On the Equilibrium Orbit and Linear Oscillations of Charged Particles in Axisymmetric E x B Fields and Application to the Electron Ring Accelerator.

M. Reiser +)

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Abstract

The first-order theory of charged particle motion in axisymmetric E x B fields is reviewed and extended to include both nonrelativistic as well as relativistic effects. The fields may be produced externally or by the particles themselves; the only restrictions are that they are independent of time and azimuth angle θ, have a plane of symmetry (median plane) and that E_θ and B_θ are zero. Results are applied to the electron ring accelerator. The self-fields of an electron ring with stationary ions are calculated for large aspect ratio; problems arising from the polarization of the ring are discussed and possible solutions suggested. Expressions for the betatron frequencies ν_r and ν_z for the electron motion in the ring are derived and compared with the results of Ivanov et al.[1] and Laslett[2].

I. Introduction

In connection with electron ring accelerator work I.N. Ivanov et al. [1] and L.J. Laslett [2] have calculated the particle oscillation frequencies about the equilibrium orbit taking into account the self fields of the beam as well as certain boundary conditions outside or inside the toroidal beam. They first derived expressions for the oscillation frequencies in a general axially symmetric field configuration for the ultra-relativistic case and then applied the results to the case of an electron ring in an external magnetic field.

In the following section of the present report we extend the theory of Ivanov and Laslett to the more general case which includes both the ultra-relativistic as well as the nonrelativistic limit. An additional focusing term is found in the expression for the radial frequency which is quite significant at nonrelativistic particle energies, goes to zero in the ultra-relativistic limit, and vanishes when either the magnetic or the electric field is zero. J.D.Lawson gave a general formula for the radial frequency in a recent report [3] which did not include this term; however, on checking his calculations he independently found that such an additional term arises although the exact form does not quite agree with the results presented here (private communication).

In Section 3 of this report we derive approximate expressions for the self fields of an electron ring loaded with a fraction f of ions. The results are in agreement with those given by Laslett for $f = 0$ and show a minor difference for the case $f \neq 0$. In addition we comment briefly on the polarization effects that occur when positive ions are loaded into the electron ring.

Lastly, in Section 4, expressions for the radial and axial betatron frequencies for an ion-loaded electron ring are presented and compared with the results given by Laslett.

II. Equilibrium Orbit and Oscillation Frequencies in a General Static E x B Field with Axial Symmetry.

We assume combined electric and magnetic fields with axial symmetry and which possess a median plane. The fields may be produced by charges and currents in conductors outside the region of interest as well as by the fields arising from the charges and currents of the particles that constitute the beam, i.e. the divergence of E and the curl of B need not be zero. In addition the direction of the field vectors, or, more precisely, the force vectors acting on a particle, must be such that local force balance with the centrifugal force can be achieved and an "equilibrium orbit" can be defined. Lastly we assume that particle displacements and velocities perpendicular to the equilibrium orbit are small compared to the orbit radius and the velocity along the equilibrium orbit, respectively.

The equations of motions (in cylindrical coordinates) will be used, and in the interest of clarity we present the derivation for the nonrelativistic case adding relativistic effects afterwards. The radial force equation may be written for electrons (q = -e) as

$$m\ddot{r} - mr\dot{\theta}^2 = -eE_r - er\dot{\theta}B_z , \qquad (1)$$

where $E_r = E_r(r,z)$, $B_z = B_z(r,z)$.

Define equilibrium orbit, $r = R$, in median plane ($z = 0$, $E_r = E_0$, $B_z = B_0$) such that balance exists between centrifugal, electric and magnetic force, i.e. $\ddot{r} = 0$ and

$$mR\dot{\theta}_0^2 = eE_0 + eR\dot{\theta}_0 B_0 ,$$

hence

$$\dot{\theta}_0^2 = \frac{eE_0}{mR} + \frac{eB_0}{m}\dot{\theta}_0 ,$$

or

$$\omega^2 = \omega_e^2 + \omega_m \omega . \qquad (2)$$

ω is the radian frequency of the particle at the equilibrium orbit, $\omega_e = (eE_0/mR)^{1/2}$ is the frequency associated with the particle motion in a purely electrostatic field, and $\omega_m = eB_0/m$ is the magnetic (or cyclotron) frequency. Solving (2) for ω yields

$$\omega = \dot{\theta}_0 = \frac{1}{2}\left[\omega_m \pm \left(\omega_m^2 + 4\omega_e^2\right)^{1/2}\right] . \qquad (3)$$

Since the velocity of the equilibrium orbit particle is $v_0 = R\dot{\theta}_0$ we can solve (2) for the orbit radius R and obtain

$$R = \frac{mv_0^2}{eE_0 + ev_0B_0} = \frac{mv_0}{eB_0}\left(1 + \frac{E_0}{v_0B_0}\right)^{-1} . \qquad (4)$$

The angular velocity of a particle off the equilibrium orbit is determined by the azimuthal force equation from which one gets (Busch's theorem!):

$$mr^2\dot{\theta} - \left(mr^2\dot{\theta}\right)_{r=R} = e\int_R^r B_z(r)\, r\, dr \qquad (5)$$

In linear approximation, writing $r = R + x$ ($x \ll R$), one gets $(mr^2\dot{\theta})_{r=R} = mR^2\dot{\theta}_0$ and $\int_R^r B_z r\, dr = B_0Rx$.

