilibria where the potentials do not depend on, say, y, the subdomains are characterised in addition by p_y. Nevertheless we consider one-dimensional equilibria of form (14) only.

3. Formulation of the stability problem

We are interested in the stability of equilibria of the kind described in section 2. The perturbations are assumed to be two-dimensional with

$$\frac{\partial}{\partial z} = 0 \tag{15}$$

This means that we study the subset $k_z = 0$ of all possible modes, where k_z is the wave number associated with a Fourier expansion with respect to the z-coordinate. We assume that all perturbations vanish on the boundary of a suitable domain in phase space x, y, v_x, v_y, v_z.

We will restrict our consideration to modes which vary with time sufficiently slowly. The precise meaning of that assumption will be given at the end of this section.

With $\underline{x}$ and $\underline{p}$ being canonical conjugate coordinates the Liouville equation for the particle distribution function $f(\underline{x}, \underline{p}, t)$ for each particle species separately can be written in the form

$$\frac{\partial f}{\partial t} + \{H, f\} = 0, \qquad \{,\} \quad \text{Poisson brackets}$$
$$H = \frac{1}{2m} (\underline{P} - e\underline{A})^2 + e\phi \tag{16}$$

It is convenient to introduce the linear differential operator $\hat{H}$ by

$$\hat{H}f = \{H, f\} \tag{17}$$

It is easy to show that $\hat{H}$ is antihermitean with respect to the scalar product

$$(u, v) \equiv \int u^* v \; d^3x d^3v \tag{18}$$

where the integration extends over the full phase space because by partial integration one finds

$$(u,\hat{H}v) = -(\hat{H}u,v) \tag{19}$$

Any distribution function $F_o(H_o,p_z)$ is an eigenfunction of $\hat{H}_o$, because H_o, p_z are constants of the motion

$$\hat{H}_o F_o(H_o,p_z)=0 \tag{20}$$

It follows immediately from (17) and (20) that $\hat{H}$ commutes with F_o.

In terms of the eigenvalue problem

$$\hat{H}_o f = \lambda f \tag{21}$$

F_o is an eigenfunction associated with the eigenvalue $\lambda = 0$.

We shall restrict the analysis to the linear stability problem. By linearising Vlasov's equations around the equilibrium $f_o = F_o(H_o,p_z)$, $A = A_o$, $\phi = \phi_o$, we find from the Liouville equation

$$\frac{\partial f_1}{\partial t} + \hat{H}_o f_1 = - \hat{H}_1 F_o \tag{22}$$

where

$$\hat{H}_1 F_o = \{H_1, F_o\} \;,\; H_1 = e\phi_1 - e\underline{v} \cdot \underline{A}_1 \tag{23}$$

the subscript 1 denoting perturbations.

Using the fact that F_o depends on H_o and p_z only, we write

$$\{H_1,F_o\} = F_o'\{H_1,H_o\} = -F_o'\{H_o,H_1\} = -F_o'\hat{H}_o H_1 \tag{24}$$

Thus (22) becomes

$$\frac{\partial f_1}{\partial t} + \hat{H}_o f_1 = F_o{}' \hat{H}_o H_1 \qquad (25)$$

We consider (25) as an initial value problem and obtain a formal solution by a Laplace-transform in time. With the initial condition

$$f_1 \Big|_{t=0} \equiv f_1(0)$$

we find

$$f\ (x, \underline{v}, s) = \frac{\hat{H}_o}{s+\hat{H}_o} F_o{}' H_1 + e \frac{\partial f_o}{\partial \underline{p}} \cdot \underline{A}_1 + \frac{1}{s+\hat{H}_o} f_1(0) \qquad (26)$$

Since, from here on, we are mostly concerned with the Laplace transformed quantities, there is little danger of confusion in not introducing a different notation for the transformed quantities. The second term on the rhs of (26) arises from switching from $\underline{p}$ to $\underline{v}$ as independent variables.

Using

$$\frac{\partial f_o(\underline{r}, \underline{p})}{\partial \underline{p}} = F_o{}' \underline{v} + \frac{\partial F_o}{\partial p_z} \underline{e_z} \qquad (27)$$

(26) can be written as

$$f_1(x, \underline{v}, s) = - \frac{s}{s+\hat{H}_o} F_o{}' H_1 + eF_o{}' \phi_1 + e \frac{\partial F_o}{\partial p_z} A_{1Z} + \frac{1}{s + \hat{H}_o} f_1(0) \qquad (28)$$

From (28) we find the perturbation of the charge density

$$\rho_1 = - \Sigma\ e \int d^3 v \frac{s}{s+\hat{H}} F_o{}' H_1 + \frac{\partial \rho_o}{\partial \phi_o} \phi_1 + \frac{\partial \rho_o}{\partial A_o} A_{1Z} + \rho_1{}^{(in)} \qquad (29)$$

$$\underline{j}_1 = -\Sigma e \int d^3v \; \underline{v} \; \frac{s}{s+\hat{H}_o} F_o{}'H_1 + \underline{e}_z \frac{\partial j_o}{\partial \phi_o} \phi_1 + \underline{e}_z \frac{\partial j_o}{\partial A_o} A_{1z} \tag{30}$$

Here $\rho_1^{(in)}$ and $\underline{j}_1^{(in)}$ are the contributions arising from the initial condition imposed on f_1. If we introduce (29) and (30) into Maxwell's equations we obtain a set of integro-differential equations for $\underline{A}_1$ and ϕ_1

We are interested in the time-asymptotic behaviour of unstable solutions. We therefore can drop the terms involving the initial conditions. A simple formulation of the resulting homogeneous equations can be achieved if we introduce a matrix notation. We shall ignore the perturbations A_{1x}, A_{1y}; a discussion of this point is given at the end of this section.

