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STOCHASTIC ACCELERATION
- A REVIEW - +)

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A b s t r a c t

In this paper the motion of charged particles in externally applied (or otherwise specified) random electric fields is reviewed. Several existing results on stochastic acceleration are rederived as simple consequences of the random walk process in the velocity space. Experimental evidence as well as the possibility of using stochastic heating for thermonuclear ignition is discussed.

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I. INTRODUCTION

In the nth step of the random walk (Fig. 1) problem

$$\Delta x = \lambda \cos\theta_n$$

$$\Delta y = \lambda \sin\theta_n$$

where λ is the step size and θ is the angle with the x-axis. After N steps

$$\begin{aligned} z^2 = x^2 + y^2 &= \left(\sum^N \Delta x\right)^2 + \left(\sum^N \Delta y\right)^2 \\ &= \lambda^2 \left\{ \left(\sum^N \cos\theta_i\right)^2 + \left(\sum^N \sin\theta_i\right)^2 \right\} \\ &= N\lambda^2 + \lambda^2 \sum_{i\neq j}\sum (\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j) \end{aligned} \qquad (1.1)$$

or

$$z^2 = N\lambda^2 + \lambda^2 \sum_{i\neq j}\sum \cos(\theta_i - \theta_j) \qquad (1.2)$$

If the walk is "truly random", then for a large ensemble of such walks each lasting N steps, the average contribution from the last term of (1.2) vanishes so that

$$\langle z^2 \rangle = N\lambda^2 \qquad (1.3)$$

where the angle brackets denote expected or average value. The result is readily generalized to include randomly varying step size λ_n so that (1.3) becomes

$$\langle z^2 \rangle = N \langle \lambda^2 \rangle \ . \qquad (1.4)$$

Furthermore, the summations of the type $(\sum^N \Delta x)^2$ in (1.1) possess a "gaussian" or a normal distribution with a well defined variance as can be shown using the "central limit theorem" of statistics.[1,2] This is the classical problem of "drunken walk" or Brownian motion studied by Einstein[3] who formulated the elegant relationship between diffusion and mobility.

There are several subtle assumptions implied in the above derivation. An ensemble is a collection of realizations of a process, which although differing in detail have certain common macroscopic statistical properties. Thus, in deriving (1.4), the average step size in each realization or "trial" is assumed constant. This is the statement of "stationarity". The qualification "truly random" demands that the process be memoryless, i.e. after each step, the process starts anew. Such a process is called a Markov process. We shall later show how seemingly innocuous appearing constraints can render a process non-Markovian and care must be exercised in detecting such correlations while studying random processes. This is best accomplished by the use of correlation functions which automatically help to determine the "state of the drunk" or the degree of statistical independence.

Before we formulate the problem of stochastic acceleration using correlation techniques, it is instructive to gain some physical insight into the problem by studying it as a random walk in velocity space. A stochastic electric field may be regarded as a sinusoidal waveform dephased at random intervals. Consider a segment of this stochastic field with a duration of two cycles and at the <u>cyclotron frequency</u> applied to a charged particle in a direction transverse to the static magnetic field. The particle undergoes either acceleration and spirals outwards (Fig. 2a) or it suffers deceleration and spirals inwards (Fig. 2b) depending upon its relative phase φ with the electric field. The force on the particle is $qE\cos\theta$ and its energy change is given by the product of this force with the distance moved,

$$\Delta \mathcal{E} = q E \cos\varphi \, \Delta s \tag{1.5}$$

Assuming that φ can equally likely have any value between 0 and 2π, it is at once clear that on probability basis there is net energy imparted to the particle because Δs for an accelerating particle is greater than that for a decelerating particle. If the electric field is switched on repeatedly but each time with a randomly shifted phase, the particle will continually extract energy from the field. From the "central-limit-theorem" it is once more evident that the expected energy distribution follows the well known "gaussian" or Maxwellian curve associated with thermodynamical equilibrium in statistical mechanics. The above observations are independent of the magnetic field strength and are valid even for the case $B_o=0$. Stochastic heating viewed in this manner, is a differential process with the inphase particle staying a little longer in the field than the out-of-phase particle and is sometimes compared with Landau damping. Instead of switching the electric field phase we could cause the particle to have a collision which would result in identical energy absorption. For this reason stochastic heating is often compared with collisional absorption. This is a perfectly valid viewpoint so long as we confine our attention to an ensemble of realizations of a single particle and concern ourselves with the energy absorption rate only. A hasty extension of this conclusion, however, to an ensemble of particles (as opposed to an ensemble of realizations of a single particle) is frought with serious pitfalls. Although, the electric field of Fig. 2a would impart net energy to the ensemble of particles initially spinning with random phases, a repetitive application of such field segments with random dephasing would cause exactly the same velocity change in all the particles which could hardly be regarded as heating. Collisions, on the contrary, cause individual members of the ensemble to select their own dephasing times and angles so that each particle experiences an almost independent realization of the electric field ensemble so that the heating results almost certainly in a Maxwellian velocity distribution. The qualifications "almost" are genuine and no qualitative proof is advanced for either one of the two assertions which find tacit acceptance through usage and experience. In stochastic heating theories this irksome problem is either ignored or masked in complex models using spatially varying random fields and Fokker-Planck formalism or patched up with the equally irritating assumption that each particle manages to experience an independent electric field through some unspecified mechanism. Clearly, such theories would remain of little practical significance unless it could be shown that in normally occurring situations, the particles indeed move independently of one another (note that interparticle interactions have been neglected and are not pertinent to this discussion) without benefit of selective application of a separate field to each one of them. One obvious possibility is the spatial or temporal variations in the confining magnetic field either deliberate or through flucutations. A fractional deviation ϵ in the assumed magnetic field would randomize a particle in $1/\epsilon$ gyrations. As in the case of its thermodynamical (collisional) counterpart, the degree of independence imparted to the particle motion is difficult to guage quantitatively but fairly plausible arguments can be advanced (see § III) to show that a spatially inhomogeneous magnetic field acts as a thermodynamic stirrer serving the function normally appropirated by collisions. There is still an outstanding remaining difference between the nature of randomization introduced by collisions and magnetic field inhomogeneities respectively. While the former gives rise to isotropic scattering, the latter is limited to scattering in two dimensions only. Thus, a plasma heated through stochastic cyclotron acceleration would be susceptible to temperature anisotropy instabilities.

With these reservations in mind, we shall assume in the treatment that follows that all the particles "somehow" manage to experience independent realizations of the random electric field ensemble. In (1.5) writing $\Delta s = v\tau$, where τ is the interval between successive randomizations and $\Delta\epsilon = mv\Delta v$, one obtains ($\eta$ is the charge to mass ratio)

$$\Delta v = \eta E \tau \cos\varphi$$

so that

$$\Delta v_x = \eta E \tau \cos\varphi \cos\theta$$

$$\Delta v_y = \eta E \tau \cos\varphi \sin\theta$$

and after N randomizations

$$\Delta v^2 = \eta^2 E^2 \tau^2 \left\{ \left(\sum^N \cos\varphi \cos\theta \right)^2 + \left(\sum^N \cos\varphi \sin\theta \right)^2 \right\}$$

which on performing an ensemble average yields

$$\langle \Delta v^2 \rangle = \frac{1}{2} \eta^2 E^2 T^2 N$$

Writing $t = N T$ and $T = 1/\Delta\omega$, where $\Delta\omega$ is the bandwidth of the applied electric field, finally gives,

$$\frac{d\langle v^2 \rangle}{dt} = \frac{\eta^2 E^2}{2 \Delta\omega} \tag{1.6}$$

which resembles the well known collisional heating formula if $2\Delta\omega$ is replaced with the collision frequency ν .

In this derivation the frequency of the random electric field segments was assumed to be the particle gyrofrequency. In general, the electric field may possess random, complicated frequency and amplitude modulations. It can be shown that such a field would also produce heating although recourse must be taken to more sophisticated correlation techniques for quantitative results. It then becomes tempting to call this kind of heating non-resonant[4]. We shall see in the next section that such nomenclature is misleading because for any energy absorption to take place, a resonance must necessarily exist and only the component of the electric field in phase with the gyrating particle motion contributes towards energy absorption.

Another way to look at stochastic heating is by considering the case of a charged particle in a longitudinal random electric field $E(t)$ acting along the magnetic field direction. The particle velocity at time t is given by

$$v(t) = v(0) + \eta \int_0^t E(t)\, dt. \tag{1.7}$$

Since $E(t)$ is random, the integral in (1.7) can equally likely have a positive or a negative value so that

$$v(t) = v(0) \pm \Delta v$$

with equal probability. Hence the probable energy of the particle is given by

$$\frac{1}{2} m v^2(t) = \frac{1}{2} m v^2(0) + \frac{1}{2} m (\Delta v)^2 \tag{1.8}$$

which is an alternative way to regard the stochastic heating as a random walk in the velocity space. A possible scheme to test this result experimentally would be to attempt to heat a plasma by applying a random electric field between two capacitor plates. A little reflection, however, shows that if the capacitor plates are initially and finally free of charge, the integral in (1.7) vanishes identically and no energy transfer is possible. This is one example where the integral constraint that the plates may not be charged endlessly, renders the process non-Markovian.

The above examples illustrate that, if properly interpreted, random electric fields could impart energy to charged particles. This possibility was first recognized by Fermi[5] in connection with the cosmic ray phenomenon. Later this principle was suggested by Burshtein, Veksler and Kolomenskii[6] for high energy particle accelerators and successfully tested by Keller, Dick and Fidecaro[7] who accelerated a beam of protons to energies in excess of 5 MeV in a cyclotron driven by a noise source.

In the context of thermonuclear ignition and energetic particle production in astrophysical plasmas, stochastic heating exists as an adjunct to the more general problem of turbulent interactions either existing naturally or driven by externally injected currents and charged particle beams. Since several papers specifically on turbulence are being presented in this symposium, we shall enter into the subject of turbulence only to the extent of showing its relationship to the topic presently under consideration.

After it became apparent that rf energy may not be easily coupled and absorbed in a plasma through collisions and other known (linear) mechanisms, it was proposed to inject energy directly into the plasmas by means of charged particle beams or induced currents. In either case the injected energy is large enough to preclude linearized treatments and gives rise to instabilities and turbulence. A large numberof unstable modes are excited resulting in waves with broad frequency spectra. The theoretical treatment must also take into account wave-wave interactions by including the non-linear terms in the Boltzmann-Vlasov equation. At the same time the evolution of the particle velocity distribution has to be followed with the particles already in a state of unmanageable unrest. The solution of the problem involving a simultaneous analysis of the origin of the turbulent spectrum and its interaction on the plasma becomes too involved to be tackled theoretically with any degree of generality.

This was recognized by Bass, Fainberg and Shapiro[4] who sought an artificial separation of the problem of generation of the turbulent spectrum and its subsequent interaction with the charged particles. Even after this controversial separation, we shall see that the simplified problem still poses considerable theoretical challenge.