Hence $$\dot{\Theta} = \frac{mR^2\dot{\Theta}_0 + eB_0Rx}{mR^2(1+\frac{x}{R})^2} = \left(\dot{\Theta}_0 + \frac{eB_0}{m}\frac{x}{R}\right)\left(1-2\frac{x}{R}\right),$$

or
$$\dot{\Theta} = \dot{\Theta}_0 + \left(\frac{eB_0}{m} - 2\dot{\Theta}_0\right)\frac{x}{R} = \omega\left[1+\left(\frac{\omega_m}{\omega}-2\right)\frac{x}{R}\right]. \qquad (6)$$

From the equilibrium condition one has
$$\frac{\omega_m}{\omega} - 2 = 1 - \frac{\omega_e^2}{\omega^2} - 2 = -1 - \frac{\omega_e^2}{\omega^2} = -1 - \frac{eE_0R}{mv_0^2};$$
hence we may also write
$$\dot{\Theta} = \omega\left[1 - \left(1 + \frac{eE_0R}{mv_0^2}\right)\frac{x}{R}\right]. \qquad (7)$$

For the radial motion one obtains in linear approximation
$$\ddot{x} - R\omega^2\left(1+\frac{x}{R}\right)\left[1-2\left(1+\frac{eE_0R}{mv_0^2}\right)\frac{x}{R}\right] = -\frac{eE_0}{m} - \frac{eE'}{m}x - \frac{eR\omega}{m}\left(1+\frac{x}{R}\right)\left[1-\left(1+\frac{eE_0R}{mv_0^2}\right)\frac{x}{R}\right]\left[B_0+B'x\right],$$
where $E' = \left.\frac{\partial E_r}{\partial r}\right|_{r=R}$ and $B' = \left.\frac{\partial B_z}{\partial r}\right|_{r=R}$.

Dropping all quadratic terms in x after multiplication yields[+]
$$\ddot{x} - R\omega^2 + \omega^2\left(1+2\frac{eE_0R}{mv_0^2}\right)x = -\frac{eE_0}{m} - \frac{eE'}{m}x - \frac{eR\omega B_0}{m} - \frac{\omega e^2E_0RB_0}{m^2v_0^2}x + \frac{eR\omega}{m}B'x.$$

The encircled terms cancel (from the equilibrium condition) and we get
$$\ddot{x} + \omega^2\left[1 + 2\frac{eE_0}{mv_0^2} - \frac{e^2B_0E_0R}{m^2v_0^2\omega} + \frac{eE'}{m\omega^2} + \frac{eRB'}{m\omega}\right]x = 0 \qquad (8)$$

[+] The same result is obtained if one writes (correct to first order) $r\dot{\Theta} = v = v_0\left(1+\frac{dv}{v_0}\right)$, $r^2\dot{\Theta} = \frac{v^2}{r} = \frac{v_0^2}{R}\left(1+2\frac{dv}{v_0}\right)\left(1-\frac{x}{R}\right)$ in Equation (1), and substitutes
$\frac{dv}{v_0} = -\frac{eE_0x}{mv_0^2}$.

The terms inside the brackets represent the square of the radial focusing frequency, ν_r.
From the equilibrium condition

$$\frac{e^2 B_o E_o R}{m^2 v_o^2 \omega} = \frac{e E_o B_o R^2}{m v_o^2} = \frac{e E_o R}{m v_o^2}\left(1 - \frac{e E_o R}{m v_o^2}\right).$$

Employing this relation one finds

$$\nu_r^2 = 1 + 2\frac{e E_o R}{m v_o^2} - \frac{e E_o R}{m v_o^2} + \frac{e^2 E_o^2 R^2}{m^2 v_o^4} + \frac{e E' R^2}{m v_o^2} + \frac{e B' R^2}{m v_o},$$

or, in view of $\frac{m v_o^2}{R} = eE_o + e v_o B_o$:

$$\nu_r^2 = 1 + \frac{E_o^2}{(E_o + v_o B_o)^2} + \frac{E_o}{E_o + v_o B_o} + \frac{R\left(\frac{\partial E_r}{\partial r} + v_o \frac{\partial B_z}{\partial r}\right)}{E_o + v_o B_o} \tag{9}$$

(Nonrelativistic case).

The analysis for the general relativistic case follows the same pattern. For the mass m one has to write γm_o, and in linear approximation the fractional change of γ for a particle off the equilibrium orbit is $\frac{d\gamma}{\gamma} = -\frac{e E_o x}{\gamma m_o c^2}$. Thus the mass m has to be replaced by $\gamma m_o\left(1 - \frac{e E_o x}{\gamma m_o c^2}\right)$ in our previous equations. The derivation is straightforward, though a little tedious, and one obtains the following results:

$$\dot{\Theta} = \omega\left[1 - \left(1 + \frac{e E_o R}{m_o c^2 \gamma^3 \beta^2}\right)\frac{x}{R}\right] \tag{10}$$

$$\nu_r^2 = 1 + \frac{E_o^2(1-\beta^2)}{(E_o + \beta c B_o)^2} + \frac{E_o}{E_o + \beta c B_o} + \frac{R\left(\frac{\partial E_r}{\partial r} + \beta c \frac{\partial B_z}{\partial r}\right)}{E_o + \beta c B_o} \tag{11}$$

Eq. (11) is identical with the nonrelativistic expression (9) except for the factor $(1 - \beta^2)$ in the second term on the right.