We define the vector

$$\underline{\xi} = \begin{pmatrix} \phi_1 \\ A_{1z} \end{pmatrix} \tag{31}$$

and the matrices

$$N = \begin{pmatrix} -\frac{\partial \rho_o}{\partial \phi_o} & -\frac{\partial \rho_o}{\partial A_o} \\ \frac{\partial j_o}{\partial \phi_o} & \frac{\partial j_o}{\partial A_o} \end{pmatrix}$$

$$K = \begin{pmatrix} \varepsilon_o \nabla^2 & 0 \\ 0 & -\frac{1}{\mu_o} \nabla^2 \end{pmatrix} \tag{32}$$

$$C = \frac{1}{2} \begin{pmatrix} 1 & -v_z \\ 1 & -v_z \end{pmatrix}$$

N and K are hermitean with respect to the scalar product (different from (18))

$$[u,v] = \int u\, v d^2x \qquad (33)$$

where the integral is extended over the full spatial domain considered.

With these notations, Maxwell's equations assume the form

$$K\underline{\xi} = N\underline{\xi} + \Sigma\, e^2 \int d^3v\, C^+ \frac{s}{s + \hat{H}_o} F_o' \, C\, \underline{\xi}; \qquad (34)$$

here we have chosen the gauge $\nabla \cdot \underline{A_1} = 0$. The boundary conditions require that $\underline{\xi}$ vanish at the boundaries. The case of periodic functions can be treated analogously. Multiplying (34) by $\underline{\xi}$ and integrating over $\underline{x}$-space gives

$$[\underline{\xi}, (K-N)\, \underline{\xi}] - \Sigma\, e^2 (C\underline{\xi}, \frac{s}{s + \hat{H}_o} F_o' C\underline{\xi}) = 0 \qquad (35)$$

This quadratic form will play an important role in our stability analysis. We shall use Hilbert space methods, the scalar product being given by (18), where u, v are elements of continuous L^2 space.

We now discuss why we have ignored the perturbations A_{1x} and A_{1y}. This is justified asymptotically by our assumption that the modes considered vary sufficiently slowly. Suppose

$$|s_o| \tau << 1$$

where τ is a suitable characteristic time associated with the motion of the particles and s_o^{-1} a typical time constant of the mode. In typical situations envisaged (see section 1) τ may be the time a particle needs to cross the system, i.e. $\tau = L / v_{th}$ where L is the characteristic length of the equilibrium and v_{th} the thermal velocity. Under those circumstances it is appropriate to discuss our problem from the multiple time scale point of view, where $|s_o|\tau$ is the smallness parameter. We find (Biskamp and Schindler 1971)

$$f_1 = f_1^{(o)}(p_z, \Omega(H, p_z, t)) + O(|s_o|\tau)$$

for the non-Fourier-transformed distribution function where Ω is the phase space volume associated with the domain traced out by the functions H and p_z. Clearly $f_1^{(o)}$ does not yield current density components j_{1x} and j_{1y}. Thus j_{1x}, j_{1y} and consequently A_{1x} and A_{1y} or of order $|s_o|\tau$, whereas A_{1z} is of order 1. It is in that asymptotic sense that ignoring A_{1x}, A_{1y} is justified. Note however, that that property will not be needed in section 7.

In the case of strictly one-dimensional equilibria, the "adiabatic" part $f_1^{(o)}$ reduces to $\frac{\partial F_o}{\partial A_o} A_1 + \frac{\partial F_o}{\partial \phi_o} \phi_1$ (Pfirsch 1962).

4. Properties of unstable modes

The main purpose of this section is to show that unstable modes are purely growing, that is that s in (34) is real. We assume an unstable mode, i.e. Re(s) > 0 and split (35) into real and imaginary parts obtaining

$$\left[\underline{\xi}, (K-N)\underline{\xi}\right] = \Sigma\, e^2 \left(C\,\underline{\xi},\ \frac{ss^*}{(s+\hat{H}_o)(s-\hat{H}_o)}\, F_o'\, C\underline{\xi}\right) \tag{36}$$

$$0 = \left(C\underline{\xi},\ \frac{\hat{H}_o}{(s+\hat{H}_o)\ (s-\hat{H}_o)}\, F_o' C\underline{\xi}\right) \tag{37}$$

The property Im(s) = 0 follows from equation (37). To show this we write $s = \gamma + i\nu$ and consider the function

$$G(\nu) = \left(C\ \underline{\xi},\ \frac{\hat{R}}{\gamma^2 + (\nu + \hat{R})^2}\, F_o'\ C\ \underline{\xi}\right) \tag{38}$$

where

$$i\hat{R} =: \hat{H}_o$$

Since $\hat{H}_o$ is antihermitean, $\hat{R}$ is hermitean. G(ν) is an alternative way of writing the rhs of equ. (37), such that

$$G(\nu) = 0. \tag{39}$$

Let $\hat{S}$ be a transformation which transforms

$$v_x \to -v_x,\quad v_y \to -v_y$$

all other variables remaining unchanged. It is easy to see that this operation is hermitean and unitary

$$\hat{S} = \hat{S}^+,\quad \hat{S}\hat{S}^+ = \hat{1} \tag{40}$$

Furthermore the following relation holds

$$\hat{S}\,\hat{H}_o = -\hat{H}_o\,\hat{S} \tag{41}$$

If $g(\hat{H}_o)$ is a function of $\hat{H}_o$ we find

$$\hat{S}\, g(\hat{H}_o) = g\,(-\hat{H}_o)\,\hat{S} \tag{42}$$

Since $F_o(H_o, p_z)$ contains v_x and v_y only in the combination $v_x^2 + v_y^2$ the unperturbed distribution function is invariant with respect to $\hat{S}$ Clearly $\phi_1 - v_z A_{1z}$ is also invariant. With the aid of these properties we will see that

$$G(-\nu) = -\,G(\nu). \tag{43}$$

We write

$$\begin{aligned} G(-\nu) &= \left(C\underline{\xi}, \frac{\hat{R}}{\gamma^2 + (-\nu + \hat{R})^2}\, F_o'\, C\underline{\xi}\right) \\ &= \left(\hat{S}C\underline{\xi}, \frac{\hat{R}}{\gamma^2 + (-\nu + \hat{R})^2}\, \hat{S}\, C\,\underline{\xi}\right) \\ &= \left(C\underline{\xi}, -\frac{\hat{R}}{\gamma^2 + (-\nu - \hat{R})^2}\, \hat{S}\hat{S}C\underline{\xi}\right) \\ &= -\,G(\nu) \end{aligned} \tag{44}$$

which proves (43). In equating $\hat{S}C\underline{\xi}$ with $C\underline{\xi}$ we made use of our assumption that A_{1x} and A_{1y} are negligible.