In this paper we limit ourselves to the aspect of turbulence dealing with the interaction of externally applied or otherwise specified electric fields with the charged particles. In accordance with common usage we shall call this problem "stochastic heating".

In the next two sections we rederive the stochastic heating formulae in spatially uniform and non-uniform fields respectively using correlation functions and power spectra which are briefly reviewed in Appendix A. It is sought to keep the development cohesive and systematic rather than chronological; nor are all the existing works referenced. A more complete bibliography is, however, appended. The experimental work is reviewed in § IV. The present status of research on this topic and the pertinence of stochastic heating to thermonuclear ignition are discussed in § V.

II. STOCHASTIC HEATING IN A UNIFORM MAGNETIC FIELD

Three cases of successively increasing complexity are studied with the assumption that the fluctuating electric field is specified as a function of time and space and the flucuating magnetic field produces negligible contribution. The treatment is non-relativistic and the random electric field is assumed to be stationary in time and space with correlation time τ_c and correlation distance ℓ_c . The results are applicable either to the expected acceleration of a particle in the random field or to a collection of particles, provided that each particle experiences an independent realization of the electric field ensemble and produces no reaction on the electric field itself, i.e. the interparticle interactions are ignored. The results are valid only for time scale $T >> \tau_c$. Further assumptions are listed under the individual subheadings.

a) Electric Field Prescribed at the Particle's Position

Let $E'_y = E'(t)$ be the electric field at the instantaneous position occupied by the particle moving in a uniform magnetic field $B_z = B_o$. Then

$$\dot{v}_x(t) = \omega_c v_y(t) \tag{2.1}$$

$$\dot{v}_y(t) = -\omega_c v_x(t) + \eta E'(t) \tag{2.2}$$

where $\omega_c = \eta B_o$.

The electric field $E'(t)$ is a stationary random process. We can treat $E'(t)$ as an aperiodic function if we consider the particle motion as an initial value problem with the field extending from time 0 to t for a duration T = t. T can have an arbitrarily large but finite value. Combining (2.1) and (2.2)

$$\dot{v}_r(t) = -j\,\omega_c v_r(t) + j\,\eta E'(t) \tag{2.3}$$

where $$v_r = v_x + j\, v_y$$

The solution of (2.3) is

$$v_r(t) = \exp(j\,\omega_c t)\int_0^T j\eta\, E'(t)\exp(j\omega_c t)\,dt + v_r(0)\exp(-j\,\omega_c t) \tag{2.4}$$

And since $E'(t)$ vanishes outside the limits of integration (2.4) becomes

$$v_r(t) = \exp(-j\omega_c t)\int_{-\infty}^{\infty} j\eta\, E'(t)\exp(j\omega_c t)\,dt + v_r(0)\exp(-j\,\omega_c t) \tag{2.5}$$

which simplifies to

$$v_r(t) = \left[2\pi j\eta\,\tilde{E}'(\omega_c) + v_r(0)\right]\exp(-j\,\omega_c t) \tag{2.6}$$

where $\tilde{E}'(\omega)$ is the Fourier-transform of $E'(t)$. Taking magnitudes of both sides of (2.7)

$$v^2(t) = v^2(0) + 2\pi\eta^2\,\Phi^{T\prime}(\omega_c) - 2\pi\eta\left\{v_x(0)\,I_m[\tilde{E}'(\omega_c)] - v_y(0)\,Re[\tilde{E}'(\omega_c)]\right\} \tag{2.7}$$

where $\Phi'^{T}(\omega_c)$ is the energy density spectrum at the cyclotron frequency for the entire aperiodic function extending through time T. If the reader is unfamiliar with spectral analysis, it is helpful to associate $\Phi'^{T}(\omega) = 2\pi E^{2\prime}(\omega)$ with the total energy density in analogy with the Fourier-series expansion. As $T \to \infty$, $\Phi'^{T}(\omega) \to T\,\Phi'(\omega)$ where $\Phi'(\omega)$ is the power spectrum of the random electric field. For a large ensemble of particles, each seeing the field independently and having independent initial conditions, the last two terms in (2.7) contribute no expected value and

$$\frac{d\langle v^2\rangle}{dt} = 2\pi\eta^2\,\Phi'(\omega_c)\,. \tag{2.8}$$

The switch from aperiodic to random fields is justified because the expression in (2.8) needs information about the power spectrum only. All the phase sensitive terms in (2.7) cancelled out in the averaging process.

The result of (2.8) is very general and is applicable for electric field of arbitrary field strngth and spatial variation. The principal difficulty lies in determining $E'(t)$ at the particles' position. This difficulty, however, is not peculiar to stochastic heating and is a rather familiar problem in theoretical plasma physics. Part (b) of this section is devoted to the determination of $E'(t)$ in spatially varying electric fields. The results obtained are necessarily more complicated but all the physics is already contained in the above derivation.

One case, in which the result of (2.8) is perfectly valid without any further ado is when the electric field possesses no spatial variation[8]. Since the particle always stays "in resonance" and produces no reaction (or radiation) on the source implying a zero impedance generator, it can gain energy endlessly without limit. In practice, an energetic particle will not only radiate energy but will also tend to move out of the resonance due to the doppler shift in the frequency spectrum due to finite wavelength of the electric field.

For a random electric field the autocorrelation function decays approximately exponentially andthe power spectrum has the form

$$\Phi(\omega) \approx \langle E^2 \rangle / 4\pi \Delta\omega$$

so that (2.8) becomes

$$\frac{d\langle v^2 \rangle}{dt} \approx \frac{\eta^2 \langle E^2 \rangle}{2\Delta\omega} \qquad (2.9)$$

which is the result of (1.6) obtained from the random walk theory. Despite this apparent similarity there are essential differences in the two forms of the result expressed by (2.8) and (2.9) respectively. Unlike (2.9), equation (2.8) makes it unequivocally plain that it is only the component of the electric field spectrum at the cyclotron frequency (a so called particle resonance) which contributes to energy absorption. If an ideal band-stop filter with center frequency $\omega = \omega_c$ is introduced between the rf source and the plasma, no energy absorption would be possible irrespective of the electric field strength and the extent of randomness present. Thus a stochastic field merely acts to broaden an already existing resonance where energy absorption would in any case occur. It is incapable of transferring energy where no resonance previously existed. Therefore, it would be incorrect to label stochastic heating as non-resonant energy absorption.

b) Electric Field with Spatial Variation

i) Zero gyroradius limit

In this case, the doppler shifted frequency ω' at the particles position is given by

$$\omega'(t) = \omega \pm k_z v_z(t) \tag{2.1o}$$

for the electric field component with the wave numbers $\pm k_z$. In the limit of a weak electric field $E(t) \rightarrow 0$, $v_z(t) \rightarrow v_z(o) = v_z$ so that

$$\omega'(t) = \omega' = \omega - k_z v_z \tag{2.11}$$

Using the transformation

$$\Phi(\omega) \rightarrow \frac{1}{2}\left[\Phi(\omega + k_z v_z) + \Phi(\omega - k_z v_z)\right] \tag{2.12}$$

and summing over all k_z we obtain from (2.8)

$$\frac{d\langle v^2\rangle}{dt} = \pi\eta^2 \int_{-\infty}^{\infty} dk_z\, \Phi(k_z, \omega_c - k_z v_z)\,. \tag{2.13}$$

This is the result of Sturrock[9] derived using the Fokker-Planck prescription. This is not surprising since the Fokker-Planck equation follows the time development of a Markov process and incorporates all the assumptions stated for the random walk analysis.

The above result is derived using straightforward linear equations with zero-order particle trajectories. The similarity of these results with those derived from the quasi-linear theory is due to the fact that in the quasi-linear calculations too, the change in particle distribution function is calculated from the zero-order particle trajectory with perfectly linear analysis.

All these remarks are also valid for the finite gyroradius case described below.

ii) Finite gyroradius case

The only additional complication in the calculation of $E'(t)$ arises from the transverse field variations encountered by the particle during its cyclotron motion. This is easily handled by following the standard approach used in the derivation of the hot plasma dielectric tensor. In conformity with the method employed in that derivation, the electric field is allowed to have both right and left circular polarizations.

The zero-order particle orbit admits of the static magnetic field only

$$v_x(t) = -v_{\perp o} \cos(\omega_c t + \varphi) \tag{2.14}$$

$$v_y(t) = v_{\perp o} \sin(\omega_c t + \varphi) \tag{2.15}$$

$$x(t) = r_g \sin \omega_c t \tag{2.16}$$

$$y(t) = r_g \cos \omega_c t \tag{2.17}$$

where r_g is the gyroradius. The phase angle has been dropped to avoid needless clutter. The phase angle ψ of the propagation vector, on the other hand, has to be included in order to allow an arbitrary distribution of wave-polarization vector. Writing

$$k_x = k_\perp \cos\psi \tag{2.18}$$

$$k_y = k_\perp \sin\psi \tag{2.19}$$

From (2.16) to (2.19)

$$k_x x = r_g k_\perp \sin\omega_c t \cos\psi \tag{2.20}$$

$$k_y y = r_g k_\perp \cos\omega_c t \sin\psi \tag{2.21}$$

and

$$k \cdot r = k_z v_z t + r_g k_\perp \sin(\omega_c t + \psi) \tag{2.22}$$

The electric field component (with wave number k) at the particle position is

$$E'(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{j(\omega t - k\cdot r)}\, d\omega \tag{2.23}$$

Note that while $E'(t)$ is the field expressed at the particle's position, the unprimed quantity $\tilde{E}(\omega)$ is the Fourier-transform of the actual field component $E(t)$. From (2.22) and (2.23)

$$E'(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega) \exp j\left[\omega t - k_z v_z t - r_g k_\perp \sin(\omega_c t + \psi)\right] d\omega \tag{2.24}$$

Using the Bessel function identity

$$\exp\left[j\, r_g k_\perp \sin(\omega_c t + \psi)\right] = \sum_{-\infty}^{\infty} J_m(k_\perp r_g) \exp\left[j m (\omega_c t + \psi)\right]$$

we obtain from (2.24)

$$E'(t) = \sum_m J_m(k_\perp r_g) \int_{-\infty}^{\infty} \tilde{E}(\omega) \exp\left[j(\omega - m\omega_c - k_z v_z)t - jm\psi\right] d\omega . \tag{2.25}$$