If neither E_o nor B_o is zero we can introduce the electric and magnetic field index

$$k_e = \frac{R}{E_o}\left.\frac{\partial E_r}{\partial r}\right|_{r=R}, \quad k_m = \frac{R}{B_o}\left.\frac{\partial B_z}{\partial r}\right|_{r=R}; \text{ also,}$$

following Lawson [3], we define $\beta_o = \frac{E_o}{cB_o}$. With this notation the expression for ν_r^2 may be written in the form

$$\nu_r^2 = 1 + \frac{\beta_o^2(1-\beta^2)}{(\beta_o+\beta)^2} + \frac{\beta_o}{\beta_o+\beta} + \frac{\beta_o k_e}{\beta_o+\beta} + \frac{\beta k_m}{\beta_o+\beta}. \quad (12)$$

Of interest are the following limits:

a. $E_o = 0$, $k_e = 0$ (magnetostatic case)

Here

$$\nu_r^2 = 1 + k_m \quad (13)$$

b. $B_o = 0$, $k_m = 0$ (electrostatic case)

$$\nu_r^2 = 3 - \beta^2 + k_e \quad (14)$$

c. Ultrarelativistic limit ($\beta^2 = 1$)

$$\nu_r^2 = 1 + \frac{E_o}{E_o + cB_o} + \frac{R\left(\frac{\partial E_r}{\partial r} + c\frac{\partial B_z}{\partial r}\right)}{E_o + cB_o}, \quad (15)$$

or

$$\nu_r^2 = 1 + \frac{\beta_o}{1+\beta_o} + \frac{\beta_o k_e}{1+\beta_o} + \frac{k_m}{1+\beta_o}.$$

Equations (13) and (14) are in agreement with the well known solutions for a purely magnetic or purely electric field. Eq. (15) agrees with both Laslett's [2] as well as Lawson's [3] results; the main difference in this analysis is the additional term $\frac{E_o^2(1-\beta^2)}{(E_o+\beta cB_o)^2}$ in the general expression, Eqs.(11) and (12) for ν_r^2. This term may be quite significant at low (nonrelativistic) kinetic energies, but in electron ring accelerator application, where $\beta^2 \simeq 1$, it can be neglected for most practical purposes.

The frequency ν_z of the axial motion can be derived from the axial force equation (in relativistic form):

$$m\ddot{z} = \gamma m_o \ddot{z} = -eE_z + er\dot{\theta}B_r \quad (16)$$

In the midplane (z = 0) one has $r = R$, $\dot{\theta} = \dot{\theta}_0 = \omega$, $E_z(R,0) = 0$ and $B_r(R, 0) = 0$. The field components outside the midplane are then in linear approximation $E_z(R,z) = \left.\frac{\partial E_z}{\partial z}\right|_{z=0} z$, $B_r(R,z) = \left.\frac{\partial B_r}{\partial z}\right|_{z=0} z$ and one obtains

$$\ddot{z} = -\frac{e}{\gamma m_0}\frac{\partial E_z}{\partial z} z + \frac{e R\dot{\theta}_0}{\gamma m_0}\frac{\partial B_r}{\partial z} z,$$

or

$$\ddot{z} + \omega^2\left[\frac{e}{\gamma m_0 \omega^2}\frac{\partial E_z}{\partial z} + \frac{eR}{\gamma m_0 \omega}\frac{\partial B_r}{\partial z}\right] z = 0 . \tag{17}$$

From the equilibrium condition $R\gamma m_0\omega^2 = eE_0 + e\beta c B_0$; hence

$$\ddot{z} + \omega^2 \frac{R\left(\frac{\partial E_z}{\partial z} - \beta c \frac{\partial B_r}{\partial z}\right)}{E_0 + \beta c B_0} z = 0,$$

from which

$$\nu_z^2 = \frac{R\left(\frac{\partial E_z}{\partial z} - \beta c \frac{\partial B_r}{\partial z}\right)}{E_0 + \beta c B_0} . \tag{18}$$

In the purely magnetostatic case, $E_z = 0$, $\frac{\partial E_z}{\partial z} = 0$, and

$$\nu_z^2 = -\frac{R}{B_0}\frac{\partial B_r}{\partial z} . \tag{19}$$

On the other hand, if $B_0 = 0$, $\frac{\partial B_r}{\partial z} = 0$ (electrostatic case), one gets

$$\nu_z^2 = \frac{R}{E_0}\frac{\partial E_z}{\partial z} . \tag{20}$$

For the special case where the self fields of the particles can be neglected, i.e. the fields are produced by external sources only, one gets from $\operatorname{div}\vec{D} = \rho = 0$:

$$\frac{\partial E_z}{\partial z} = -\frac{E_0}{R} - \frac{\partial E_r}{\partial r} . \tag{21}$$

Also $\operatorname{curl}\vec{B} = \vec{J} = 0$, and therefore

$$\frac{\partial B_r}{\partial z} = \frac{\partial B_z}{\partial r} . \tag{22}$$

Under these conditions Eq. (18) may be written in the form

$$\nu_z^2 = - \frac{E_0 + R\frac{\partial E_r}{\partial r}}{E_0 + \beta c B_0} - \frac{R\beta c \frac{\partial B_z}{\partial r}}{E_0 + \beta c B_0} . \tag{23}$$