Thus we can write

$$\begin{aligned} 2\,G(\nu) &= G(\nu) - G(-\nu) \\ &= -\,4\nu\,\left(C\underline{\xi}\ \frac{\hat{R}^2}{(\gamma^2 + (\nu + \hat{R})^2)(\gamma^2 + (\nu - \hat{R})^2)}\, F_o'\, C\underline{\xi}\right) \end{aligned} \tag{45}$$

Since $F_o' \neq 0$ the scalar product in (45) can vanish only if $\hat{R}C\underline{\xi} = 0$. As shown below that is impossible. Thus, from (39) and (45) we find

$$\nu = 0 \tag{46}$$

We now show that $\hat{R}\,C\,\underline{\xi} \neq 0$ for $\underline{\xi} \neq 0$. Suppose $\hat{R}\,C\,\underline{\xi} \neq 0$.

Then $\phi_1 - v_z A_1$ must be a function of H_o and p_z. The only possibility is

$$\phi_1 - v_z A_{1z} = \alpha\, p_z$$

where $\alpha \neq 0$ is a constant. Then

$$A_{1z} = - \alpha m, \quad \phi_1 = e\alpha A_o$$

This however leads to a contradiction because A_1 has to vanish at the boundary. Thus $\hat{R}C\underline{\xi} \equiv 0$ is impossible.

Since $\nu = 0$, unstable modes are purely growing. Therefore, the transition from stability to instability, as a result of changing a continuous parameter, can occur only at $s = 0$. In other words, marginal modes are necessarily neighbouring equilibria.

For one-dimensional equilibria this property was first shown by Pfirsch (1962).

5. The marginal mode

The result of the preceding section suggests that the marginal mode $s \to o$ will be of particular interest in discussing stability. In this section we therefore consider the limit $s \to o$ of (34) and (35) explicitely.

We introduce the spectral representation of the resolvent of the hermitean operator $\hat{R}$:

$$\frac{1}{\omega + \hat{R}} = \int_{-\infty}^{+\infty} \frac{d\hat{E}_\lambda}{\omega+\lambda} \qquad \mathrm{Im}(\omega) \neq 0. \tag{47}$$

$\hat{E}_\lambda$ is a projection operator, projecting on all eigenspaces of $\hat{R}$ with eigenvalues $\mu \leq \lambda$. Note however that $\hat{E}_\lambda$ exists even if eigenfunctions and eigenvalues cannot be properly defined (see e.g. Achieser and Glasmann 1960). If $\hat{E}_\lambda$ is discontinuous

$$\hat{P}_\lambda = \hat{E}_{\lambda+0} - \hat{E}_\lambda$$

is the projection operator, projecting on the eigenspace with the (discrete) eigenvalue λ. Setting $s = i\omega$ we determine the limit $s \to o$ of the operator

$$\hat{Q} = \frac{s}{s + \hat{H}_o} = \frac{\omega}{\omega + \hat{R}} \tag{48}$$

This limit is defined in terms of

$$\lim_{s \to o} (U,\hat{Q}\, U) = \lim_{s \to o} \int \frac{is}{is-\lambda}\, d\sigma(\lambda) \tag{49}$$

$$\sigma(\lambda) = (U,\hat{E}_\lambda U)$$

for any $U \in L^2$. We write (49) in the form

$$\lim_{s \to o} (U,\hat{Q}U) = \lim_{s \to o} \left[- i\, s \int \frac{\lambda}{s^2+\lambda^2}\, d\sigma(\lambda) + s^2 \int \frac{1}{s^2+\lambda^2}\, d\sigma(\lambda)\right] \tag{50}$$

The first term of the rhs of (50) vanishes the second term becomes equal to $\lim_{s\to o} (\sigma(o+s) - \sigma(o))$. With the aid of the projection operator $\hat{P}_o$ associated with the eigenvalue $\lambda = o$ this quantity can be written as $(U, \hat{P}_o U)$. Thus we obtain

$$\lim_{s \to o} \frac{s}{s+\hat{H}_o} = \hat{P}_o \tag{51}$$

Hence we find from (34) and (35) that the marginal mode satisfies the equations

$$(K-N)\underline{\xi} + \Sigma e^2 \int d^3v \; C^+ |F_o'| \; \hat{P}_o C\underline{\xi} = 0 \tag{52}$$

$$\left[\underline{\xi}, (K-N)\;\underline{\xi}\right] + \Sigma e \;(C\underline{\xi}, |F_o'| \;\hat{P}_o C\underline{\xi}) = 0.$$

The terms containing $\hat{P}_o$ can be cast into a more explicit form by making use of the definition of $\hat{P}_o$ as a projection operator.

The projection $h = \hat{P}_o g$ of an arbitrary element of L^2 satisfies $\hat{H}_o h = o$, that is h will have the form

$$h = h_i\,(H_o, p_z) \tag{53}$$

where in accordance with (14), i counts non-overlapping regions.