Writing the first-order particle orbit as

$$v_x(t) = -v_\perp(t) \cos \omega_c t \tag{2.26}$$

$$v_y(t) = v_\perp(t) \sin \omega_c t \tag{2.27}$$

$$\dot{v}_x(t) = \eta E_x'(t) + \omega_c v_y(t) \tag{2.28}$$

$$\dot{v}_y(t) = \eta E_y'(t) - \omega_c v_x(t) \tag{2.29}$$

From (2.26) to (2.29)

$$\dot{v}_\perp(t) = -\tfrac{1}{2}\eta \left[E_r'(t) e^{j\omega_c t} + E_\ell'(t) e^{-j\omega_c t}\right] \tag{2.3o}$$

where

$$E_r = E_x + j E_y$$

$$E_\ell = E_x - j E_y$$

We treat the random electric field $E'(t)$ as an aperiodic function if we consider the particle motion as an initial value problem with the field extending from time 0 to t for a duration T = t. On integrating (2.3o)

$$v_\perp(t) = -\frac{1}{2}\eta\int_0^T [E_r'(t)e^{j\omega_c t} + E_\ell'(t)e^{-j\omega_c t}]dt \qquad (2.31)$$

Since $E'(t)$ vanishes outside the limits of integration

$$v_\perp(t) = -\frac{1}{2}\eta\int_{-\infty}^{\infty} [E_r'(t)e^{j\omega_c t} + E_\ell'(t)e^{-j\omega_c t}]dt \qquad (2.32)$$

Combining (2.32) with (2.25)

$$v_\perp(t) = -\frac{1}{2}\eta\sum_m J_m(k_\perp r_g)\iint_{-\infty}^{\infty}\{\tilde{E}_r(\omega)\exp[j(\omega+\omega_c-m\omega_c-k_z v_z)t$$

$$-jm\psi] + \tilde{E}_\ell(\omega)\exp[j(\omega-\omega_c-m\omega_c-k_z v_z)t-jm\psi]\}\,d\omega\,dt \qquad (2.33)$$

On ω integration

$$v_\perp(t) = -\frac{1}{2}\eta\sum_m J_m(k_\perp r_g)\int_{-\infty}^{\infty}\{E_r(t)\exp[j(\omega_c-m\omega_c-k_z v_z)t-jm\psi]$$

$$+ E_\ell(t)\exp[j(-\omega_c-m\omega_c-k_z v_z)t-jm\psi]\}\,dt \qquad (2.34)$$

Note that in the above equation we have secured the goal of obtaining the results in terms of the actual electric field $E(t)$ instead of the doppler shifted field $E'(t)$. Integration with respect to t yields

$$v_\perp(t) = -\pi\eta \sum_m J_m(k_\perp r_g)\Big\{ \tilde{E}_r\left[(m-1)\omega_c - k_z v_z\right] e^{-jm\psi}$$

$$+ \tilde{E}_\ell\left[(m+1)\omega_c - k_z v_z\right] e^{-jm\psi} \Big\} \tag{2.35}$$

or

$$v_\perp(t) = -\pi\eta \sum_m \Big\{ J_{m-1}(k_\perp r_g)\,\tilde{E}_r(m\omega_c - k_z v_z)\, e^{-j(m+1)\psi}$$

$$+ J_{m+1}(k_\perp r_g)\,\tilde{E}_\ell(m\omega_c - k_z v_z)\, e^{-j(m-1)\psi} \Big\} \tag{2.36}$$

On taking magnitude square followed by ensemble averaging (2.36) gives

$$\langle v^2(t)\rangle = \frac{1}{2}\pi\eta^2 \sum_m \Big\{ J_{m-1}(k_\perp r_g)\, J_{m+1}(k_\perp r_g)$$

$$\left[\Phi^T_{r\ell}(m\omega_c - k_z v_z)\, e^{-2j\psi} + \Phi^{*T}_{r\ell}(m\omega_c - k_z v_z)\, e^{2j\psi}\right]$$

$$+ J_m^2(k_\perp r_g)\left[\Phi^T_{rr}(m\omega_c - k_z v_z) + \Phi^T_{\ell\ell}(m\omega_c - k_z v_z)\right]\Big\}$$

(2.37)

where the asterik denotes complex conjugate and $\Phi^T(\omega)$ is the energy density spectrum of E(t). In the limit $T \rightarrow \infty$ and on integrating over all possible values of k finally yields the result (the arguments of J and Φ have been dropped)

$$\frac{d\langle v^2\rangle}{dt} = \frac{1}{2}\pi\eta^2 \sum_m \int d^3k \left\{ J_{m-1} J_{m+1} \left[\Phi_{r\ell} e^{-2j\psi} + \Phi^*_{r\ell} e^{2j\psi} \right] + J_m^2 \left[\Phi_{rr} + \Phi_{\ell\ell} \right] \right\}, \tag{2.38}$$

where $\Phi_{ii}(\omega)$ is the autocorrelation power spectrum of $E_i(t)$ while $\Phi_{ij}(\omega)$ is the cross-correlation power spectrum of $E_i(t)$ and $E_j(t)$. Except for algebraic corrections (2.38) is the result of Hall and Sturrock[10] who derived it using quasi-linear techniques. In the zero-gyroadius limit $J_o = 1$ while $J_{m\neq o} = 0$ and the result of (2.13) is immediately obtained. For $k \rightarrow 0$, the result of (2.8) follows similarly.

Hitherto, only the turbulent electric field has been considered so that the results are valid only if the fluctuating magnetic field has negligible contribution. This would be the case for electrostatic turbulence or for relatively low frequency fields produced between capacitor plates. Hall and Sturrock[10] have tried to include the effect of the fluctuating magnetic field. Although, their analysis is correct in principle, the results are erroneous due to the neglect of correlations imposed by Maxwell's equations between the longitudinal magnetic fields and the transverse electric fields and vice-versa. This is easily seen by considering particle motion in a uniform solenoid driven by a random current. The particle experiences random electric and magnetic fields and the results of Ref. 10 would predict stochastic acceleration. But due to the conservation of magnetic moment, the particle could not extract energy from the random fields. This is another example of integral constraints rendering a process non-Markovian and closely resembles the earlier example mentioned in the introduction.

If, in addition, the particle is paraxial, it can be rigorously proved[11] that it would forever encircle the magnetic axis and can not possibly suffer diffusion due to the random electromagnetic field in the solenoid which is once again in contradiction with the results of Ref. lo for reasons already described. Precisely the same error vitiates the enhanced diffusion results obtained by Puri[12].

The situation can be remedied either by including the cross correlation terms between longitudinal and transverse electric and magnetic fields or by writing all magnetic field quantities in terms of the electric field using Faraday's law as has been done by Kennel and Engelmann[13].

III. STOCHASTIC HEATING IN A MAGNETIC MIRROR SYSTEM[14]

In this section the heating rate of a particle contained in a magnetic mirror and subject to a transversely applied random electric field is derived. Unlike the case of a uniform magnetic field where exact heating rates were obtained, we shall be content to derive approximate bounds on the heating rate. The results obtained in this section have the important significance that they are capable of experimental verification, because a laboratory plasma in order to be confined must exist in inhomogeneous magnetic fields, the simplest among them being the magnetic mirror.

Before we start with the actual analysis, it is instructive to make some guesses as to the expected outcome. It is obvious that the electric field spectrum should contain the cyclotron frequencies existing in the mirror system. If $\Phi(\omega_{c1})$ is the maximum value of the maximum value of the electric field power spectrum and $\Phi(\omega_{c2})$ its minimum value where ω_{c1} and ω_{c2} are two gyrofrequencies somewhere in the mirror, we may expect that the actual heating rate would lie somewhere between the two values

$$2\pi\eta^2\,\Phi(\omega_{c1}) > \frac{d\langle v^2\rangle}{dt} > 2\pi\eta^2\,\Phi(\omega_{c2}) \qquad (3.1)$$

provided the actual detail of the magnetic field variation is unimportant. The following analysis would confirm these observations provided certain conditions are fulfilled.

We consider an externally applied fluctuating electric field in a direction transverse to the magnetic field. The electric field is assumed to be uniform in space and stationary in time. The effects of the first order magnetic field as well as the reaction of the particles is neglected.

The method of derivation would also necessitate the assumption that each particle experiences an independent realization of the electric field ensemble. This assumption is too stringent because the spatially inhomogneous magnetic field acts as a "thermodynamic stirrer" as was pointed out in the introduction. The extent of this randomization is hard to calculate but its plausibility would be demonstrated both analytically and with the aid of numerical calculation of the particles' trajectories in an ideal parabolic mirror.

The important new assumption is that no marked correlation exists between the applied electric field and the static magnetic field at the instantaneous position occupied by the particle. This is reasonable if the time taken by the particle between successive reflections from one of the mirrors exceeds the electric field correlation time, i.e.

$$4L/v_z > 1/\Delta\omega \tag{3.2}$$

or

$$\theta_{\parallel} \lesssim \frac{2\,\omega_{co}^2 L^2}{\eta}\left(\frac{R-1}{R+1}\right)^2 \tag{3.3}$$

where $\theta_{\parallel}$ is the parallel temperature, ω_{co} is the average value of ω_c, L is half the distance between the mirrors and R is the mirror ratio.

a) <u>Formal Solution</u>

Let $B_x(x,y,z)$, $B_y(x,y,z)$ and $B_z(x,y,z)$ be the magnetic field due to the magnetic mirror used to contain the particle. Let $B_x[x(t),y(t),z(t)]$, $B_y[x(t),y(t),z(t)]$ and $B_z[x(t),y(t),z(t)]$, henceforth referred to as $B_x(t)$, $B_y(t)$ and $B_z(t)$ be the magnetic field at the instantaneous position of the particle. Then the non-relativistic equations of motion are

$$\dot{v}_r(t) = -j\,\omega_c(t)\,v_r(t) + j\,\omega_{cr}(t)\,v_z(t) + j\,\eta\,E_y(t) \tag{3.4}$$

and

$$\dot{v}_z(t) = \eta\left[B_y(t)\,v_x(t) - B_x(t)\,v_y(t)\right] \tag{3.5}$$

where $\omega_c(t) = \eta\,B_z(t)$ and $\omega_{cr}(t) = \eta\left[B_x(t) + j\,B_y(t)\right]$.

The analysis is at once simplified if one neglects the second term on the right-hand side of (3.4) by considering the paraxial particles only. However, the analysis need not be so restricted for it can be shown that this term produces a negligible contribution towards particle acceleration, compared with the electric field's contribution. This term represents the back-and-forth energy exchange between the parallel and perpendicular motion of the particle; both $\omega_{cr}(t)$ and $v_z(t)$ vary at the frequency of the longitudinal particle motion of the particle already assumed small (see (3.2)) in comparison with the gyrofrequency. Hence, the contribution towards particle acceleration from the term $j\omega_{cr}(t)v_z(t)$ may be neglected compared to that of $j\eta E_y(t)$, if the electric field is applied near the cyclotron frequency.