If $E_0 \neq 0$ and $B_0 \neq 0$ we can introduce the field indices k_e and k_m and write

$$\nu_z^2 = - \frac{(1 + k_e) E_0}{E_0 + \beta c B_0} - \frac{k_m \beta c B_0}{E_0 + \beta c B_0} \tag{24}$$

or, with $\beta_0 = \dfrac{E_0}{B_0 c}$

$$\nu_z^2 = \frac{-(1 + k_e)\beta_0 - k_m \beta}{\beta_0 + \beta} . \tag{25}$$

It should be noted that the general expression for ν_z^2 given in Eq. (18) is in agreement with Laslett's results (for $\beta = 1$) while Eq. (25) is in the form given by Lawson (private communication).

III. Electric and Magnetic Self Fields of an Electron Ring Loaded with Positive Ions.

The determination of the self fields of a toroidal electron beam in free space involves complete elliptical integrals, i.e. one does not get convenient analytical expressions but, in general, has to employ the computer and present the results in tabulated form, as was done by Laslett [4] and Luccio [5], for instance. For a rough evaluation of self field effects under various conditions, however, analytical expressions or approximations are almost indispensable.
In the case of the electron ring such approximate analytical expressions can be obtained when the minor dimensions of the toroidal beam are small compared to the diameter of the ring. Laslett [2] derived such simple expressions and used them in the calculation of ν_r and ν_z for a relativistic electron ring. In the following we present a derivation which avoids the elliptical integrals and perhaps illuminates the physical picture a little better from a somewhat more direct angle.

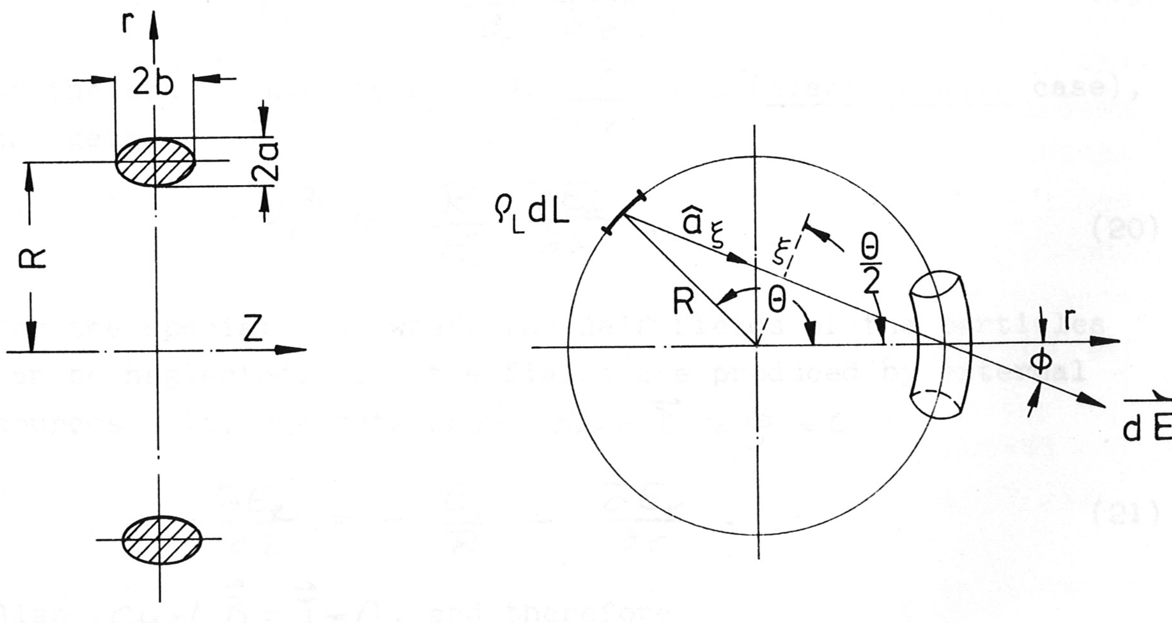

Fig. 1

Consider a toroidal beam of N_e electrons (moving in azimuthal direction with velocity $v = \beta c$) and N_i stationary ions, major radius R and elliptical cross section with minor radii a and b as shown in Fig. 1. Assume $a \ll R$, $b \ll R$ and uniform charge and current density.

For a straight beam ($R \rightarrow \infty$) of elliptical cross section, semi - axis a in x direction, b in y direction, total current I and uniform charge density $\rho = I/ab\pi v$, the two components of the electric field inside the beam are

$$E_x = \frac{\rho}{\varepsilon_0}\frac{bx}{a+b} = \frac{I}{\varepsilon_0 ab\pi v}\frac{bx}{a+b}, \tag{26}$$

$$E_y = \frac{\rho}{\varepsilon_0}\frac{ay}{a+b} = \frac{I}{\varepsilon_0 ab\pi v}\frac{ay}{a+b}. \tag{27}$$

The magnetic field components are given by

$$B_x = -\frac{\mu_0 I}{ab\pi}\frac{ay}{a+b} = -\frac{v}{c^2}E_y, \tag{28}$$

$$B_y = \frac{\mu_0 I}{ab\pi}\frac{bx}{a+b} = \frac{v}{c^2}E_x. \tag{29}$$

If the beam is bent into a toroidal shape, the above field distribution will be distorted. The charge and current elements in distant parts of the ring produce at any given point a "bias" field, as Laslett calls it, which crudely speaking is superimposed on the straight beam configuration. Indeed, if the major radius R of the ring is large in comparison to the minor dimensions, we can approximate the (axisymmetric!) field at any given point (r, z) within the beam by a superposition of the straight-beam term and the bias effect; thus, for instance, we may write for the radial field component $E_r(r,z) = E_r^{bias} + E_r^{straight}$.