On the other hand, for a given set of constants of the motion $h(H_o, p_z)$ can be defined by

$$||h-g|| = \inf_{h' \; L^2} ||h'-g|| \tag{54}$$

where $||U||$ denotes the norm of U associated with (18). This leads to the variational problem

$$\delta \int |h-g|^2 d^2x d^3v = 0, \tag{55}$$

or

$$\int (h-g)\ \delta h^* \ d^2x d^3v = 0 \tag{56}$$

Making use of (53) we rewrite (56) as

$$\Sigma \int dH_o \ dp_z \ \delta h^* \int_{\Omega_i(H_o, p_o)} d\Omega \ (h-g) = 0 \tag{57}$$

where Ω_i is the domain of phase space traced out the constants of the motion H_o, p_z, i. We find

$$h = \hat{P}_o g = \frac{1}{\Omega_i(H_o, p_z)} \int_{\Omega_i} d\Omega_i g \tag{58}$$

which has the form of a phase space average. If, as in the case of $C\underline{\xi}$ the function g does not depend on v_x, v_y, we can write

$d\Omega \sim dH_o \ dp_z \ d^2x$ such that

$$h = \langle g \rangle \tag{59}$$

where

$$\langle g \rangle = \frac{1}{D_i(H_o, p_z)} \int_{D_i} g d^2x \tag{60}$$

i.e. the average takes place in x-space only. Clearly $\langle g \rangle$ depends on H_o, p_z and i.

Using these relations we can write

$$P_o C\underline{\xi} = \langle C\underline{\xi} \rangle \tag{61}$$

With the aid of the definitions of $\underline{\xi}$, N, K, C in (31) and (32) together with (42) and (61) we find the explicit form of the marginal mode equations (52)

$$\varepsilon_o \Delta\phi_1 + \frac{\partial\rho_o}{\partial\phi_o}\phi_1 + \frac{\partial\rho_o}{\partial A_o} A_1 - \Sigma e^2 \int d^3v\, F_o' < \Psi_1 > = 0 \tag{62}$$

$$\frac{1}{\mu_o} \Delta A_1 + \frac{\partial j_o}{\partial\phi_o}\phi_1 + \frac{\partial j_o}{\partial A_o} A_1 - \Sigma e^2 \int d^3v\, v_z\, F_o' < \Psi_1 > = 0 \tag{63}$$

$$W_o(A_1,\phi_1) = 0 \tag{64}$$

where

$$\begin{aligned} W_o(A_1,\phi_1) = \int d^2x \Big[& \frac{1}{\mu_o} | \nabla A_1 |^2 - \varepsilon_o |\nabla\phi_1|^2 + \frac{\partial\rho_o}{\partial\phi_o} |\phi_1|^2 \\ & - \frac{\partial j_o}{\partial A_o} |A_1|^2 + \frac{\partial\rho_o}{\partial A_o}(A_1\phi_1^* + \phi_1 A_1^*) \\ & + \Sigma\, e^2 \int d^3v |F_o'|\, |< \Psi_1 >|^2 \Big] \end{aligned} \tag{65}$$

and

$$\Psi_1 = \phi_1 - v_z A_{1z}$$

In the case of quasi-neutrality, the term $\varepsilon_o |\nabla\phi_1|^2$ can be dropped, the other ϕ_1-terms however have to be retained.

For two-dimensional perturbations of one-dimensional equilibria, the procedure of this paragraph requires a modification because h may depend on an edditional constant of the motion, say p_y. In that case (60) has to be replaced by

$$h = \int \frac{g\ dx\ dy}{\sqrt{H - \frac{1}{2m}(p_z - eA)^2 - e\phi}} \Bigg/ \int \frac{dx\ dy}{\sqrt{H - \frac{1}{2m}(p_z - A)^2 - e\phi}}$$

This expression vanishes because a two-dimensional perturbation of an equilibrium which depends on x only has the form $g \sim e^{iky}$, $k \neq o$. This result is consistent with the treatment chosen by Pfirsch (1962).

6. Sufficient stability criteria

We shall recover a sufficient stability criterian obtained by Schindler (1970) in a different way. We compare equation (36) with its marginal version, i.e. the second of equations (52).

By returning to spectral representation we write

$$(C\underline{\xi}, \frac{ss^{*}}{(s+\hat{H}_o)(s-\hat{H}_o)} F'_o C\underline{\xi}) = (C\underline{\xi}, \int \frac{s^2}{s^2+\lambda^2} d\hat{E}_\lambda F'_o C \underline{\xi}) \tag{66}$$

From the shape of the integrand on the rhs of (66) it is obvious that

$$\int \frac{s^2}{s^2+\lambda^2} d\sigma \geq \lim_{s\to o} \int \frac{s^2}{s^2+\lambda^2} d\sigma = (C\underline{\xi}, |F'_o| \hat{P}_o C\underline{\xi}) \tag{67}$$

where the equality sign applies to the case $s = o$ only. Thus for $s \neq o$ we find from (36) with (61)

$$[\underline{\xi}, (K-N)\underline{\xi}] + \Sigma e^2(\langle C\underline{\xi}\rangle, |F'_o| \langle C\underline{\xi}\rangle) < 0 \tag{68}$$

or using the explicit form (65)

$$W_o(A_1, \phi_1) < 0$$

Since we have assumed instability (e.g. by ignoring the contributions from the initial conditions) the following stability criterion is evident:

$W_o(A_1, \phi_1) > 0$ for all $A_1, \phi_1 \in L^2_n$ is sufficient for stability. (69)

The subscript n refers to the fact that a suitable normalisation is imposed on A_1 and ϕ_1 to exclude the trivial solution $A_1 = \phi_1 = o$. Weaker sufficient stability criteria follow by dropping positive definite terms in the expression for W_o. An obvious and interesting possibility is to leave out the contribution of the averages $<\Psi_1>$ (Schindler 1970). Since $W_o(A_1, \phi_1)$ vanishes for marginal modes we don't expect that it is possible to improve the criterion (69) further. That in fact this is true will be shown in section 8.