Once again, we may treat the random electric field as a transient function if the particle motion is considered as an initial value problem with the fields extending from time 0 to T. Then the solution of (3.4) is

$$V_r(T) = \left\{ \int_0^T j\eta\, E_r(t) \exp\left[j\int_0^t \omega_c(t)\,dt \right] dt \right\} \exp\left[-j\int_0^T \omega_c(t)\,dt \right] \tag{3.6}$$

Since $E_r(t)$ and $\omega_c(t)$ can be made to vanish outside the limits of integration we may write (3.6) as

$$V_r(T) = \left[\int_{-\infty}^{\infty} F(t) \exp(j\bar{\omega}_c t)\,dt \right] \exp\left[-j\int_{-\infty}^{\infty} \omega_c(t)\,dt \right] \tag{3.7}$$

where $\omega_c(t) = \bar{\omega}_c + \hat{\omega}_c(t)$, $\bar{\omega}_c = \eta\,\bar{B}_{zo}$ (3.8)

$$\hat{\omega}_c(t) = \eta\,\hat{B}_{zo}(t)$$

$$G(t) = \exp\left[j\int_0^t \hat{\omega}_c(t)\,dt \right] \tag{3.9}$$

and

$$F(t) = j\eta\, E_r(t)\, G(t)\,. \tag{3.10}$$

We refer to G(t) as the "spectrum selection function" and presently show that it has the rate of determining which part of the electric field spectrum contributes towards particle acceleration. From (3.7),

$$v_r(T) = 2\pi \tilde{F}(\bar{\omega}_c) \exp\left[-j\int_{-\infty}^{\infty} \omega_c(t)\, dt\right] \tag{3.11}$$

where $\tilde{F}(\omega_c)$ is the Fourier-transform of F(t). Proceeding as before we obtain from (3.11)

$$\frac{d\langle v^2\rangle}{dt} = 2\pi\, \Phi_F(\bar{\omega}_c)\,. \tag{3.12}$$

Since F(t) involves a product of two independent functions $E_r(t)$ and G(t)

$$\varphi_F(T) = \eta^2\, \varphi_{E_r}(T)\, \varphi_G(T) \tag{3.13}$$

where $\varphi_G(T)$ is the autocorrelation function of G(t) defined as

$$\varphi_G(T) = \langle G(t) G^*(t+T)\rangle = \langle \cos\int_t^{t+T} \bar{\omega}_c(t)\, dt \rangle \tag{3.14}$$

From (3.12) to (3.14) one finally obtains the formal result

$$\frac{d\langle v^2\rangle}{dt} = 2\pi\eta^2 \int_{-\infty}^{\infty} \Phi_G(\omega)\, \Phi_E(\bar{\omega}_c - \omega)\, d\omega\,. \tag{3.15}$$

The effect of the inhomogeneous magnetic field as compared to the constant magnetic field is best seen by comparing (3.15) with (2.8). Instead of Φ_{Er} the particle acceleration rate in a changing magnetic field depends upon the quantity obtained by convolving Φ_{Er} with Φ_G. Although the actual acceleration is produced by the energy imparted by the electric field to the particle, the relative contribution of each frequency component towards particle acceleration is assigned by the "spectrum weighting function" Φ_G, the Fourier-transform of the autocorrelation of the spectrum selection function G(t).

b) Role of the Magnetic Field in Particle Acceleration

We shall show how the role of Φ_G, in assigning the relative contribution of each component of the electric field towards particle acceleration, becomes increasingly critical as the electric field bandwidth becomes narrower in comparison with the width of gyrofrequencies.

i) White noise electric field

Integrating $\Phi_G(\omega)$ over the entire frequency range we obtain using (3.14)

$$\int_{-\infty}^{\infty} \Phi_G(\omega)\, d\omega = \varphi_G(0) \equiv 1. \tag{3.16}$$

Hence the area occupied by $\Phi_G(\omega)$ is an invariant equal to unity, a change in the magnetic field merely changing the shape of $\Phi_G(\omega)$. An interesting consequence of this property of $\Phi_G(\omega)$ follows, namely, that the particle acceleration rate in an electric field possessing a "white noise" spectrum is independent of the magnetic field. Since $\Phi_{Er}(\omega)$ is a constant, one readily obtains from (3.15)

$$\frac{d\langle v^2\rangle}{dt} = 2\pi\eta^2 \Phi_{E_r}(0) = \text{constant} \tag{3.17}$$

ii) Broadband electric field

Whenever the electric field half-power spectrum spans the particle gyrofrequencies, the upper and lower bounds of the statistical acceleration rate are given by the inequality

$$2\pi\eta^2 \Phi_{E_r}(\omega_0) \geq \frac{d\langle v^2\rangle}{dt} \gtrsim \pi\eta^2 \Phi_{E_r}(\omega_0) , \tag{3.18}$$

provided

$$\omega_0 \pm \Delta\omega \gtrsim \overline{\omega_0} \pm |\hat{\omega}_c(t)|_{max} \tag{3.19}$$

and

$$|\hat{\omega}_c(t)|_{max} \gg \nu_2 , \tag{3.20}$$

where $2\Delta\omega$ is the half-power bandwidth and $\Phi_{Er}(\omega_o)$ the peak value of the power spectrum of the random electric field; $|\hat{\omega}_c(t)|_{max}$ and ν_2 are the maximum value and the average zero-crossing frequency, respectively, of the time varying part of the particle gyrofrequency at the particles position.

Proof:

From (3.14) one obtains

$$T \geq \pi/2\,|\hat{\omega}_c(t)|_{max} \qquad (3.21)$$

where T is the time at which the autocorrelation function $\varphi_G(T)$ has its first zero. Hence from (2o) and (21), the maximum angular frequency $\hat{\Omega}_c$ associated with $\Phi_G(\omega)$ is

$$\hat{\Omega}_c = \frac{2\pi}{4T} = |\hat{\omega}_c(t)|_{max} \qquad (3.22)$$

i.e. the maximum frequency associated with the spectrum selection function $\Phi_G(\omega)$ is the maximum value of the time varying part of the gyrofrequency. From (3.15) and (3.22) using (3.19) one obtains,

$$2\pi\eta^2\Phi_{E_r}(\omega_o) \gtrsim \frac{d\langle v^2\rangle}{dt} \gtrsim \pi\eta^2\Phi_{E_r}(\omega_o)\int_{-\hat{\Omega}_c}^{\hat{\Omega}_c}\Phi_G(\omega)\,d\omega. \qquad (3.23)$$

Using (3.22) and an approximation of (3.16) one finally arrives at (3.18). This result is readily extended to include the case when the n^{-1}th power bandwidth of the electric field spans the particle gyrofrequencies in the mirror, and one obtains

$$2\pi\eta^2\Phi_{E_r}(\omega_o) \gtrsim \frac{d\langle v^2\rangle}{dt} \gtrsim 2\pi n^{-1}\eta^2\Phi_{E_r}(\omega_o) \qquad (3.24)$$

Approximating

$$\Phi_{E_r}(\omega_o) \approx \frac{\langle E^2\rangle}{4\pi\Delta\omega}$$

we finally get

$$\frac{\eta^2 \langle E^2 \rangle}{2\,\Delta\omega} \gtrsim \frac{d\langle v^2 \rangle}{dt} \gtrsim \frac{\eta^2 \langle E^2 \rangle}{2n\;\Delta\omega} \qquad (3.25)$$

iii) Narrow band electric field

We derive a lower bound for the statistical acceleration rate for the case when the electric field bandwidth available is smaller than the width of particle gyrofrequencies. For this case (3.15) gives

$$\frac{d\langle v^2 \rangle}{dt} \gtrsim \pi\eta^2 \Phi_{E_r}(\omega_o) \int_{\omega_o-\Delta\omega}^{\omega_o+\Delta\omega} \Phi_G(\bar{\omega}_c - \omega)\, d\omega \qquad (3.26)$$

From (3.16) and (3.22) the average value of $\Phi_G(\omega)$ is given by

$$\Phi_G(\omega) \approx \frac{1}{2} \left| \hat{\omega}_c(t) \right|_{max} \qquad (3.27)$$

If in (3.26), ω_o is adjusted to maximize the integral, using (3.27) we obtain

$$\frac{d\langle v^2 \rangle}{dt} \gtrsim \pi\eta^2 \Phi(\omega_o) \frac{\Delta\omega}{|\omega_c(t)|_{max}} \qquad (3.28)$$

c) Particle velocity distribution

In this section the evolution of the distribution function of a plasma consisting of noninteracting "test particles" with all the particles experiencing the s a m e spatially uniform electric field will be studied. It will be shown that, quite unaided by the randomizing influences of interparticle interactions and electric field spatial decoherence, a nonuniform magnetic field tends to Maxwellianize the particle velocity distribution. Seidl[15] has shown that (in the absence of an externally applied electric field) $\omega_o(t)$ for paraxial particles in a parabolic mirror is given by

$$\hat{\omega}_c(t) \approx \bar{\omega}_c A \cos(pt + \psi) \tag{3.29}$$

where $A = (R-1)/(R+1)$, R is the mirror ratio, $p \approx (av_{\perp o}/\pi L)$, $\alpha = (R-1)L^2$, L is the mirror half-length, $v_{\perp o}$ is the perpendicular velocity of the particle on the mirror axis. Although the externally applied electric field would change $\omega_o(t)$ in a complicated manner, we shall assume the form of $\omega_o(t)$ given by Eq.(3.29) to establish the thermalizing properties of the magnetic field. It will be evident from later analysis that a more complicated variation of $\omega_o(t)$ will lead to a more marked Maxwell-ionization.

From eqs.(3.4) and (3.29) the approximate equation of motion, of a particle in such a magnetic mirror,

$$\dot{v}_r(t) = -j\bar{\omega}_c\left[1 + A\cos(pt + \psi)\right]v_r(t) + j\eta E_y(t) \tag{3.30}$$

has the solution

$$\begin{aligned} v_r(t) = {} & v_{ro}\exp\left[-j\bar{\omega}_c t - j\beta\sin(pt + \psi) + j\beta\sin\psi\right] \\ & + \exp\left[-j\bar{\omega}_c t - j\beta\sin(pt + \psi)\right]\int_0^t j\eta E_y(t) \\ & \cdot \exp\left[j\bar{\omega}_c t + j\beta\sin(pt + \psi)\right]dt, \end{aligned} \tag{3.31}$$

where $\beta = A\omega_c/p$. Since $E_y(t)$ vanishes outside the limits of integration, eq.(3.31) can be reduced to the form

$$v_r(t) = v_{ro}\exp[-j\bar{\omega}_c t - j\beta\sin(pt+\psi) + j\beta\sin\psi]$$

$$+ 2\pi j\eta\exp[-j\bar{\omega}_c t - j\beta\sin(pt+\psi)]\{J_o(\beta)\tilde{E}(\omega_c)$$

$$+ \sum_1^\infty J_n(\beta)[\tilde{E}(\bar{\omega}_c + np)\exp(jn\psi)$$

$$+ \tilde{E}(\bar{\omega}_c - np)\exp(-jn\psi + n\pi)]\} \qquad (3.32)$$

where $\tilde{E}(\omega)$ is the Fourier transform of E(t).