In carrying out the analysis along this line, we shall restrict ourselves to the two orthogonal surfaces $r = R$ and $z = 0$. For the straight-beam terms we shall use Equations (26) to (29) substituting $r - R$ for x and z for y. The bias fields will be evaluated at $r = R$, $z = 0$ by approximating the toros with a circular line charge or filamentary current loop (neglecting the finite width 2a and 2b in the two directions). The bias field at any other point $(r, 0)$ or (R, z) is then obtained by linear expansion using the field gradients at (R, 0). Thus one obtains for the contribution to the electrical bias field at point $r = R$ on the r-axis (see Fig. 1) resulting from the line charge element $\rho_L \, dL$ at distance ξ

$$d\vec{E}^{\,bias} = \frac{\rho_L \, dL \, \hat{a}_\xi}{4\pi\varepsilon_0 \xi^2} .$$

Due to symmetry only the radial component $dE_r = dE \cos\phi$ will survive in the summation. Now $dL = R d\theta$, $\cos\phi = \sin\frac{\theta}{2}$, $\xi = 2R \sin\frac{\theta}{2}$, and the integral may be written as

$$E_r^{bias}(R,0) = \int_0^{2\pi} \frac{\rho_L R \, d\theta \sin\frac{\theta}{2}}{4\pi\varepsilon_0 \xi^2} = 2\int_0^{\pi} \frac{\rho_L \, d\theta}{16\pi\varepsilon_0 R \sin\frac{\theta}{2}} , \tag{30}$$

which yields

$$E_r^{bias} = \frac{\rho_L}{4\pi\varepsilon_0 R} \ln \tan\frac{\theta}{4} \Big|_0^{\pi} . \tag{31}$$

To avoid the singularity problem at the lower limit we replace $\theta = 0$ by $\theta = \alpha \frac{a}{R}$ (where $0 < \alpha < 1$), i.e. we exclude a small part of the ring (of width $2\alpha \frac{a}{R}$) near the field point (R, 0) from the integration. This is justified as the charges in this section of the ring near the field point do not contribute to the "bias" field and are taken into account in the straight beam term. If one puts $\alpha = \frac{1}{2}$ and $\tan\frac{\theta}{4} = \tan\alpha\frac{a}{4R} = \tan\frac{a}{8R}$, one obtains from Eq. (31)

$$E_r^{bias} = \frac{\varrho_L}{4\pi\varepsilon_0 R}\ln\frac{8R}{a}, \tag{32}$$

which agrees with the result that Laslett derived [2] from evaluating the elliptical integral (if one replaces a by $\bar{b} = \frac{a+b}{2}$).

The magnetic bias field can be calculated in similar fashion. For the contribution due to a current element $I\vec{dL}$ one obtains at the field point (R, 0) from Biot-Savart' Law

$$d\vec{B}^{bias} = \mu_0 \frac{I\,\vec{dL} \times \hat{a}_\xi}{4\pi\,\xi^2}$$

Only the z-component survives the integration which yields

$$B_z^{bias}(R,0) = \frac{\mu_0 I}{4\pi R}\ln\frac{8R}{a}, \tag{33}$$

i.e. $B_z^{bias} = \frac{v}{c^2}E_r^{bias}$ (since $\varrho_L v = I$, $\frac{1}{\mu_0\varepsilon_0} = c^2$).

To obtain the radial gradients due to the bias field at the point (R, 0) we simply differentiate E_r^{bias} and B_z^{bias} with respect to R. In doing so we have to recognize that the fields according to Eqs. (32) and (33) exhibit only a $1/R$ dependence, i.e. $\ln\frac{8R}{a}$ has to be considered as a constant geometry factor. Then $\frac{\partial E_r}{\partial r} = -\frac{1}{R}E_r$ and $\frac{\partial B_z}{\partial r} = -\frac{1}{R}B_z$, i.e.

$$\left.\frac{\partial E_r^{bias}}{\partial r}\right|_{r=R} = -\frac{\varrho_L}{4\pi\varepsilon_0 R^2}\ln\frac{8R}{a}, \tag{34}$$

$$\left.\frac{\partial B_z^{bias}}{\partial r}\right|_{r=R} = -\frac{\mu_0 I}{4\pi R^2}\ln\frac{8R}{a}. \tag{35}$$