The quadratic form (59) takes a particularly simple form if we assume quasi neutrality, that is ignore $\varepsilon_o \Delta\phi$, in (62) and $\varepsilon_o (\nabla\phi)^2$ in (64). Using (64) we can then express ϕ_1 in terms of A_1 and $<\Psi_1>$

$$\phi_1 = - \frac{\partial\rho_o / \partial A_o}{\partial\rho_o / \partial\phi_o} A_{1z} + \overline{< \Psi_1 >} \tag{70}$$

where $\bar{g}$ denotes an average which for arbitrary g is defined by

$$\bar{g} = \frac{\Sigma e^2 \int d^3v |F_o'| \, g}{\Sigma e^2 \int d^3v |F_o'|} \tag{71}$$

Inserting (70) into the quasi-neutral version of (65) yields

$$W_{qn}(A_1, \phi_1) = \int d^2x \Big[\frac{1}{\mu_o} |\nabla A_1|^2 - \Big(\frac{\partial j_o}{\partial A_o} + \frac{(\partial\rho_o / \partial A_o)^2}{\partial\rho_o / \partial\phi_o} \Big) |A_1|^2$$

$$+ \Big| \frac{\partial\rho_o}{\partial\phi_o} \Big| \; |<\Psi_1> - \overline{\Psi}_1|^2 \Big] \tag{72}$$

where the subscript qn is supposed to remind of the quasi-neutrality assumption.

In deriving (72) we have used (7) and the fact that

$$\overline{\langle \Psi_1 \rangle} = \overline{\Psi_1} \tag{73}$$

which is another form of quasi-neutral Poisson's equation as follows from (70).

Since we have used (73) in deriving (72) the stability criterion takes the form:

$W_{qn} > o$ for all $A_1, \phi_1 \in L^2_n$ satisfying (73) is sufficient for stability of quasi-neutral systems. (74)

Noting that the last term in (72) is positive we also find the following (weaker) criterion:

$$\int d\overset{2}{\mathbf{x}} \left[\frac{1}{\mu_o} |\nabla A_1|^2 - \frac{dj_o}{dA_o} |A_1|^2 \right] > 0 \text{ for all } A_1 \in L^2 \tag{75}$$

is sufficient for stability of quasi-neutral systems. The constraint (73) no longer applies because ϕ_1 has disappeared from the quadratic form in (75); $\frac{dj_o}{dA_o}$ is the derivative of $j_o(A)$ as defined in (8).

As shown above, the average $\langle\Psi_1\rangle$ and consequently ψ_1 vanish for two-dimensional perturbations of one-dimensional equilibria. Then (74) coincides with (75). This case was studied by Schindler and Soop (1968). The criterion (74) was obtained by Biskamp and Schindler (1971) using a different approach.

7. Variational principle

The stability criterion (69) can obiously be expressed in terms of the following variational principle:

$$\inf_{A_1,\ \phi_1 \in L^2_n} W_o(A_1,\ \phi_1) > 0 \tag{76}$$

is sufficient for stability.

If we normalise by

$$\int |\underline{\xi}|^2\ d^2x \ = \ \kappa \tag{77}$$

where κ is a constant we obtain as the Euler-Lagrange equations for the minimising function $\underline{\xi}_m$

$$(K-N)\ \underline{\xi}_m + \Sigma\ e^2 \int d^3v\ C^+ |F'_o| P_o C \xi_m = \lambda\ \underline{\xi}_m \tag{78}$$

The lowest eigenvalue λ_o coincides with $\frac{\kappa}{x} \inf W_o$.

Since $\lambda_o = 0$ reproduces (52) we find

$$\inf_{A_1,\ \phi_1 \in L^2_n} W_o(A_1,\phi_1) = 0 \tag{79}$$

is necessary and sufficient for marginal stability.

If W_{qn} is used instead of W_o, the variation in (76) and (79) must be carried out under the constrainr (73).

We now show that the variational principle (76) can be obtained in different way. This is of interest because two of the restrictions we had to impose so far can be relaxed:

We will no longer restrict the consideration to

(i) the time asymptotic regime

(ii) slow modes, i.e. we allow for $A_{1x}, A_{1y} \neq 0$

For simplicity quasi-neutrality is assumed, the generalisation to the exact case ($\varepsilon_o \neq 0$) is straight forward.

Generalising an approach developed by Schindler (1966) for one-dimensional equilibria we start from energy conservation in the form

$$\Sigma \int \left[\frac{m}{2}(v_x^2 + v_y^2) + \frac{1}{2m}(p_z - eA_o)^2\right] f d\Omega + \int \frac{B^2}{2\mu_o} d^2x = C \tag{80}$$

where

$$d\Omega = d^2x \, d\tau, \quad d\tau = dv_x dv_y d(p_z/m) \tag{81}$$

Proceeding along the lines of the paper by Schindler (1966) we find from (80) to second order in the perturbations

$$Z(A_1, \phi_1) \equiv \Sigma \int d\Omega \left[-\frac{f_1^2}{F_o'} - \frac{2eA_{1z}f_1}{m}(p_z - eA_o) + \frac{e^2}{m} A_{1z}{}^2 F_o\right] + \int d^2x \frac{1}{\mu_o} B_1^2 = C_2 \tag{82}$$

where the constant C_2 is determined by the initial conditions.

We minimize the lhs of (82), subject to the following two constraints

$$\Sigma \int e f_1 \, d\tau = 0 \tag{83}$$

i.e. quasi-neutral Poisson's equation. In addition we impose

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where the constant C_2 is determined by the initial conditions.

We minimize the lhs of (82), subject to the following two constraints

$$\Sigma \int e\, f_1\, d\tau = 0 \tag{83}$$

i.e. quasi-neutral Poisson's equation. In addition we impose

$$\int Y(H_o,p_z) f_1 d\Omega = 0 \text{ for all } Y(H_o,p_z) \tag{84}$$

which follows from the fact that f_1 satisfies the linearised Vlasov equation. (84) can be derived by multiplying (25) by $Y(H_o,p_z)$ and integrating over phase space. Variation of (82) under these constraints determines the minimising f_1 for each species

$$f_1 = F_o' \left[\chi_1(\underline{x},p_z) + e\varphi(\underline{x}) + X(H_o,p_z) \right] \tag{85}$$

with

$$\chi_1 = -\frac{e}{m}(p_z - eA_o)A_{1z} \tag{86}$$

φ and X are the Lagrange multipliers associated with the constraints (83) and (84).