In the case of a uniform magnetic field, R = 1, $\beta = 0$, and eq. (3.32) becomes

$$v_r(t) = v_{ro}\exp(-j\bar{\omega}_c t) + 2\pi j\eta\tilde{E}(\omega_c)\exp(-j\bar{\omega}_c t). \qquad (3.33)$$

The second term on the right-hand side of eq. (3.33) represents the increment in particle velocity due to the electric field, which would be independent of the initial position of the particle in the magnetic field.

In contrast with the above situation we note from eq.(3.32) that for a mirror magnetic field, the particle velocity at time t consists of an infinite summation of "quasi-independent" modes if the particles are assumed to possess independent initial positions along the mirror axis given by angle ψ. The central limit theorem predicts an approach toward the Gaussian distribution for the probability density of $v_r(t)$ under such conditions.

A complex magnetic field variation as opposed to the simpler sinusoidal variation would cause a greater randomization of a $v_r(t)$ as may be confirmed by using the new variation in eq. (3.3o) and following the above analysis.

To measure the extent of randomization,particle trajectories were followed exactly by numerical integration using a computer. The magnetic field was taken to be an ideal parabolic mirror of (3.29) and all the particles were at rest at $t = 0$, but occupying random positions along the magnetic field axis. The electric field was uniform in space and in the direction transverse to the magnetic field axis. The resultant particle distribution was indeed close to being Gaussian which should lend support to the arguments of this section. In this manner it is now possible to relax one of the most irritating assumption in stochastic heating theories namely that each particle should be subject to an independent realization of the electric field ensemble.

IV. EXPERIMENTAL VERIFICATION

A great many experiments on turbulent heating with impressive results have undoubtedly been described already in this meeting. In keeping with the objectives of this paper, we restrict our attention to the experiments where the stochastic heating occurs due to externally applied random electric fields. Of the two experiments discussed in this section, the first one is aimed at checking the theoretical results of the previous section while in the second experiment it is sought to heat a plasma significantly to demonstrate the effectiveness of stochastic heating.

a) Stochastic Heating of a Tenuous Plasma[16]

In this experiment the plasma density is kept low enough (10^7 cm^{-3}) to approximate the theoretical requirements of a collisionless, test particle model. The electron-electron collision time is 5oo μsec which far exceeds the particle containment time so that collisional heating effects can be assumed to be absent. We expect the plasma to heat at a rate calculated from the test-particle theory until the average particle has been lost and replaced, at which time the temperature reaches its equilibrium value.

A tenuous plasma (10^7 cm^{-3}) in thermodynamic equilibrium is created by contact ionization of hot (214o ^oK) tantalum surfaces shown schematically in Fig. 3. The half inch diameter cathodes are spaced 5 inches apart. The low background pressure of 3 x 10^{-7} mm Hg, coupled with low plasma density, precludes interparticle interactions and the plasma is "collisionless". The mean-free-path for electron-electron collisions is several meters, so there is no heat conduction between the plasma and the end plates.

The plasma is situated in a magnetic mirror of mirror ratio 1.5; the mirror regions roughly coincide with the location of the cathodes. The rf voltage for heating the plasma is applied transverse to the plasma column by a pair of copper plates situated outside the vacuum envelope. The rf signal generated in a voltage tunable magnetron operating in a noisy mode[17] and amplified by a travelling-wave-amplifier has the noise spectrum of Fig. 4. A combination of balun, taper line and terminating resistors are used to match the source to the load. The plasma absorbs only about 10^{-8} W and has an effective shunt impedance several orders of magnitude larger than the impedance of the resistors shunting the copper plates.

In the absence of any loss processes we can assume that the energy given to the plasma by the external fields goes entirely to plasma heating until the particles are lost. Our task is then to measure the electric field strength, its spectral shape, and the rate of change of temperature. The applied voltage was measured with an rms voltmeter and the field strength then estimated from the geometry of the plates. With the maximum available noise power, we were able to create an electric field of o.25 V/cm. From Fig. 4, the 3 db and 6 db bandwidths were 2.1×10^8 and 3.2×10^8 Hz, respectively.

The plasma heating time was measured in two ways. In the first measurement, Langmuir probe curves (Fig. 5) were obtained on an oscilloscope. The probe voltage was swept from - 2.5 to + 2.5 V in 2o sec as rf noise source was pulsed on and off at a frequency of 7oo cps, with a pulse width varying between 0 and 1oo μsec (pulse rise time $<$ o.1 μsec). The probe trace alternated between the "heated" and the "unheated" plasma and the lower and the upper traces in Fig. 5 correspond, respectively, to the cases with and without the applied noise field. As the pulse width is gradually increased, the temperature rises steadily, as indicated by the decreasing slope of the lower curve. This trend continues until the duration of the noise pulse reaches 3o μsec. A further widening of the pulse produced no additional change in the plasma temperature. The experiment was repeated at several values of the applied noise field strength. Each time, the plasma temperature reached a maximum for a noise pulse duration of 3o μsec. The final temperature attained by the plasma increased, of course, with the increasing field strength of the random electric field. This would imply that the plasma particles stay in the interaction region for 30 μsec and the small noise input does not alter the existing particle loss rate. The ion transit time, from the center of the column to the ends, is also of the order of 30 μsec, hence an electron might be admitted every time an ion reaches the electron sheath and is swept into the cathode.

In the second method of measuring plasma heating time, 100 μsec pulses of rf noise were applied across the plasma. The probe current monitored on an oscilloscope (at a fixed probe voltage) saturated in 30 μsec, as shown in Fig. 6. The same results were obtained on repeating the experiment with several different values of the probe voltage and noise field strengths. The plasma starts "cooling" as soon as the 100 μsec pulse switches off. It may be seen from Fig. 6 that the plasma cooling time is 30 μsec, supporting our conclusion that the plasma particles stay for 30 μsec in the interaction region.

Having shown that the plasma temperature reaches its final value in 30 μsec for all power inputs, we can measure it more accurately by using continuous signals on an x-y recorder. Figure 7 shows a set of Langmuir probe curves for different values of the applied rf noise field (the curves are staggered for clarity) using a mirror ratio of 1.5. The magnetic field varied between 230 and 340 G. The plasma temperature and density near the center of the tube determined from the uppermost curve, taken without a noise input are 0.2 eV and 1.0×10^7 electrons cm^{-3}, respectively. The plot of temperature vs. field strength is shown in Fig. 8 and is linear as would be expected. From the slope of Fig. 8 and the heating time measurement of 30 μsec we calculate the plasma heating rate,

$$\frac{d\theta_{eV}}{dt} = 1.0 \times 10^{5} \langle E^{2} (V/cm)^{2} \rangle \; eV/sec. \qquad (4.1)$$

The theoretical heating rate using (2.9) is

$$1.7 \times 10^{5} \langle E^{2} \rangle < \frac{d\theta}{dt} < 6.7 \times 10^{5} \langle E^{2} \rangle \; eV/sec \qquad (4.2)$$

which is in satisfactory agreement with (4.1). In deriving (4.2) use was made of the fact that the 6 db (1/4 power) band width of the electric field spans the gyrofrequencies in the mirror.

b) Stochastic Heating Using Larger Power

In this experiment we do not seek a verification with theory but demonstrate that an electron plasma of $\sim 10^{11}$ cm^{-3} density and one liter volume can be heated to a temperature of several hundred electron volts with loo watt applied noise power.

Part of a 6 cm diamter quartz tube is inclosed in a multi-mode copper cavity 39 cm long and 15 cm in diameter as shown in Fig. 9. Gas is admitted into the quartz tube at one end while the other end is evacuated by a molecular diffusion pump. Gas pressure is continuously monitored with a Penning ionization gauge and the output signal is used to regulate the pressure in conjunction with an automatic pressure controller. A 34-cm-long magnetic morror with the field strength varying between 3 kG in the middle and 4.5 kG at the ends is used for confining the plasma. A set of four Ioffe bars (not shown in the figure) can produce a minimum magnetic field with a well depth of 1.25. Plasma formation and electron heating are accomplished by feeding loo watt of broadband (8 - 12 GHz, corresponding to the electron cyclotron frequencies in the mirror) microwave power. The microwave power is derived by amplification of noise from a gas discharge tube using a travelling wave amplifier. The power spectrum of the output of the TWA is shown in Fig. lo. There is also a provision to heat the ions by applying a broad-band rf noise (spanning the ion cyclotron frequencies in the mirror) across the plasma to a pair of copper plates. Both the electron and ion heating sources can be either pulsed or used continuously.

The principal diagnostics to date have been performed with an 8 mm microwave interferometer and a movable diamagnetic probe.

The diamagnetic probe consists of two copper coils of loo and 2o turns respectively, electrostatically shielded from each other and enclosed in a thin stainless steel can. In order to calibrate the diamagnetic probe, current pulses of known shape and magnitude are passed alternately through the 2o-turn coil and through a solenoid (of the dimensions of the plasma) placed inside the quartz tube. On integrating the signal from the loo-turn coil, the original pulses are recovered. This procedure allows the calibration of the diamagnetic signal independently of the eddy current losses. During the actual operation of the experiment, it is further necessary to remove the low-frequency pickup due to the ripple in the magnetic field using a combination of high pass filters and averaging the results of several thousand shots using a signal averager.

The experiment is performed in the pulsed mode with the pulse frequency between loo and 25o Hz and a pulse duration of 5oo - looo μsec. The gas used is hydrogen and the microwave power is loo watt. Fig. 11 shows the plasma density and electron temperature as a function of pressure read directly on the penning ionization gauge. Remembering that the plasma pressure at the pumping end is typically one-third of this value and on applying the correction factor for hydrogen, the calculated ionization degree is typically hundred percent at the lower gas pressures used. The plasma density decays within 2o μsec at the higher pressures and somewhat more slowly at the lower pressures. A simple energy balance shows that all the incident microwave power is fully absorbed by the plasma. The variation of density and temperature is shown in Fig. 12 and typical plasma buildup and decay signals in Fig. 13.

The use of Ioffe-bars is rather critical in this experiment. The Ioffe-bars reduce the density fluctuations from over 3o % to practically zero. Without the Ioffe bars it was in fact quite difficult to perform the experiment at all with hydrogen. The results using argon are given in ref. 18.

V. DISCUSSION

The principal theoretical result is that in a uniform magnetic field stochastic heating is proportional to the power spectrum at cyclotron frequency of the electric field seen by the particle at its instantaneous position. Further theoretical effort boils down to the determination of this doppler shifted field in terms of the actual field in the stationary frame. If only the zero-order particle motion is considered for calculating this doppler shift, then finite gyroradius and wave number effects lead to an absorption spectrum that is reminiscent of Bernstein resonances. All the theoretical results derived in this paper follow from perfectly linear analysis. The resemblence with quasi-linear theories results due to the fact that the quasi-linear theoreis themselves employ only linearized equations to calculate the change in the distribution function due to a spectrum of waves. Although, the effects of fluctuating magnetic fields were disregarded, they may be readily included[10,13] as was pointed out in § II. Similarly, the extension to higher order particle orbits in calculating the doppler-shifted spectrum presents no great formal difficulty[19,20]. But to quote Orszag[21],

> "The essential difficulty does not involve finding an exact mathematical description of turbulence but rather involves extracting useful information from a formally exact solution."