To find the gradients of the bias force in the z direction, $\frac{\partial E_z}{\partial z}$ and $\frac{\partial B_r}{\partial z}$, we make use of the divergence and curl conditions. Here we have to recognize that the bias fields are generated by the charges and currents in the distant parts of the ring. (The local singularity was excluded in the integration.) Consequently, we have div $\vec{E}^{bias} = 0$ and curl $\vec{B}^{bias} = 0$ and we get

$$\left.\frac{\partial E_z}{\partial z}\right|_{r=R}^{bias} = -\frac{E_r}{R} - \frac{\partial E_r}{\partial r} = 0, \tag{36}$$

and

$$\left.\frac{\partial B_r}{\partial z}\right|_{r=R}^{bias} = \frac{\partial B_z}{\partial r}^{bias} = -\frac{\mu_0 I}{4\pi R^2} \ln \frac{8R}{a}. \tag{37}$$

In applying the previous results to an electron ring containing N_e electrons (velocity βc) and $N_i = f\, N_e$ stationary ions we have to substitute $\rho_L = -\frac{eN_e(1-f)}{2\pi R}$ and $I = -\frac{eN_e}{2\pi R}\beta c$.

The total radial bias force acting on an electron at $r = R$, $z = 0$ may be written as

$$F_r^{bias} = -eE_r^{bias} - e\beta c\, B_z^{bias}$$

$$F_r^{bias} = \frac{e^2 N_e(1-f+\beta^2)}{8\pi^2 \varepsilon_0 R^2} \ln \frac{8R}{a}. \tag{38}$$

This force is radially outward and hence tends to enlarge the electron equilibrium orbit in comparison to the single-particle cyclotron orbit in an applied external magnetic field. The corresponding orbit radius is readily obtained from the balance equation between centrifugal and bias force on one side (radially outward) and the inward Lorentz force due to the applied field on the other hand. It should be pointed out, however, that this radius is not identical with the major (or mean) radius R of the torus. This is due to the fact that the positive ions experience an inward radial force due to the bias field. The net result is that the center of mass of the electron distribution and that of the ions are radially separated; in other words, the assumption that the ion density at any given point is proportional to the electron density is not exactly valid in this case. The mean radius R of the torus will have a value which should be less than that obtained from Eq. (38). We can calculate it approximately by treating the ring like a rigid body where the electron-ion binding force reduces the bias force by a factor 1 - f. Hence instead of eE_r (electrostatic force on an electron) one has $(1 - f)eE_r$, and instead of Eq. (38) one has to write

$$F_r^{bias} = \frac{e^2 N_e \left[(1-f)^2 + \beta^2\right]}{8\pi^2 \varepsilon_0 R^2} \ln \frac{8R}{a} . \tag{39}$$

This result is in agreement with the expressions for the ring equilibrium derived by Linhart[6]; in other models[7] on ring equilibria the ion concentration appears in the form of a linear term (1 - f).

The addition of positive ions to a ring beam of relativistic electrons in free space thus results in the formation of two subrings which are radially separated. The net result of this effect is a decrease in the holding power and an intrinsic polarization which leads to the dipole oscillations of the two subrings treated theoretically by Koshkarev and Zenkevich[8]. The separation of the center of mass of the two beams and the associated dipole oscillations were observed in a numerical computer study by Boris and Lee[9] on ring beams in which the initial density of positive ions was proportional to that of the electrons.

It should be pointed out that the radial polarization of the ring can be avoided by applying an external electric field which shifts the negative potential minimum of the beam from the inner edge to the mean radius R. In this case the center of mass of the two subrings coincide. Such an external field can be provided by putting a conducting rod inside and a cylindrical boundary outside of the beam. The electric bias field at $r = R$ can then be cancelled either by a suitable choice of the radii of the two conductors (if they are both at the same potential) or by applying a potential difference between them. An inner conductor will be employed in the Maryland ERA system[10] and in the Garching experiment when a B_Θ field[11] is generated by an axial current.

After these remarks we shall now proceed with our analysis of the self-field terms needed to compute the radial and axial oscillation frequencies of the electrons. In doing so we shall ignore the polarization effect, i.e. we will use Eq. (38) for the bias force acting on the electrons at the mean radius R and expand linearly about R to obtain the force on an electron displaced from R in radial or axial direction. For the latter case we need the radial and axial gradients of the bias force at $r = R$ and $z = 0$. First we have

$$\frac{\partial F_r^{bias}}{\partial r} = -\frac{1}{R} F_r^{bias} = -\frac{e^2 N_e (1 - f + \beta^2)}{8\pi^2 \varepsilon_0 R^3} \ln \frac{8R}{a} . \qquad (40)$$

This is in agreement with Laslett[2], ERAN-30, p.3, if we put $f = 0$ and $\beta^2 = 1$. The axial gradient of the bias force is

$$\frac{\partial F_z^{bias}}{\partial z} = \frac{e^2 N_e \beta^2}{8\pi^2 \varepsilon_0 R^3} \ln \frac{8R}{a} . \qquad (41)$$

The factor 1 - f is absent in this expression due to the fact that $\frac{\partial E_z^{bias}}{\partial z} = 0$, from Eq. (36). Note that the bias field gives rise to a focusing force gradient in radial direction and a defocusing gradient in axial direction.