Introducing (85) into (84) yields with (14)

$$\sum_i \int dH_o dp_z \, Y_i(H_o,p_z) F_{oi}'(H_o,p_z) \int (\chi_1 + e\varphi + X) d^2x = 0 \tag{87}$$

Considering that $Y_i(H_o,p_z)$ is arbitrary one obtains for each of the non-overlapping regions

$$X(H_o,p_z) = -<\chi_1> - e<\varphi> \tag{88}$$

Inserting (88) into (85) gives

$$f_1 = F_o' (\chi_1 + e\varphi - <\chi_1 + e\varphi>) \tag{89}$$

We will identify φ with a physical quantity only in the marginal state. It is easy to see that there $\varphi = \phi_1$. Thus we can write

$$f_1 = eF_o' (\Psi_1 - <\Psi_1>) \tag{90}$$

Inserting (90) into Z as defined in (82) we obtain

$$\begin{aligned} \inf Z = \int d^2x \Big[& \frac{B_1^2}{\mu_o} + \frac{\partial\rho_o}{\partial\phi_o}\phi_1^2 \\ & - \frac{\partial j_o}{\partial A_o} A_{1z}^2 + 2\frac{\partial\rho_o}{\partial A_o} A_{1z}\phi_1 \Big] + \\ & + \Sigma\, e^2 \int d\Omega |F_o'| <\Psi_1>^2 \end{aligned} \tag{91}$$

Noting that we have minimised the lhs of (82) and that the contributions from A_{1x} and A_{1y} in (91) are positive we can write (91) in the form

$$U + W = c_2 \tag{92}$$

where $U > 0$ and

$$W = \int d^2x \left[\frac{1}{\mu_o} (\nabla A_1)^2 + \frac{\partial \rho_o}{\partial \phi_o} \phi_1^2 - \frac{\partial j_o}{\partial A_o} A_1^2 + 2 \frac{\partial \rho_o}{\partial A_o} A_1 \phi_1 + \Sigma e^2 \int d^3v \, |F_o'| \, \langle \Psi_1 \rangle^2 \right] \tag{93}$$

Clearly unlimited growth is impossible if $W > 0$. Therefore we find

$$\inf_{A_1,\ \phi_1 \in L^2_{n(r)}} W(A_1, \phi_1) > 0 \tag{94}$$

is sufficient for stability, $L^2_{n(r)}$ denotes real L^2 space with a suitable normalisation.

W coincides with the quasineutral version of (65) for real A_1, ϕ_1. Since all coefficients (factors and differential operators) of the quadratic form in (65) are real the variational principles (65) (with $\varepsilon_o = 0$) and (94) are equivalent.

8. Necessary stability criteria

In the preceding sections we have shown that the minimum

$$W_o^* \equiv \inf_{A_1,\ \phi_1 \in L_n^2} W_o \tag{95}$$

determines stability in the sense that $W_o^* > o$ implies stability while $W_o^* = o$ is necessary and sufficient for the system to be in a marginal state. In this section we shall discuss the case $W_o^* < o$.

For that purpose we carry the expansion of equation (36) with respect to s one step beyond the limit $s \to o$. We split the spectrum of the operator $\hat{R} = -i\hat{H}_o$ for a given equilibrium orbit into a continuous part (subscript c) and a discontinuous part (subscript d)

$$\begin{aligned}
&(C\underline{\xi}, \frac{s^2}{(s+\hat{H}_o)(s-\hat{H}_o)} |F_o'| C\underline{\xi}) \\
&= (C\underline{\xi}, \int \frac{s^2}{s^2 + \lambda^2} \frac{d\hat{E}_c}{d\lambda} d\lambda |F_o'| C\underline{\xi}) \\
&+ (C\underline{\xi}, \int_{(y)} \frac{s^2}{s^2+\lambda^2} d\hat{E}_d(\lambda) |F_o'| C\underline{\xi})
\end{aligned} \tag{96}$$

To lowest order the first term gives, with the aid of $\lambda/s = t$, $d\hat{E}_c/d\lambda = \hat{E}_c'$

$$(C\underline{\xi}, s \int \frac{\hat{E}_c'(st)}{1+t^2} dt |F_o'| C\underline{\xi})$$

$$\to \pi\ s\ (C\underline{\xi}, |F_o'|\ \hat{E}_c'(o) C\underline{\xi})$$

the next higher order term being of 3rd order. The second term of the rhs of (96) gives the zero order term $(C\underline{\xi}, |F_o'| \hat{P}_o C\underline{\xi})$ as was already shown in section 5. The next higher order comes from the first non-vanishing discrete eigenvalue with smallest absolute value, λ_1. For $|s| << |\lambda_1|$ one finds

$$s^2 \left(C\underline{\xi}, F_o' \frac{\hat{P}_1}{\lambda_1^2} C\underline{\xi}\right)$$

where $\hat{P}_1$ is the corresponding projection operator. Thus we find from (36) asymptotically for small s

$$W_o(A_1,\phi_1) = -\pi s(C\underline{\xi}, |F_o'| \hat{E}_c'(o)C\underline{\xi}) - s^2(C\underline{\xi}, |F_o'| \frac{\hat{P}_1}{\lambda_1^2} C\underline{\xi}) \tag{97}$$

$\hat{E}_c'$ and $\hat{P}_1$ are self-adjoint operators which commute with F_o'. Therefore both scalar products appearing on the rhs of (97) are positive.