This precisely defines one of the directions of future research in this field.

An even more inportant effect that must be included in a more sensible treatment of stochastic heating is the collective effects due to interparticle interactions. This effect is intimately linked with the question of penetration of externally applied rf fields into the plasma. Apart from modifying the cyclotron absorption (particle resonance) results, the interparticle interactions would introduce collective resonances like the hybrid resonances as additional absorption mechanisms. This is simpler than may, at first, appears to be the case. In Fig. 14 let I(t) be the current supplied by the generator for a duration 0 to T. Then

$$I(t) = \int_{-\infty}^{\infty} \tilde{I}(\omega)\, e^{j\omega t}\, d\omega \tag{5.1}$$

In principle, it is a straightforward exercise in algebra, to determine the response of the linearized system consisting of a waveguide partially filled with a plasma which is homogeneous in the longitudinal direction. Far away from the source, assuming that only the fundamental waveguide mode is excited, the response may be written as

$$\tilde{E}(\omega)\, e^{j(\omega t - k\cdot r)} = \tilde{I}(\omega)\, \tilde{Z}(\omega)\, e^{j\omega t} \tag{5.2}$$

so that

$$E(t) = \int_{-\infty}^{\infty} \tilde{I}(\omega)\, \tilde{Z}(\omega)\, e^{j(\omega t - k\cdot r)}\, d\omega \tag{5.3}$$

Using Maxwell´s equations we get

$$\tilde{H}(\omega) = \tilde{\xi}(\omega)\, \tilde{E}(\omega) \tag{5.4}$$

If

$$\tilde{P}_{diss}(\omega) = -\frac{1}{4}\int d^3r\ \nabla\cdot\left(\tilde{E}(\omega)\times\tilde{H}^*(\omega)\right) \tag{5.5}$$

then the total power dissipated using Parseval´s theorem is given by

$$P_{diss} = \int_{-\infty}^{\infty} \tilde{P}_{diss}(\omega)\, d\omega \tag{5.6}$$

Writing $P_{diss}(\omega)$ in terms of $\tilde{I}(\omega)$ and $\tilde{z}(\omega)$ and on performing an average over the electric field ensembles, it can be readily seen that the total power dissipated may be written in terms of the power spectrum of the driving current and no phase information is necessary. In case, the plasma was collisionless $\tilde{E}(\omega)$ will diverge at the resonances. The situation is easily remedied by using the Landau approach in evaluating the integral in (5.5).

Although, the actual evaluation of $\tilde{z}(\omega)$ is not easy, it can be shown that the energy is readily coupled to and absorbed by a plasma contained in a waveguide even in the presence of extremely weak absorption processes e.g. collisions at thermonuclear temperatures.[22] The penalty for this weak absorption mechanisms is the buildup of enormous electric fields, dissipation densities and volume forces in an extremely narrow region of the plasma. An obvious bid to avoid such a situation would be to use broadband noise sources in order to enlarge the volume over which the energy absorption occurs. Another instance where stochastic heating may be important in thermonuclear research is in the broadening the class of particles gaining energy from the field as has been suggested by Hall and Sturrock[23] in connection with transit-time magnetic pumping.

In the absove treatment including collective effects, as well as in all the previous results derived in this paper, the stochastic field was treated as a transient function with a well defined Fourier spectrum except when performing ensemble averages. It is then at once clear that the problem of rf penetration into the plasma, contrary to the earlier expectations expressed by this author[14], is exactly the same whether the field is stochastic or sinusoidal.

Thus, the only possible thermonuclear application of stochastic heating using externally applied rf fields is in broadening the resonances or the class of particles being heated as was quite correctly concluded by Bol[24] as early as 1966.

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APPENDIX A

DEFINITIONS OF SPECTRUM FUNCTIONS

We define below the various spectrum functions used in this paper . These definitions are consistent with the ones used in ref.25.

a) Aperiodic Functions (or Transient Functions)

For an arbitrary function E(t) of finite duration (0 to T) satisfying the condition

$$\int_{-\infty}^{\infty} |E(t)|\, dt < \infty \qquad (A.1)$$

the following pair of Fourier transforms may be defined

$$E(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{j\omega t}\, d\omega \qquad (A.2)$$

$$\tilde{E}(\omega) = \frac{1}{2\pi} \int_{0}^{T} E(t)\, e^{-j\omega t}\, dt$$

$$\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t)\, e^{-j\omega t}\, dt \qquad (A.3)$$

The autocorrelation function $\varphi^{T}(\tau)$ and energy density spectrum $\Phi^{T}(\omega)$ of E(t) are defined as

$$\varphi^{T}(\tau) = \int_{0}^{T} E(t)\, E(t+\tau)\, dt$$

$$\equiv \int_{-\infty}^{\infty} E(t)\, E(t+\tau)\, dt \qquad (A.4)$$

$$\Phi^T(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi^T(\tau)\, e^{-j\omega\tau}\, d\tau \tag{A.5}$$

It can be shown that

$$\Phi^T(\omega) = 2\pi\, |\tilde{E}(\omega)|^2 \tag{A.6}$$

b) Random Functions

Autocorrelation function $\varphi(\tau)$ and power spectrum $\Phi(\omega)$ of a random function E(t) are defined as

$$\varphi(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E(t)\, E(t+\tau)\, dt \tag{A.7}$$

$$\Phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\tau)\, e^{-j\omega\tau}\, d\tau \tag{A.8}$$

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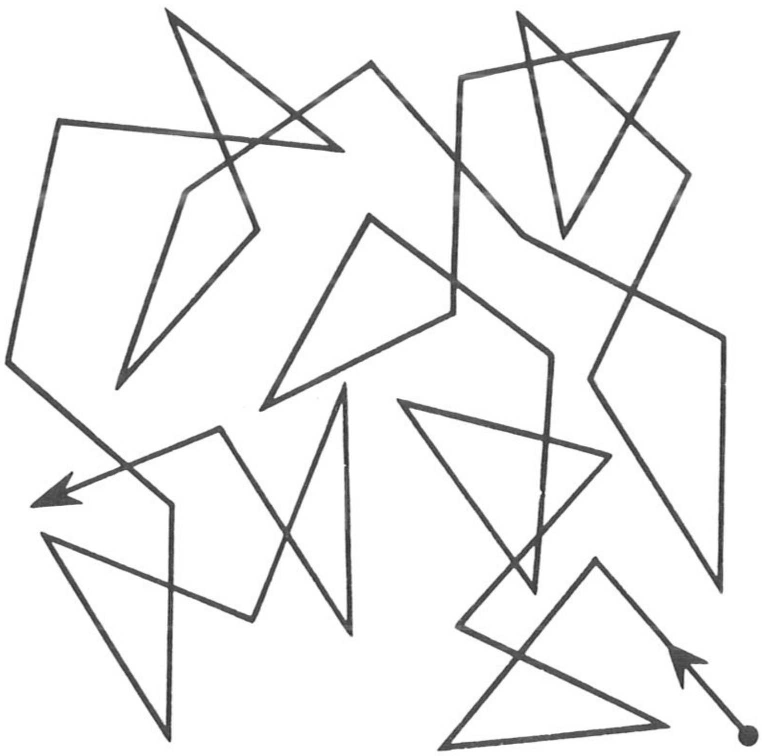