The "straight" beam effect produces a force $F_r^{straight}$ on an electron at radius r= R + x which from Eqs. (26) and (29) may be written as

$$F_r^{straight} = \frac{e^2 N_e (1-f-\beta^2)\, x}{2\pi^2 \varepsilon_0 R\, a(a+b)} \quad (42)$$

For the "straight" beam force in axial direction (on an electron at distance z from the midplane) one obtains from Eqs. (27) and (28), putting y = z:

$$F_z^{straight} = \frac{e^2 N_e (1-f-\beta^2)\, z}{2\pi^2 \varepsilon_0 R\, b(a+b)} \quad (43)$$

Both force terms are focusing whenever $f > 1 - \beta^2$ or $f > \frac{1}{\gamma^2}$ (Bennett - Budker condition). It should be pointed out that the factor 1 - f implies singly charged positive ions. If the ions are multiply charged, i.e. Z electrons removed from the neutral atom, we must replace 1 - f by 1 - Zf in all previous equations. Since the bias force according to Eq. (41) is defocusing, the amount of positive ions must be increased sufficiently above the Bennett-Budker limit of $Zf = 1 - \beta^2$ in order to obtain satisfactory focusing in axial direction.

IV. Focusing Frequencies in an Electron Ring.

Following the notation of Soviet authors and Laslett we introduce the dimensionless parameters $\mu = \nu/\gamma$ and $P = 2 \ln \frac{8R}{\bar{b}}$, where $\bar{b} = \frac{a+b}{2}$ is substituted for a in $\ln \frac{8R}{a}$ and (MKS units):

$$\mu = \frac{\nu}{\gamma} = \frac{e^2 N_e / 2\pi R}{4\pi\varepsilon_0 m_0 c^2} \frac{1}{\gamma} = \frac{e^2 N_e}{8\pi^2 \varepsilon_0 R m_0 c^2 \gamma} . \tag{44}$$

We also define the total guide field, B_g, from the force-equilibrium condition:

$$\begin{aligned} \frac{\gamma m_0 v^2}{R} = e v B_g &= e E_0 + e v B_0 \\ &= e\left(E_r^{bias} + E_r^{appl.}\right) + e v\left(B_z^{bias} + B_z^{appl.}\right). \end{aligned} \tag{45}$$

Hence

$$B_g = \frac{\gamma m_0 \beta c}{e R} . \tag{46}$$

The applied electric field, $E_r^{appl.}$, depends on the potential difference between inner and outer conductors. In the following we shall assume that $E_r^{appl.} = 0$ and that no conductors or other boundaries are near the ring. The bias field is then unaffected by image charges and we can use Eq. (32); introducing $\rho_L = \frac{e N_e (1-f)}{2\pi R}$ and the parameters P, μ and B_g we can write

$$E_r^{bias} = -\frac{c}{\beta} B_g \mu P \frac{1-f}{2} . \tag{47}$$

Likewise

$$B_z^{bias} = -B_g \frac{\mu P}{2} , \tag{48}$$

and the total bias force

$$F_r^{bias} = \frac{ec}{\beta} B_g \mu P \frac{1-f+\beta^2}{2}. \quad (49)$$

Substituting $E_r^{appl.} = 0$ and Eqs. (47), (48) into (45) we may write the applied magnetic field in the form

$$B_z^{appl.} = B_g \left[1 + \mu P \frac{1-f+\beta^2}{2\beta^2} \right]. \quad (50)$$

The gradients due to the bias field may be written as

$$R\left(\frac{\partial E_r^{bias}}{\partial r} + \beta c \frac{\partial B_z^{bias}}{\partial r} \right) = \frac{c}{\beta} B_g \mu P \frac{1-f+\beta^2}{2}, \quad (51)$$

and

$$R\left(\frac{\partial E_z^{bias}}{\partial z} - \beta c \frac{\partial B_r^{bias}}{\partial z} \right) = -c\beta B_g \frac{\mu P}{2}. \quad (52)$$

The gradients due to an applied electric field we assume to be zero. For the applied magnetic field we introduce the field index $n = -\frac{R}{B_z^{appl.}} \frac{\partial B_z^{appl.}}{\partial r}\Big|_{r=R} = -k_m$. (53)

Then

$$R\beta c \frac{\partial B_z^{appl.}}{\partial r} = R\beta c \frac{\partial B_r^{appl.}}{\partial z} = -\beta c n B_g \left[1 + \mu P \frac{1-f+\beta^2}{2\beta^2} \right]. \quad (54)$$

The gradients resulting from the straight beam effect are readily obtained from Eqs. (42) and (43). Accordingly

$$R\left(\frac{\partial E_r^{straight}}{\partial r} + \beta c \frac{\partial B_z^{straight}}{\partial r} \right) = -\frac{R}{e} \frac{\partial F_r^{straight}}{\partial x}$$

$$= -\frac{4 m_0 c^2 R \gamma \mu}{e} \frac{1-f+\beta^2}{a(a+b)} \quad (55)$$

or, dividing by the total guide field,

$$\beta c B_g = \frac{\gamma m_0 \beta^2 c^2}{eR} = E_0 + \beta c B_0 :$$

$$\frac{R\left(\frac{\partial E_r}{\partial r} + \beta c \frac{\partial B_z}{\partial r}\right)^{\text{straight}}}{E_0 + \beta c B_0} = -\frac{4\mu R^2}{a(a+b)} \frac{1-f+\beta^2}{\beta^2}. \tag{56}$$