We now go back to the original problem (34) considering cases where (34) contains a continuous parameter α, where α might be a parameter characterising a set of equilibria or a continuous (or piecewise continuous) wave vector k as in the cases of one-dimensional and periodic two-dimensional equilibria. We assume that there is a marginal state for $\alpha = \alpha_o$ and expand around that state in terms of a small increment $\delta\alpha$ (34) is of the form

$$M(\alpha)\,\underline{\xi} = V(\alpha,s)\underline{\xi} \tag{98}$$

where

$$M(\alpha)\underline{\xi} \equiv (K-N)\underline{\xi} + \Sigma e^2 \int C^+ |F_o'| \hat{P}_o \,\underline{\xi}\; d^3v \tag{99}$$

and $V(\alpha, s)$ has the following properties

$$V(\alpha,0)\underline{\xi} = 0$$

$$(\underline{\xi},V\underline{\xi}) = - s\pi(C\underline{\xi}|F_o'|\hat{E}_c'(o)C\underline{\xi}) \qquad (100)$$

$$- s^2(c\underline{\xi}|F_o'|\hat{P}C\xi), \ s\to 0$$

as follows from (52) and (97). We will take the second term on the rhs of equ. (100) into account only if the first one vanishes.

The lowest order equation ($\delta\alpha = 0$) is

$$M(\alpha_o)\xi_o = 0 \qquad (101)$$

assuring that α_o corresponds to a marginal state. To next order we find

$$M(\alpha_o)\underline{\xi}_1 + \frac{\partial M}{\partial\alpha}\underline{\xi}_o \ \delta\alpha = V(s,\alpha_o)\underline{\xi}_o \qquad (102)$$

By taking the scalar product with respect to $\underline{\xi}$ we find (condition of integrability)

$$(\underline{\xi}_o, \frac{\partial M}{\partial\alpha}\underline{\xi}_o) \ \delta\alpha = (\underline{\xi}_o,V(s,\alpha_o)\underline{\xi}_o) \qquad (103)$$

which gives for sufficiently small s (from (100))

$$s = - \frac{(\underline{\xi}_o,\frac{\partial M}{\partial\alpha}\underline{\xi}_o) \ \delta\alpha}{(C\underline{\xi}_o|F_o'|\hat{E}_c'(o) \ C\underline{\xi}_o)} \quad \text{for } \hat{E}_c' \ (o) \neq 0 \qquad (104)$$

$$s^2 = - \frac{(\underline{\xi}_o, \frac{\partial M}{\partial\alpha}\underline{\xi}_o)\delta\alpha}{(C\underline{\xi}_o,|F_o'| \frac{\hat{P}_1}{\lambda_1^2} C \ \underline{\xi}_o)} \quad \text{for } \hat{E}_c' \ (o) = 0 \qquad (105)$$

Both in (104) and (105) there is a sign of $\delta\alpha$ corresponding to instability.

The numerator is the change of W_o^*

$$\delta W_o^* = (\underline{\xi}_o, \frac{\partial M_o}{\partial\alpha}\underline{\xi}_o) \ \delta\alpha$$

Thus we find stability for $\delta W_o^* > 0$, as required by the results of sections 7 and instability for $\delta W_o^* < 0$. Thus we find the following criterion:

The existence of a marginal state at an inner point $\alpha = \alpha_0$ of the α-interval considered, with $(\xi_0, \frac{\partial M}{\partial \alpha} |_{\alpha_0} \xi_0) \neq 0$, is sufficient for instability.

Since a change from stability to instability and vice versa can occur only through a marginal state $W_o^* = o$, instability prevails in any continuous α-interval with $W_o^* < o$, if the interval contains a marginal point $W_o^* = o$. (see figure 1)

The following criterion is an immediate consequence.

If for a given equilibrium $W_o^* < o$, the equilibrium is unstable if it is an element of a set of equilibria S_α with the following two properties,

(i) W_o^* varies continuously with the set parameter α

(106)

(ii) S_α contains an element with $W_o^* = o$.

9. Growth rates

For sufficiently small growth rates (97) can be used to obtain an order of magnitude estimate. We find

$$s = - \frac{W_o(A_1, \phi_1)}{\pi (C\xi, |F_o'| \hat{E}_o' (o) C\xi)} ; \hat{E}_o'(o) \neq 0 \tag{107}$$

$$(s)^2 = - \frac{W_o(A_1, \phi_1)}{(C\xi, |F_o| \frac{\hat{P}_1 C\xi)}{\lambda_1^2}} ; \hat{E}_o' (o) = 0 \tag{108}$$

These formulas are particularly useful when the equilibrium contains a small parameter ε, s vanishing with some positive power of ε. We give two examples, recovering results which have been obtained by different methods.

(a) Sheet pinch.

The equilibrium magnetic field is aligned with the y-axis and $B_y(x)$ varies through zero, being constant for large x. The growth rate was computed by Laval et al. (1966) using numerical methods. An order of magnitude estimate was provided by Coppi et al. (1966) assuming that the particles do not see the magnetic field in the region $|x| < d$, $d = \sqrt{aL}$ where a is the Larmor radius in the homogeneous field region and L the characteristic length for the field variation. The result is (subscript e refers to electrons)

$$\gamma \sim k_y v_{the} \left(\frac{a_e}{L}\right)^{3/2} \tag{109}$$

where k_y is the wave number and v_{the} the electron thermal velocity.

We can expect that for $\frac{a_e}{L} << 1$(107) or (108) applies and it is in fact easy to recover (109) from (107).

To do this we consider the spectrum of the operator $\hat{R} = -i\hat{H}_o$. We note that

$$\hat{H}_o f = \frac{df}{d\tau} \tag{110}$$

where $\frac{df}{d\tau}$ is the rate of change of f that a particle on an equilibrium orbit would experience with time. The eigenvalue problem

$$\hat{R}f = \lambda f \tag{111}$$

is then readily solved by

$$f = f^{*} e^{-i\lambda\tau} \tag{112}$$

where f^{*} is constant on the equilibrium orbit considered.