Fig. 1   The drunken walk

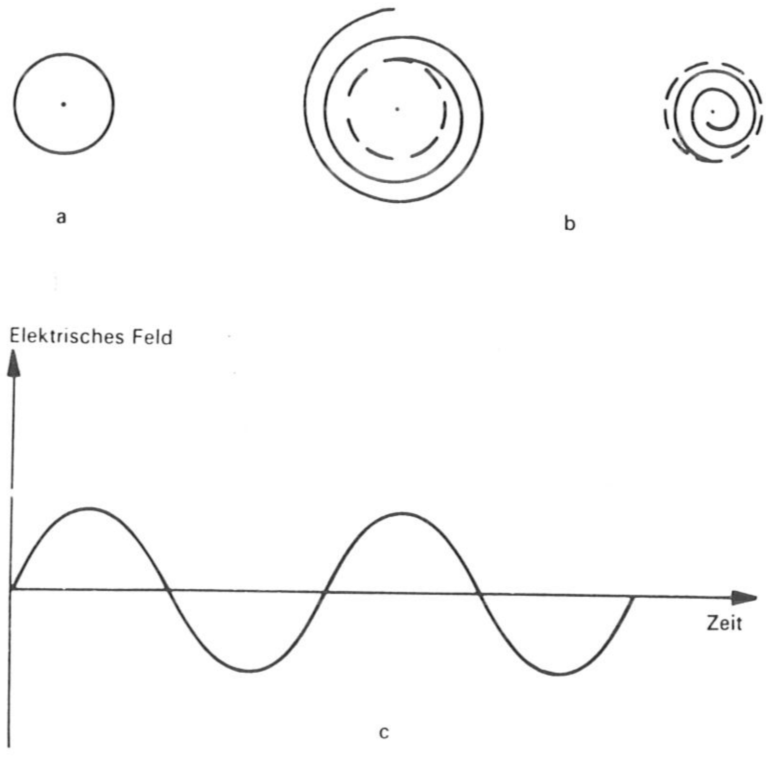

Fig. 2 Spiraling motion due to cyclotron frequency electric field

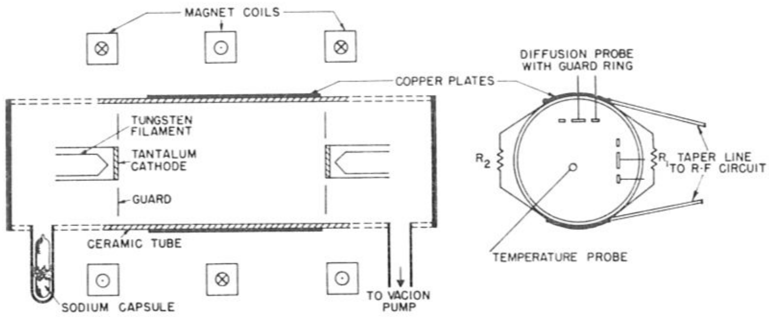

Fig. 3 The experimental tube

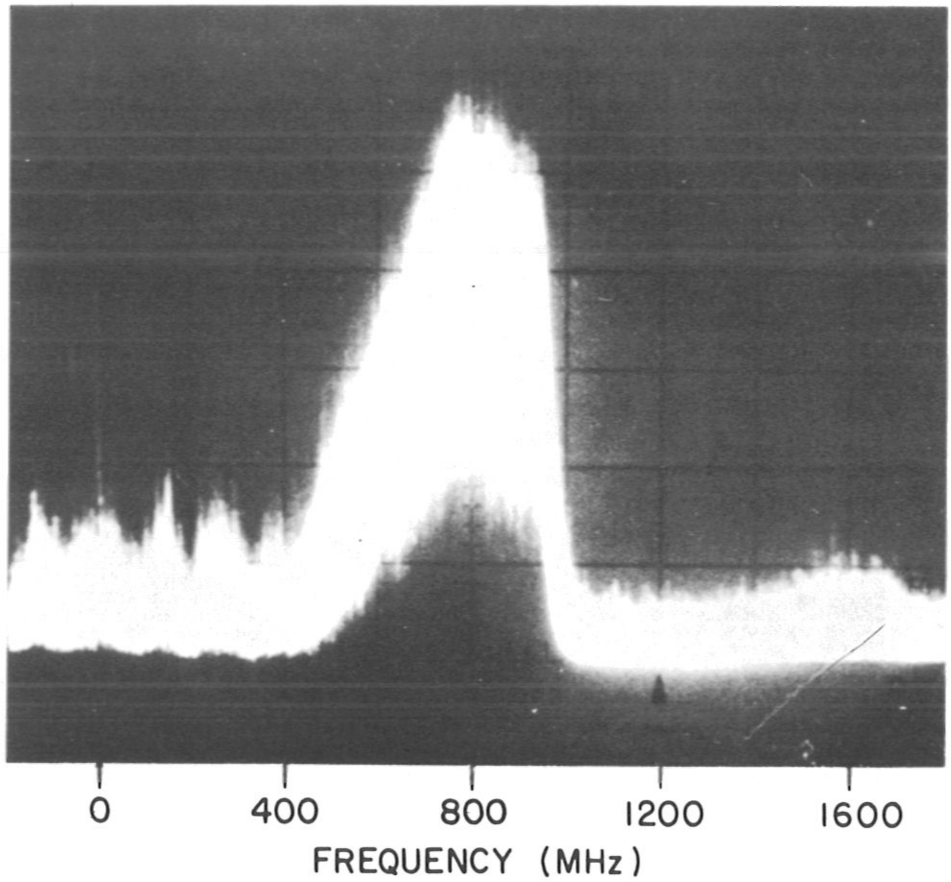

Fig. 4 The power spectrum for experiment of Fig. 3

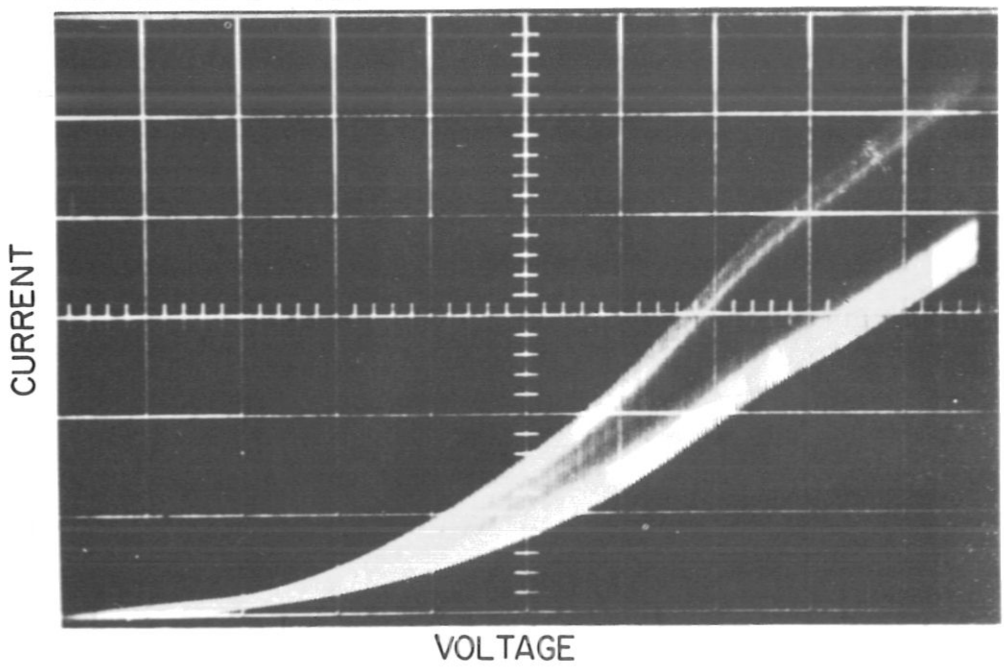

Fig. 5 Langmuir probe curves obtained on an oscilloscope

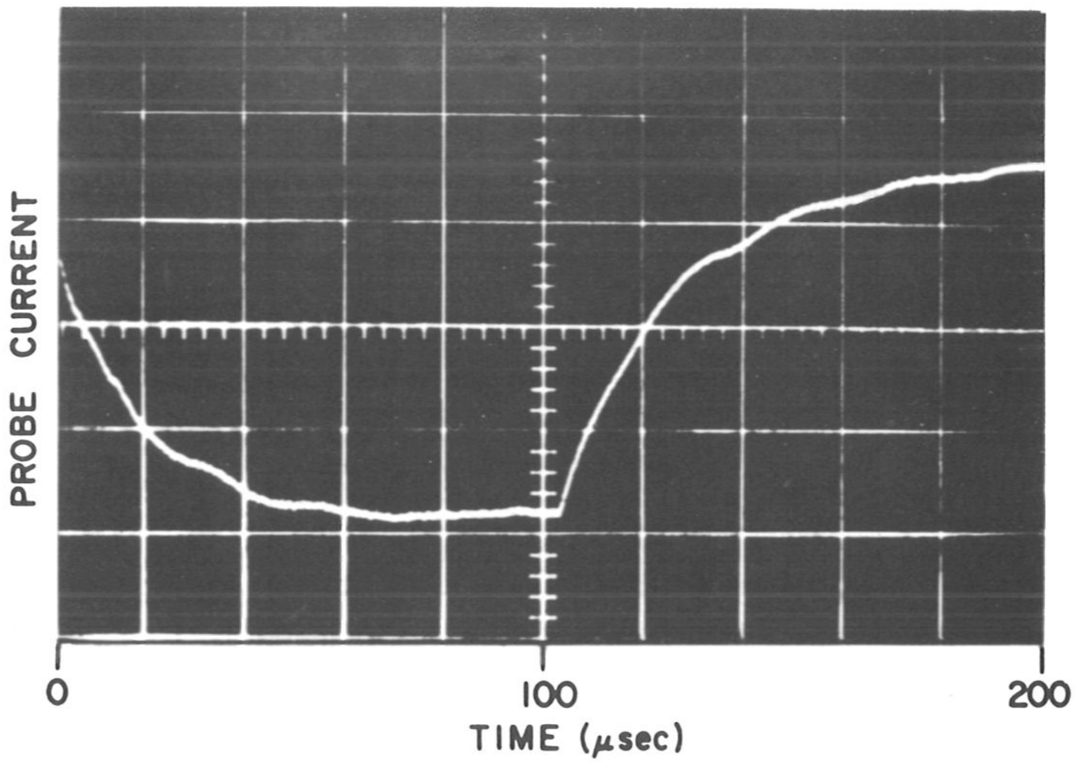

Fig. 6 Plasma heating and cooling characteristic interval on application and removal of the applied field