Likewise

$$\frac{R\left(\frac{\partial E_t}{\partial z} + \beta c \frac{\partial B_r}{\partial z}\right)^{\text{straight}}}{E_0 + \beta c B_0} = -\frac{4\mu R^2}{b(a+b)} \frac{1-f+\beta^2}{\beta^2}. \tag{57}$$

To calculate the radial frequency ν_r according to Eq. (11) we also need the bias force term

$$\frac{E_0}{E_0 + \beta c B_0} = \frac{E_r^{\text{bias}}}{\beta c B_g} = -\mu P \frac{1-f}{2\beta^2} \tag{58}$$

Putting it all together we then find for the radial oscillation frequency, ν_r, of the electrons in an ion-loaded ring the following expression:

$$\nu_r^2 = 1 + \frac{\mu^2 P^2 (1-f)^2 (1-\beta^2)}{4\beta^4} - \frac{\mu P (1-f)}{2\beta^2} +$$

$$+ \mu P \frac{1-f+\beta^2}{2\beta^2} - n\left[1 + \mu P \frac{1-f+\beta^2}{2\beta^2}\right] +$$

$$+ \frac{4\mu R^2}{a(a+b)} \frac{f-(1-\beta^2)}{\beta^2},$$

or

$$\nu_r^2 = 1 + \frac{\mu^2 P^2 (1-f)^2}{4\beta^4 \gamma^2} + \frac{\mu P}{2} - n\left(1 + \mu P \frac{1-f+\beta^2}{2\beta^2}\right)$$

$$+ \frac{4\mu R^2}{a(a+b)\beta^2}\left(f - \frac{1}{\gamma^2}\right). \tag{59}$$

The parameter μ can be calculated from the total number of electrons, N_e, in the ring, the major radius, R, and the total energy of the electrons $\gamma m_0 c^2$ according to the relation

$$\mu = 4.58 \times 10^{-14} \frac{N_e}{R_{[cm]}\,\gamma} \quad (60)$$

In the ultrarelativistic limit ($\beta^2 = 1$) the second term in Eq. (59) becomes negligible (as long as μ remains small). Our result then agrees with Laslett's [2] expression, ERAN-30, p. 22, except for the third term where he has $\frac{\mu P}{2}(1 - f)$ instead of $\frac{\mu P}{2}$. +)

For the axial oscillation frequency one obtains by substituting Eqs. (45), (52), (54) and (57) into Eq. (18):

$$\nu_z^2 = -\frac{\mu P}{2} + n\left(1 + \mu P\,\frac{1 - f + \beta^2}{2\beta^2}\right) + \frac{4\mu R^2}{b(a+b)}\,\frac{f - (1 - \beta^2)}{\beta^2}.$$

or

$$\nu_z^2 = n + \frac{4\mu R^2}{b(a+b)\beta^2}\left(f - \frac{1}{\gamma^2}\right) - \frac{\mu P}{2} + n\mu P\,\frac{1 - f + \beta^2}{2\beta^2}. \quad (61)$$

For $\beta^2 = 1$ Eq. (61) agrees with Laslett's[2] result, except for the third term where he has $\frac{\mu P}{2}(1 - f)$. +)

If boundaries and an applied electric field are present we have to add terms which account for the external field and the image effects. In ERAN-30, Laslett included image effects from boundaries outside the beam in his expressions for ν_r^2 and ν_z^2. As noted earlier, these terms would change if boundaries exist both inside and outside of the ring beam. In particular, the bias force can be cancelled and better axial focusing may be achieved by application of a potential difference between inner and outer conductor and/or by suitable design of the geometry and nature of the boundaries (conductors, dielectrics, etc.).

+) Footnote: A recent conversation with Dr. Laslett has served to confirm that he did not undertake in his work [2] to make the separation of the radial and axial bias force gradients into their electric and magnetic parts that would be required for a unique incorporation of the ion-loading factor f into the μ P term. He did this in the expectation that when $f \neq 0$ such toroidal terms in practice will be small compared to the straight-beam effects.

References

1. I.N. Ivanov *et al.*, JINR Report P9-4132 (1968).

2. L.J. Laslett, ERAN-30 (1969), Lawrence Radiation Laboratory, Berkeley.

3. J.D. Lawson, A Simple Model of a Rotating Layer of Charged Particles in a Magnetic Field, Technical Report, Dept. of Physics & Astronomy, University of Maryland, 1972 (unpublished).

4. L.J. Laslett, ERAN-43 (1969).

5. A.U. Luccio, ERAN-35 (1969).

6. S.G. Linhart, Influence of Self Fields on the Formation of a Relativistic Electron Ring, Technical Report No. 72-041, Sept. 1971, Dept. of Physics & Astronomy, University of Maryland, College Park, Md. (unpublished).

7. A comparison of various models was made by Lawson in Report RHEL/M/A4, Aug. 1971, Rutherford High Energy Laboratory (unpublished).

8. D.G. Koshkarev and P.R. Zenkevich, Particle Accelerators 3, 1 (1972).

9. J.P. Boris and R. Lee, Computations on Relativistic Electron Beams and Rings, Technical Report, Plasma Physics Division, Naval Research Laboratory, Washington, D.C. (unpublished).

10. M. Reiser, IEEE Trans. Vol. NS-18, 460 (1971).

11. L.J. Laslett and U. Schumacher, Proc. of 8th Int. Conf. on High Energy Accelerators, CERN 1971, p. 468.