It is easy to see that the contribution from those particles which move in the neutral region $|x| < d$ dominate. Since these orbits are not periodic, all values of λ in (112) are possible. Thus the spectrum is continuous and (107) will apply. Since v_y is a constant we write instead of (112)

$$f = f^{*} e^{-i\frac{\lambda}{v_y}y} \tag{113}$$

The marginal mode $\underline{\xi}_o$ has the form

$$\underline{\xi}_o \sim e^{ik_o y} \tag{114}$$

The quantity $\hat{E}_c'(\lambda)\xi_o$, which is the Fourier component (113) of $\underline{\xi}_o$ is therefore different from zero only if

$$\frac{\lambda}{v_y} = k_o .$$

Thus $\lambda \to 0$ implies $v_y \to o$ and therefore $\hat{E}'_c(o) \sim \delta(v_2)$

Under suitable conditions (Coppi et al. 1966) ϕ_1 can be neglected.

Then we find

$$(C\xi_o, |F'_o|E'_c C\xi_o) \sim \int d^3v \int_{-d_e}^{+d_e} v_z^2 \; \delta(v_y) |A_{1z}|^2 \; dx$$

The dominating term of W_o is

$$\int d^2x \; \frac{dj_o}{dA_o} \; |A_{1z}|^2 \; , \tag{115}$$

Using these expressions in an order of magnitude estimate one immediately recovers (109).

(b) Periodic neutral point equilibria.

We consider neutral point equilibria of the type (10c), the field lines are sketched in figure 2. Such configurations were discussed by Biskamp and Schindler (1971). The growth rate was estimated to be of the order

$$\gamma \sim k \; v_{the} \; \left(\frac{a_e}{L} \right)^{1/2} \tag{116}$$

In this case, it was shown that the pertinent particle orbits can be considered as being periodic. Therefore the spectrum is discontinuous ($\hat{E}_c = o$) and the lowest eigenvalue λ_1 is associated with the time a typical particle needs to go from one x-type neutral point to the next. Again, the dominating part of W_o is (115). By arguments analogous to those used in the example (a) we recover the growth rate (116) from (108).

From a rigorous view point the results of this paper provide the justification for the procedure used by Biskamp and Schindler (1971), who anticipated the instability criterion (106). A continuous change of the system into a marginal state can for instance be provided in the following way. The equilibria (10c) can be continuously transformed into a one-dimensional neutral sheet configuration by letting α approach zero. The plane sheet in turn can be stabilized by moving conducting walls is from infinity situating them sufficiently close to the sheet (Schindler 1972).

10. Summary and Discussion

We have developed a theory of stability of collision-free two-dimensional plasma equilibria. This theory is particularly suited for the discussion of stability of equilibria containing neutral points. Modes which are able to transport magnetic flux through neutral points are in typical cases sufficiently slow for the present theory to apply. By applying operator techniques we

(a) prove that the instabilities considered are purely growing

(b) recover criteria which are sufficient for stability and necessary and sufficient for the marginal state

(c) derive new sufficient criteria for instability

(d) provide general expressions for growth rates, from which we recover two special cases already published.

The theory applies to equilibria characterised by distribution functions which are arbitrary functions of the constants of the motion H and p_z. It can be expected that this class of equilibria becomes more and more representative the more complicated the particle orbits become. Clearly, the invariance of a second momentum component is already lost by going from one-dimensional to two-dimensional equilibria. More work however seems necessary to settle this question in a quantitative way.

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Figure Captions

Figure 1. Qualitative stability diagram for equilibria varying continuously with a parameter α .

Figure 2. Periodic pinch equilibrium with neutral points.

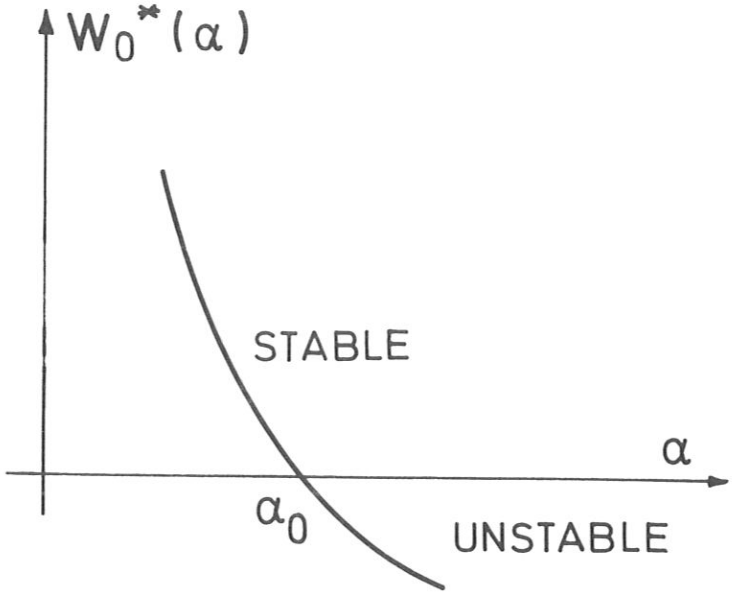

Fig. 1

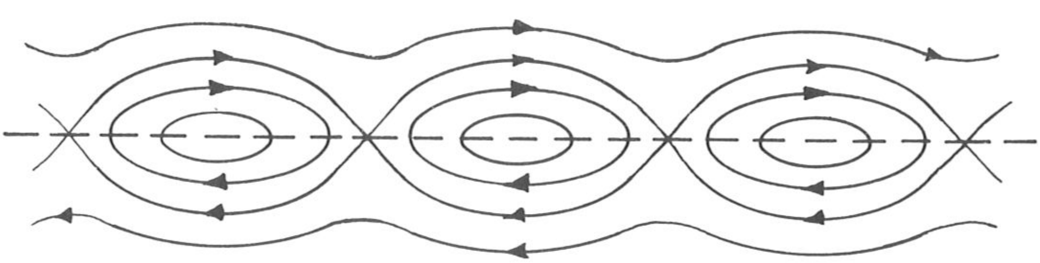

Fig. 2