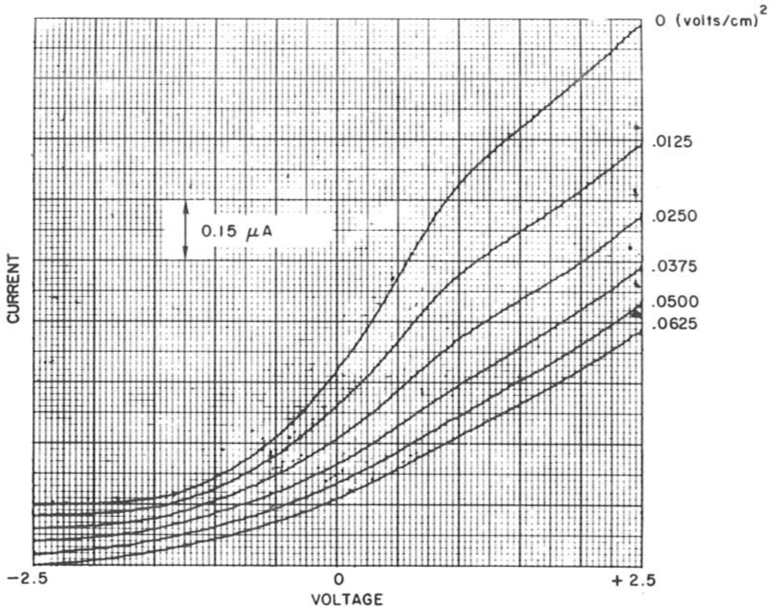

Fig. 7 Langmuir probe curves obtained on an x-y recorder

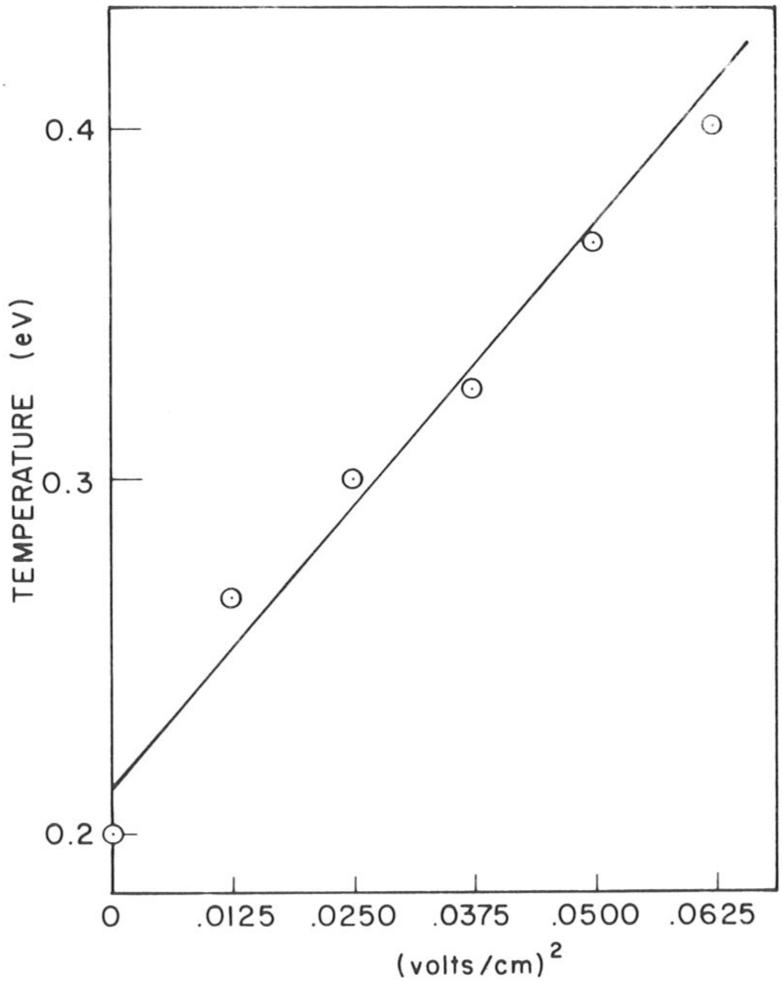

Fig. 8 Applied field strength vs plasma temperature

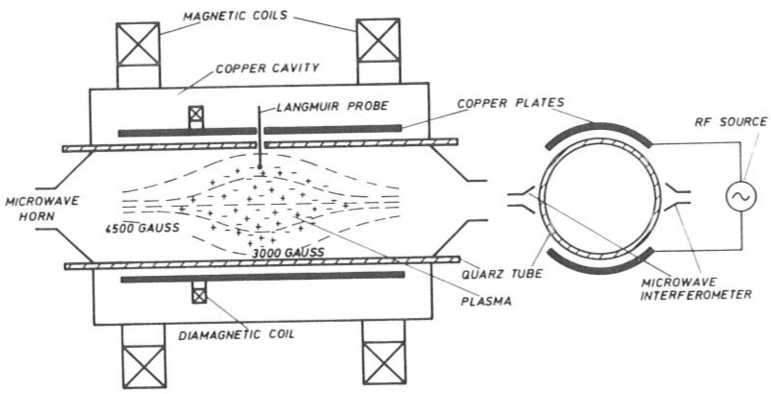

Fig. 9 Schematic of the experimental arrangement

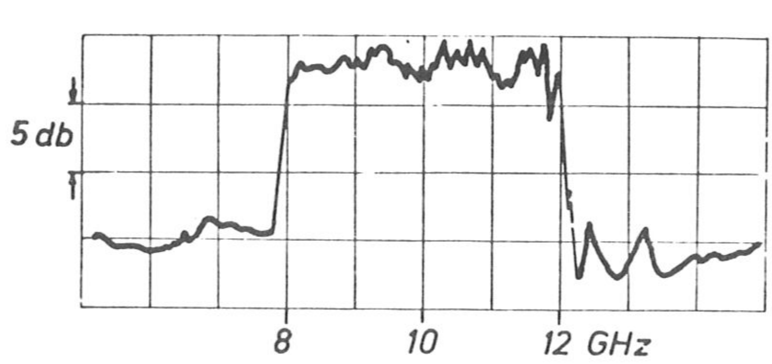

Fig. lo The power spectrum for the experiment of Fig. 9

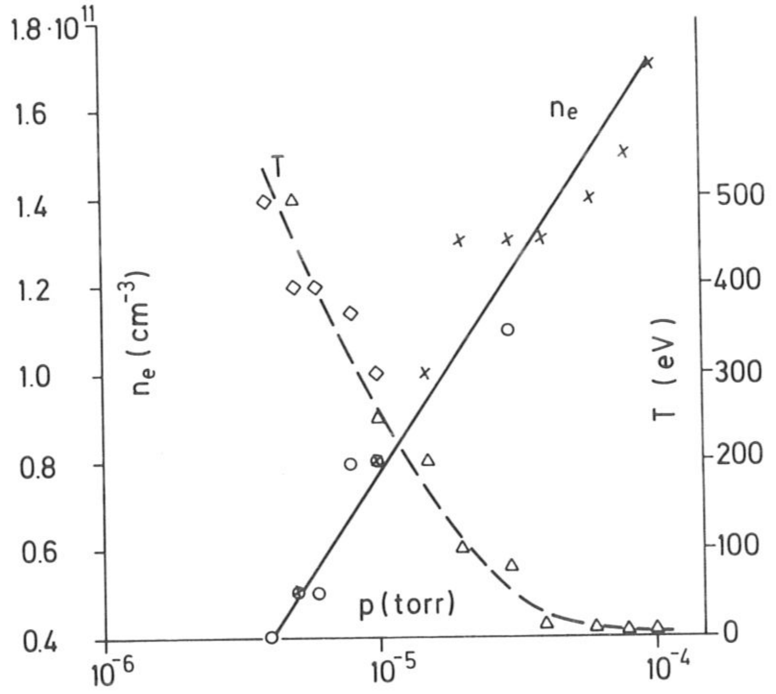

Fig. 11 Plasma density and electron temperature as a function of pressure.

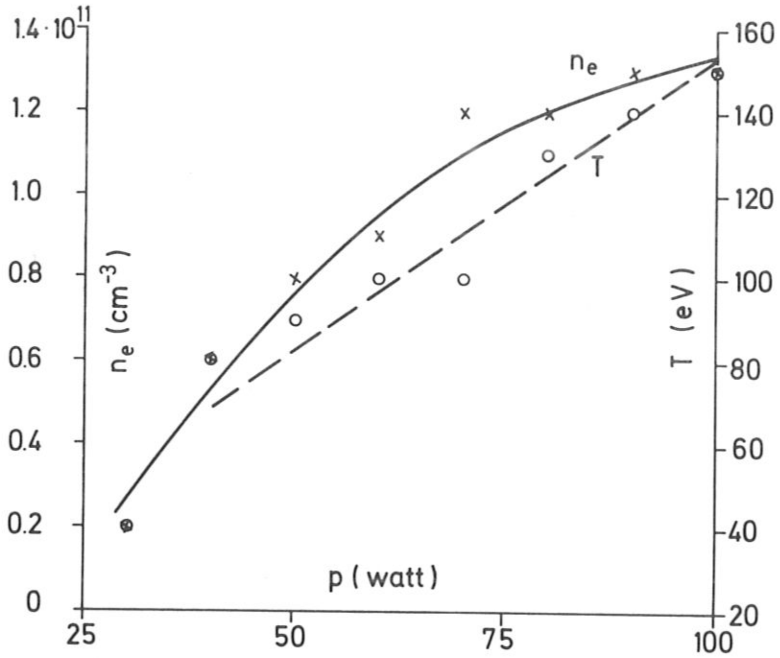

Fig. 12 Plasma density and electron temperature as a function of input noise power.

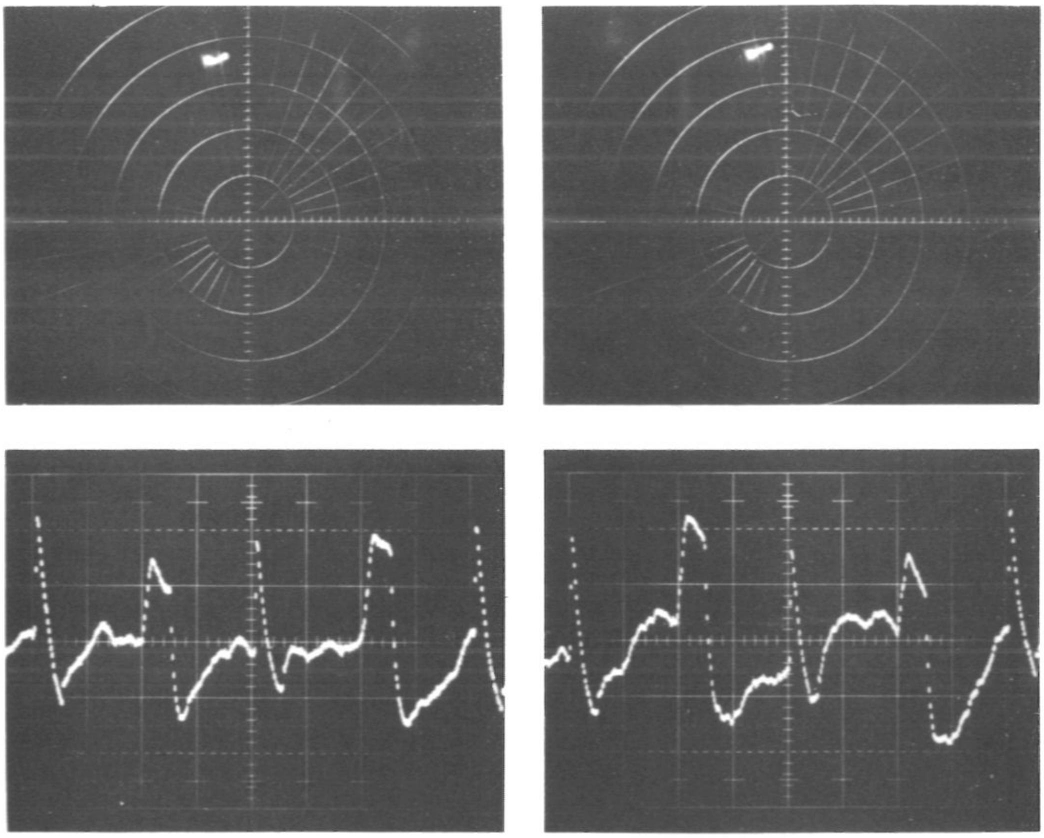

Fig. 13 Typical diamagnetic (flat topped) and calibration signals.

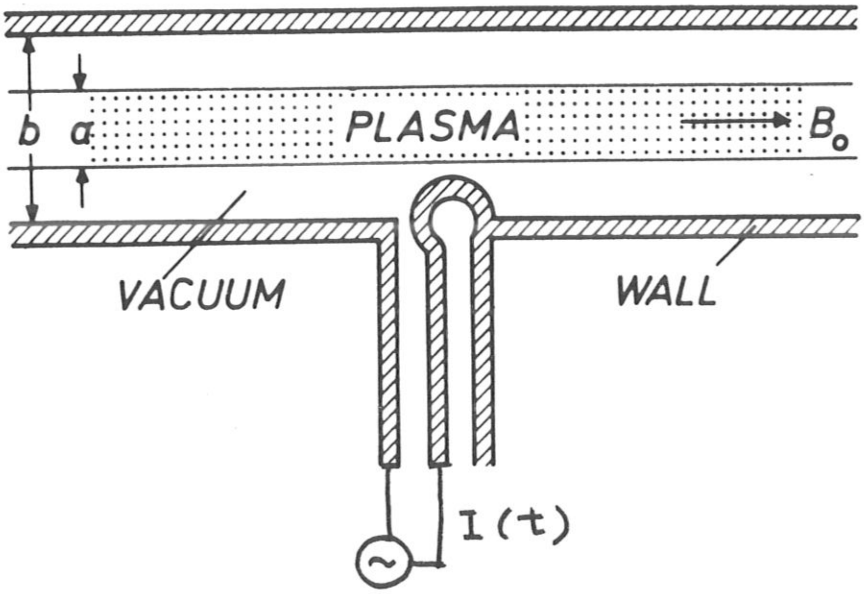

Fig. 14 A plasma filled waveguide driven by a current source.